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135 Weighted projective spaces; 2010/04/12

Duong Hoang Dung needs some results on weighted projective spaces in his lecture in the Topics in Arithmetic Geometry seminar of next week, and this text is to provide those results. We start with general stuff concerning graded rings, and will get more geometry and examples toward the end.

We begin with the construction of the *projective spectrum* of a graded ring; see Hartshorne II.2 and EGA II.2 for more details. Unfortunately, I have no references for the special case of weighted projective spaces. Maybe such references can be found in texts on the more general case of toric varieties.

The Proj construction

Let A be a ring (commutative, with unit, as always). A $\mathbb{Z}$-*grading* on A is a direct sum decomposition as $\mathbb{Z}$-modules: $A = \oplus_{i\in\mathbb{Z}} A_i$, such that for all i and j in $\mathbb{Z}$, and all f in A_i and g in A_j the product fg is in A_{i+j}. A *graded ring* is then a ring A together with a grading. A *positively graded ring* is a graded ring A with, for all $i < 0$, $A_i = 0$.

Let now A be a positively graded ring. Then one has a scheme $\mathrm{Proj}(A)$, called the projective spectrum of A. Its points are the homogeneous prime ideals p of A that do not contain $A_{>0} = \oplus_{i>0} A_i$. The topology is given by closed subsets $V_+(I)$ given by homogeneous ideals I, and regular functions are suitable elements of degree 0 in localisations of A. I find it easier to understand the construction by gluing the affine pieces $D_+(f)$ associated with homogeneous elements f of A; so, this, I describe in detail.

Let $d \in \mathbb{Z}_{>0}$, and let $f \in A_d$. Let $\psi_f\colon A \to A_f$ denote the localisation of A with respect to f. Then $A_f = A[x]/(xf-1)$ is $\mathbb{Z}$-graded, as follows: $A[x] = \oplus_{i,j} A_i x^j$, and, because we want xf to be in $A[x]_0$, we decree that $A_i x^j$ is in $A[x]_{i-dj}$. Hence:

$$A[x]_n = \bigoplus_{i-jd=n} A_i x^j.$$

Then $xf-1$ is in $A[x]_0$, hence $(xf-1)$ is a homogeneous ideal, and hence the grading of $A[x]$ induces a grading of A_f, such that $(A_f)_n$ is the set of a/f^j with $a \in A_{n+jd}$.

We let $A_{(f)}$ denote $(A_f)_0$, and define $D_+(f) := \mathrm{Spec}(A_{(f)})$.

Now, if $e \in \mathbb{Z}_{>0}$, and $g \in A_e$, then we have f^e/g^d in $A_{(g)}$, and g^d/f^e in $A_{(f)}$, and natural isomorphisms:

$$(A_{(f)})_{g^d/f^e} = A_{(fg)} = (A_{(g)})_{f^e/g^d}.$$

This means that $D_+(fg)$ is an open piece of $D_+(f)$ and of $D_+(g)$. The scheme that is obtained by gluing all the $D_+(f)$ is then $\mathrm{Proj}(A)$. In fact, if $S \subset \cup_{i>0} A_i$ is such that $V(S) = V(A_{>0})$ in $\mathrm{Spec}(A)$, then the $D_+(f)$ with $f \in S$ cover $\mathrm{Proj}(A)$.

Homogeneous rings for weighted projective spaces

Let $n \in \mathbb{Z}_{\geq 0}$, and put $A = \mathbb{Z}[x_0, \ldots, x_n]$, graded by total degree of monomials. Then $\mathrm{Proj}(A)$ is called the (ordinary) projective space $\mathbb{P}^n_{\mathbb{Z}}$. In this case, $\mathbb{P}^n$ is covered by the $D_+(x_i)$, and $D_+(x_i)$ is the spectrum of the polynomial ring over $\mathbb{Z}[\{x_j/x_i \mid j \neq i\}]$, hence $\mathbb{P}^n$ is obtained by gluing $n+1$ copies of the afine space $\mathbb{A}^n$.

We obtain weighted projective spaces if we give each variable its own weight (other word for degree). Let $n \in \mathbb{Z}_{\geq 0}$, and $d_0, \ldots, d_n$ in $\mathbb{Z}_{>0}$. Then we give $A := \mathbb{Z}[x_0, \ldots, x_n]$ the grading for which A_i has as $\mathbb{Z}$-basis the set of monomials $x_0^{m_0} \cdots x_n^{m_n}$ such that $\sum_j d_j m_j = i$. Then $\mathrm{Proj}(A)$ is called the weighted projective space $\mathbb{P} := \mathbb{P}(d_0, \ldots, d_n)_{\mathbb{Z}}$. In this case, $\mathbb{P}$ is covered by the $D_+(x_i)$, but the structure of the $A_{(x_i)}$ can be more complicated, as we will see in the examples below. In particular, weighted projective spaces can be singular.

Some geometry of weighted projective spaces

We like to see the projective space $\mathbb{P}^n$ as the quotient of the open part $U := \mathbb{A}^{n+1} - \{0\}$ of $\mathbb{A}^n$ by the action of the multiplicative group $\mathbb{G}_{\mathrm{m}}$ acting by scalar multiplication: $\lambda \cdot (a_0, \ldots, a_n) = (\lambda a_0, \ldots, \lambda a_n)$. This action induces the grading on $A := \mathbb{Z}[x_0, \ldots, x_n]$: A_i is the weight i submodule of A, i.e., consists of the f such that for all $\mathbb{Z}$-algebras B and all $\lambda \in B^\times$ and all $(a_0, \ldots, a_n) \in B^n$ we have $f(\lambda a_0, \ldots, \lambda a_n) = \lambda^i f(a_0, \ldots, a_n)$ in B.

We can indeed view $\mathbb{P}^n$ as the quotient of U. We get a morphism $U \to \mathbb{P}^n$ by observing that U is the union of the affine schemes $D(x_i)$, with ring A_{x_i}. For each i the map corresponds to the inclusion $A_{(x_i)} \to A_{x_i}$, which is indeed the subring of invariants for the $\mathbb{G}_{\mathrm{m}}$-action ("weight 0" means "invariant"). Such a quotient, where one replaces rings of affine parts by their subrings of invariants, are called *categorical quotients*, because they have the universal property of a quotient in the category of schemes (see Mumford's GIT for much much much more). In this case, the quotient map is particularly nice, as it has sections Zariski locally: $D(x_i) \to D_+(x_i)$ has the section corresponding to $A_{x_i} \to A_{(x_i)}$ that sends x_j to x_j/x_i, or, geometrically, $U \to \mathbb{P}^n$ induces an isomorphism from the closed subscheme $V(x_i - 1)$ of U to $D_+(x_i)$. This means that $U \to \mathbb{P}^n$ is also a *quotient in the sense of Grothendieck*, namely, as sheaves for some Grothendieck topology (in this case one can simply use the Zariski topology, but most often stronger topologies such as the étale topology or flat topologies are needed).

We will now show that weighted projective spaces are categorical quotients, but we will see from some examples that they need not be quotients in the sense Grothendieck.

Let $n \in \mathbb{Z}_{\geq 0}$, and $d_0, \ldots, d_n$ in $\mathbb{Z}_{>0}$. Let $A = \mathbb{Z}[x_0, \ldots, x_n]$ where x_i has degree d_i. We let $\mathbb{G}_{\mathrm{m}}$ act on $U = \mathbb{A}^{n+1} - \{0\}$ by $\lambda{\cdot}(a_0, \ldots, a_n) = (\lambda^{d_0}a_0, \ldots, \lambda^{d_n}a_n)$. We cover U by the $D(x_i)$. Let i be in $\{0, \ldots, n\}$. Then $D(x_i) = A_{x_i} = \mathbb{Z}[x_0, \ldots, x_n, x_i^{-1}]$, and $A_{(x_i)}$ is its subring of $\mathbb{G}_{\mathrm{m}}$-invariant elements. This means that $U \to \mathbb{P}$ is a categorical quotient for the $\mathbb{G}_{\mathrm{m}}$-action. On the open subset $T := \mathbb{G}_{\mathrm{m}}^{n+1}$ of U, it has a section (exercise!), and hence, on this open set, is even a quotient in the sense of Grothendieck. This big torus T whose action commutes with that of $\mathbb{G}_{\mathrm{m}}$ shows that weighted projective spaces are toric varieties: they admit an action by a torus with a dense open orbit.

Exercise: show that for any field k one has:

$$\mathbb{P}(d_0, \ldots, d_n)(k) = (k^{n+1} - \{0\})/\sim,$$

where $x \sim y$ if and only if there is a λ in $\overline{k}^{\times}$ such that for all i: $y_i = \lambda^{d_i}x_i$.

Example: $\mathbb{P}(2,3)$

We have $A = \mathbb{Z}[x_0, x_1]$ with x_0 of degree 2 and x_1 of degree 3. In this case, $\mathbb{P}$ is covered by $D_+(x_0)$ and $D_+(x_1)$. We have $D_+(x_0) = \mathrm{Spec}(\mathbb{Z}[x_0, x_1, x_0^{-1}]_0)$. A little calculation shows that $A_{(x_0)} = \mathbb{Z}[x_1^2/x_0^3]$, and that $A_{(x_1)} = \mathbb{Z}[x_0^3/x_1^2]$. This clearly shows that $\mathbb{P}(2,3) = \mathbb{P}^1$. However, the $\mathbb{C}[\varepsilon]$-point $(1 : \varepsilon)$ of $\mathbb{P}^1$ cannot be lifted to a $\mathbb{C}[\varepsilon]$-point of U (exercise); this implies that the morphism $U \to \mathbb{P}$ is not smooth. This last thing comes from the fact that, for the $\mathbb{G}_{\mathrm{m}}$-action on U, the stabilisers are not constant. On $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ it is trivial, but on $\mathbb{G}_{\mathrm{m}} \times \{0\}$ it is μ_2 and on $\{0\} \times \mathbb{G}_{\mathrm{m}}$ it is μ_3. On the positive side, the morphism $U \to \mathbb{P}$ is still flat in this case, as A_{x_i} is even a free $A_{(x_i)}$-module.

Example: $\mathbb{P}(1,2,3)$

Now $A = \mathbb{Z}[x_0, x_1, x_2]$, with weights $(1,2,3)$.

We start with $D_+(x_0)$. We have $A_{x_0} = \mathbb{Z}[x_0, x_1, x_2, x_0^{-1}]$, and x_0^{-1} has weight -1. Hence $A_{(x_0)} = \mathbb{Z}[x_1/x_0^2, x_2/x_0^3]$, hence $D_+(x_0)$ is isomorphic to $\mathbb{A}^2$, and the morphism $D(x_0) \to D_+(x_0)$ has a section and is a quotient in the sense of Grothendieck. All this is nice, because $d_0 = 1$.

But let us now consider $D_+(x_1)$. We have $A_{x_1} = \mathbb{Z}[x_0, x_1, x_2, x_1^{-1}]$, and x_1^{-1} has weight -2. A little calculation shows that $A_{(x_1)}$ is the subring of A_{x_1} generated by the three elements $u = x_0^2/x_1$, $v = x_0x_2/x_1^2$, and $w = x_2^2/x_1^3]$, and that the ideal of relations between these elements is generated by $uw - v^2$. Hence $D_+(x_1)$ is the spectrum of $\mathbb{Z}[u,v,w]/(uv - w^2)$, the "standard cone" in $\mathbb{A}^3$. In particular, it is singular, and (exercise) $D(x_1) \to D_+(x_1)$ is not flat. Also an exercise: decide wether or not for every $\mathbb{Z}$-algebra B and every B-point P of $D_+(x_1)$ there is a faithfully flat B-algebra B' so that P can be lifted to a B'-point of $D(x_1)$. If the answer is no, then $U \to \mathbb{P}(1,2,3)$ is not a quotient in the sense of Grothendieck.

Local description of $\mathbb{P}(d_0,\ldots,d_n)$ in terms of quotients by a finite group

We consider the general case of weighted projective spaces: let $n \in \mathbb{Z}_{\geq 0}$, $d_0,\ldots,d_n$ in $\mathbb{Z}_{>0}$. We will now give a useful description of $D_+(x_i)$, by first taking the quotient by μ_{d_i}. The example $\mathbb{P}(1,2,3)$ made me notice this trick; in terms of graded rings, it corresponds to replacing A by $A^{(d_i)} := \oplus_m A_{d_i m}$, which does not change the projective spectrum (no arguments needed for this, at this stage, as we have our categorical quotient interpretation). So we say:

$$A_{(x_i)} = (A_{x_i})^{\mathbb{G}_m} = ((A_{x_i})^{\mu_{d_i}})^{\mathbb{G}_m}.$$

Taking the μ_{d_i} invariants is, geometrically, taking the quotient for the action of μ_{d_i} on $D(x_i)$ given by: $\lambda\cdot(a_0,\ldots,a_n) = (\lambda^{d_0}a_0,\ldots,\lambda^{d_n}a_n)$, i.e., by the weights seen as elements of $\mathbb{Z}/d_i\mathbb{Z}$. Note that the action on the ith coordinate is trivial, so that we can describe the siuation as follows: $D(x_i) \cong \mathbb{A}^n \times \mathbb{G}_m$, and the action of μ_{d_i} comes from the action on $\mathbb{A}^n$ with weights d_j ($j \neq i$). As taking quotients commutes with flat base change, we see that:

$$\mu_{d_i}\backslash D(x_i) = (\mu_{d_i}\backslash\mathbb{A}^n) \times \mathbb{G}_m,$$

and hence:

$$D_+(x_i) = \mathbb{G}_m\backslash(\mu_{d_i}\backslash\mathbb{A}^n \times \mathbb{G}_m) = \mu_{d_i}\backslash\mathbb{A}^n.$$

Weighted projective spaces are projective, hence proper

Let us now finally state the result that Duong will need next week. We consider the general case of a weighted projective space: let $n \in \mathbb{Z}_{\geq 0}$, $d_0,\ldots,d_n$ in $\mathbb{Z}_{>0}$. Let d be the least common multiple of the d_i, and put $e_i = d/d_i$. Let $G = \prod_i \mu_{e_i}$, which we let act on $\mathbb{A}^{n+1}$ by coordinate-wise multiplication. This action commutes with that of $\mathbb{G}_m$ given by the weights d_i. Both actions preserve the $D(x_i)$. Hence we get a commutative diagram for the quotient by $G \times \mathbb{G}_m$: first take the quotient by G, and then by $\mathbb{G}_m$, or vice versa. If we first take the quotient by G we get $U \to U$, $(a_0,\ldots,a_n) \mapsto (a_0^{e_0},\ldots,a_n^{e_n})$, and the induced action of $\mathbb{G}_m$ on this quotient is the action by scalar multiplications, which gives the quotient $\mathbb{P}^n$. Taking the quotient by the $\mathbb{G}_m$-action first gives G acting on $\mathbb{P}(d_0,\ldots,d_n)$ with quotient $\mathbb{P}^n$. As the quotient for the action of a finite group is a finite morphism, and as finite morphisms are proper, we see that $\mathbb{P}(d_0,\ldots,d_n)$ is proper. It is even projective: the inverse image of $\mathcal{O}(1)$ is ample.