Short introduction to simplistic algebra, top, Homological Algebra

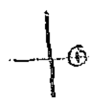
Motto Algebra is good for top, but topology gives intuition & motivation for homological algebra.

simplistic = simplex / simplicial complexes

origin: Eulers solution of the problem of the bridges of Königsberg.

EXAMPLES: $\Delta_0 = \cdot$ $\quad \Delta_1 = —$

$$\Delta_n = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} x_i \geq 0 \ \forall i \\ \sum_{i=0}^{n} x_i = 1 \end{array} \right\}$$

$\Delta_0 =$  , $\Delta_1 =$  , Δ_2 

Make compact spaces by gluing along "faces"

Nice description: Let I be a finite set and $X \subset \mathcal{P}(I)$

s.t. $\left\{ \begin{array}{l} X \text{ closed under Subsets: } \forall J \in X, \forall J' \subset J, J' \in X \\ (\text{Convenience } \bigcup_{J \in X} J = I \end{array} \right\}$

Such an X is called a simplicial complex.

Gives a topological space $|X| = \bigcup_{J \in X}$ convex hull of $\{\delta_j \mid j \in J\} \subseteq \mathbb{R}^I$

plus the induced topology from $\mathbb{R}^I$ ($|X| \subset \mathbb{R}^I$ is closed)

Convex hull: $S \subseteq V_{\mathbb{R}}$ $\quad V_{\mathbb{R}}$ is $\mathbb{R}$ vectorspace

convex hull of S = Smallest convex subset of V that cont S

$= \bigcup \{ t_0 s_0 + \cdots + t_n s_n \mid t \in \Delta_n \}$

Example 1 $|P(\{0,1,\ldots,n\})| = \Delta_n$.

Example 2 $\overset{2}{\underset{0 \quad 1}{\triangle}}$, $I = \{0,1,2\}$, $X = \{ \{0,1\}, \{0,2\}, \{1,2\}, \{0\}, \{1\}, \{2\} \}$

In case of infinite sets I : take collection of finite subsets

then $\varinjlim$ in Top

Homology groups of $|X|$ or rather of X.

take a total ordering of I (orientations are important)

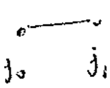
For $i \in \mathbb{Z}$

$$C_i(X) = \mathbb{Z}^{\{J \in X \mid \# J = i+1\}}$$, has $\mathbb{Z}$ basis δ_J $\forall J \in X$ with $\# J = i+1$

$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$, defined by

$$\delta_J \longmapsto \sum_{i=0}^{n} (-1)^i \delta_{J - \{j_i\}} \quad \text{where } J = \{j_0, \dots, j_n\}$$

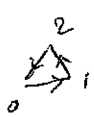
$$j_0 < j_1 < \dots < j_n$$

$\mathbb{Z}$ linear 

$\delta_{\{j_0, j_1\}} \longmapsto \delta_1 - \delta_{j_0}$

we have $\partial_{n-1} \circ \partial_n = 0$ and thus we get

chain complex $\longrightarrow C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$

Ex $X =$ 

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^3 \longrightarrow 0$$

$$\partial_1 = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$C_1(X) = \langle\{\delta_{\{0,1\}} \ \delta_{\{0,2\}} \ \delta_{\{1,2\}}\}\rangle \simeq \mathbb{Z}^3$$

$$C_0(X) = \langle\{ \delta_{\{0\}}, \delta_{\{1\}}, \delta_{\{2\}}\}\rangle$$

$$H_0(X, \mathbb{Z}) = \operatorname{im} \partial_1 = \mathbb{Z}$$

Singular homology.

For $X \in \mathrm{Top}$, $C_n(X) = \mathbb{Z}^{(\mathrm{Top}(\Delta_n, X))}$ = free $\mathbb{Z}$-mod with basis $\mathrm{Top}(\Delta_n, X)$

for any set S

$$\mathbb{Z}^{(S)} = \{ f : S \to \mathbb{Z} \mid f \text{ has finite support} \}$$

$\mathbb{Z}^{(S)}$ has a natural structure of $\mathbb{Z}$ module.

Δ_{n-1} $\quad \Delta_n$

$$\begin{tikzcd}
\{0, \dots, n-1\} \arrow[r, "\sim"] \arrow[rr, bend right, "\text{order preserving}"', "\iota_i"] & \{0, \dots, n\} - \{i\} \arrow[r, hook] & \{0, \dots, n\}
\end{tikzcd}$$

Get $\partial_{n-1} \circ \partial_n = 0$

$H_n^{\mathrm{sing}}(-, \mathbb{Z}) : \mathrm{Top} \longrightarrow \mathbb{Z}\text{-module}$.

Thm. $\exists$ isomorphism of functors

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d] & H_n(X, \mathbb{Z}) \\
{|X|} \arrow[r, mapsto] & H_n^{\mathrm{sing}}(|X|, \mathbb{Z})
\end{tikzcd}$$

Refferences P. May

Hatcher

massey.