TAG, 2010/03/01, Bas Edixhoven, 45 minutes.

<u>Classification of vector bundles on $\mathbb{P}^1_k$ via geometry & cohomology.</u>

Let $k = \bar{k}$.

$\mathbb{P}^1(k)$ as algebr. var.: set $\mathbb{P}^1(k) = \frac{k^2 - \{0\}}{k^\times}$, with Zar. top. and sheaf $\mathcal{O}$ of regular functions.

$\mathbb{P}^1 - \{\infty\} =: U_0$, $U_\infty = \mathbb{P}^1 - \{0\}$. $\mathcal{O}(U_0) = k[x]$, $\mathcal{O}(U_\infty) = k[x^{-1}]$.

As a scheme: add the generic pt. η (useful for taking stalk at η).

Example: $\mathcal{O}_\eta = k(x)$.

<u>Goal</u>: classify the loc. free $\mathcal{O}$-modules of finite rank.

<u>Rank 1.</u>

For $D = \sum_a D_a \cdot a \in \mathbb{Z}^{(\mathbb{P}^1(k))}$ we have $\underbrace{\mathcal{O}(D)}_{\text{loc. free rk 1}} \subset k(x)_{\mathbb{P}^1}$ given by:

$\forall\, U \neq \emptyset$: $(\mathcal{O}(D))U = \{ f \in k(x) \mid \forall a \in U : v_a(f) + D_a \geq 0 \}$

Example: $(\mathcal{O}(\infty))(U_0) = k[x]$, $(\mathcal{O}(\infty))(U_\infty) = x \cdot k[x^{-1}]$.

Vice versa: let $\mathcal{L}$ be loc. free of rank 1, and $0 \neq s \in \mathcal{L}_\eta$.

Then we have $\mathrm{div}(s) := \sum_a v_a(s) \cdot a$, ($v_a(s) = v_a(f)$, where $s = f \cdot s_a$, $\mathcal{L}_a = \mathcal{O}_a \cdot s_a$).

and: $\mathcal{O}(\mathrm{div}(s)) \xrightarrow{\cdot s} \mathcal{L}$ is an isom.

From this:

$$\begin{tikzcd} k(x)^\times \arrow[r, "\mathrm{div}"] & \mathrm{Div}(\mathbb{P}^1) \arrow[r] \arrow[dr, "\deg"'] & \mathrm{Pic}(\mathbb{P}^1) \arrow[d, "\sim"] \\ & & \mathbb{Z} \end{tikzcd}$$

- Each $\mathcal{L}$ is isomorphic to exactly one $\mathcal{O}(n \cdot \infty)$, $n \in \mathbb{Z}$.

<u>Thm.</u> Each loc. free $\mathcal{E}$ is isom. to exactly one

$\mathcal{O}(n_1 \cdot \infty) \oplus \cdots \oplus \mathcal{O}(n_r \cdot \infty)$, with $n_i \in \mathbb{Z}$, $n_1 \leq \cdots \leq n_r$.

Cohomology of $\mathcal{O}(n\cdot\infty)$:

$$\overset{0}{(\mathcal{O}(n\infty))(U_0) \oplus (\mathcal{O}(n\cdot\infty))(U_\infty)} \longrightarrow \overset{1}{(\mathcal{O}(n\infty))(U_0\cap U_\infty)}$$
$$\underset{\displaystyle k[x]\oplus x^n\cdot k[x^{-1}]}{\|} \qquad\qquad \underset{\displaystyle k[x,x^{-1}]}{\|}$$

$$H^0(\mathcal{O}(n\infty)) = k[x]_{\leq n}\ ; \quad H^1(\mathcal{O}(n\cdot\infty)) \xleftarrow{\sim} \bigoplus_{n<i<0} k\cdot x^i$$

In particular: $H^1(\mathcal{O}(n\cdot\infty)) \neq 0 \iff n \leq -2$. $\quad (v_\infty(dx) = -2\ !)$

Let $\mathcal{E}$ be a loc. free $\mathcal{O}$-module of finite rank > 0.
View $\mathcal{E}$ as a sub-$\mathcal{O}$-module of $(\mathcal{E}_\eta)_{\mathbb{P}^1}$, the constant sheaf with stalk $\mathcal{E}_\eta$.

Then, for $n\in\mathbb{Z}$, we have $\mathcal{E}(n\cdot\infty) \subset (\mathcal{E}_\eta)_{\mathbb{P}^1}$, defined by:
$\mathcal{E}(n\cdot\infty)_a = \mathcal{E}_a$ if $a\neq\infty$, $\quad \mathcal{E}(n\cdot\infty)_\infty = x^n\cdot\mathcal{E}_\infty$.

Let $n_r := \max\{n\in\mathbb{Z} \mid H^0(\mathcal{E}(-n\cdot\infty))\neq 0\}$, let $0\neq s\in H^0(\mathcal{E}(-n\cdot\infty))$.
Claim: s has no zero.
Proof: if $s(\infty)=0$ then $s\in H^0(\mathcal{E}(-(n+1)\cdot\infty))$, quod non.
if $s(a)=0$ and $a\neq\infty$, then $\frac{1}{x-a}\cdot s \in H^0(\mathcal{E}(-(n+1)\cdot\infty))$, quod non.

Hence: $\mathcal{O}\xrightarrow{\cdot s}\mathcal{E}(-n\cdot\infty)$ is injective, locally split, hence
$\mathcal{O}\overset{\cdot s}{\rightarrowtail}\mathcal{E}(-n\cdot\infty)\twoheadrightarrow\mathcal{E}'$, exact, with $\mathcal{E}'$ loc. free.

By induction: $\exists!\ m_1,\ldots,m_{r-1}$ in $\mathbb{Z}$, with $m_1\leq\cdots\leq m_{r-1}$
and $\mathcal{E}'\cong\mathcal{O}(m_1)\oplus\cdots\oplus\mathcal{O}(m_{r-1})$.

Claim $\forall i$: $m_i\leq 0$.
Proof. We have $\mathcal{O}(-\infty)\rightarrowtail\mathcal{E}(-(n+1)\infty)\twoheadrightarrow\mathcal{E}'(-\infty)$,
hence $$H^0(\mathcal{O}(-\infty))\rightarrowtail \underset{\displaystyle 0}{\underset{\|}{H^0(\mathcal{E}(-(n+1)\infty))}} \longrightarrow \underbrace{H^0(\mathcal{E}'(-\infty))}_{=0} \longrightarrow \underset{\displaystyle 0}{\underset{\|}{H^1(\mathcal{O}(-\infty))}} \qquad \boxtimes$$

It only remains to show that for $m \in \mathbb{Z}_{\geq 0}$: $\mathrm{Ext}^1(\mathcal{O}(-m\infty), \mathcal{O}) = 0$.

$$\mathrm{Ext}^1(\mathcal{O}(-m\cdot\infty), \mathcal{O}) = \{ \mathcal{O} \rightarrowtail \mathcal{F} \twoheadrightarrow \mathcal{O}(-m\cdot\infty) \} / \sim$$

$$= \{ \mathcal{O}(m\cdot\infty) \rightarrowtail \mathcal{F} \twoheadrightarrow \mathcal{O} \} / \sim$$

Note that for $\mathcal{F}$ an $\mathcal{O}$-module: $H^0(\mathcal{F}) = \mathrm{Hom}(\mathcal{O}, \mathcal{F})$

$$s \longmapsto (1 \mapsto s)$$

Hence: $H^1(\mathcal{F}) = \mathrm{Ext}^1(\mathcal{O}, \mathcal{F})$.

We have seen: for $m \geq -1$: $\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(m)) = 0$. $\boxtimes$

<u>To summarise</u>:

1. every locally free $\mathcal{O}$-module $\mathcal{E}$ on $\mathbb{P}^1_k$ is isomorphic to exactly one $\mathcal{O}(n_1\cdot\infty) \oplus \cdots \oplus \mathcal{O}(n_r\cdot\infty)$, $n_i \in \mathbb{Z}$, $n_1 \leq \cdots \leq n_r$.

2. for all $n, m \in \mathbb{Z}$: $\mathrm{Hom}(\mathcal{O}(n\infty), \mathcal{O}(m\infty)) = k[x]_{\leq m-n}$

3. ——————— $\mathrm{Ext}^1(\mathcal{O}(n\cdot\infty), \mathcal{O}(m\cdot\infty)) = \bigoplus_{m-n < i < 0} k\cdot x^i$

and higher Ext's are zero.