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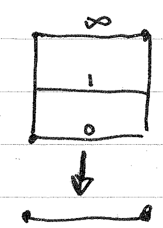
2010/11/12 (3 x 45 minutes)

$\tilde{\mathcal{P}}$ ← compatible with notation from Olsson's book. I hope

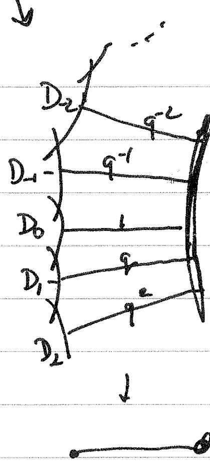
The Tate curve over $\mathbb{Z}[[q]]$.



$\mathbb{P}^1_{\mathbb{Z}[[q]]} \supset \mathbb{G}_{m,\mathbb{Z}[[q]]}$, acts $\rightsquigarrow$

$\mathbb{Z}[q, t, t^{-1}]$

picture for $\mathbb{Q}[[q]]$; imagine $\mathrm{Spec}(\mathbb{Z})$ in 3rd dim.



obtained as direct limit of inverses of blow up maps on a locus where they are defined.

Removing double points $\rightsquigarrow$ "Néron-model of $\mathbb{G}_m$".

$\mathbb{Z}$ acts: $\cdot q^{\mathbb{Z}}$; extends transl. on $\mathbb{G}_{m,\mathbb{Z}[q,q^{-1}]}$s

$(q\cdot)^*: t \mapsto q\cdot t$.

$$\mathrm{div}(q) = \sum_{i\in\mathbb{Z}} D_i \quad ; \quad \mathrm{div}(t) = \sum_{i\in\mathbb{Z}} i\cdot D_i$$

local coordinates: D_n: $q^{-n}\cdot t$, D_{n+1}: $q^{n+1}\cdot t^{-1}$ (indeed: $q^{-n}\cdot t\cdot q^{n+1}\cdot t^{-1} = q$.)

We want a $\mathbb{Z}$-equivariant line bundle; actually it is also $\mathbb{G}_{m,\mathbb{Z}[[q]]}$-invariant, whose degree is 1 on every D_i. $(D_i \cong \mathbb{P}^1_{\mathbb{Z}})$.

We try: $D = \sum_{i\in\mathbb{Z}} a_i\cdot D_i$. Note: $D_i\cdot D_j = \begin{cases} 0 & \text{if } |i-j|>1 \\ 1 & \text{if } |i-j|=1 \\ -2 & \text{if } |i-j|=0 \end{cases}$ bec. $\mathrm{div}(q) = \sum_i D_i$.

Then we must solve: $\forall n\in\mathbb{Z}$: $1 = D_n\cdot D = a_{n-1} - 2a_n + a_{n+1}$.

We choose: $a_0 = a_1 = 0$. Then $a_n = \binom{n}{2} = \frac{1}{2}\cdot n\cdot(n-1)$.

So: $D = \sum_{i\in\mathbb{Z}} \binom{i}{2}\cdot D_i$, $\mathcal{L} := \mathcal{O}(D) \subset$ constant sheaf $\mathrm{Frac}(\mathbb{Z}[q,t])$.

$\mathbb{Z}$-action on $(\tilde{\mathcal{P}}, \mathcal{L})$: just $q\cdot$: $\left(\begin{array}{l} \mathcal{L}(U) \xleftarrow{(q\cdot)^*_{\mathcal{L}}} \mathcal{L}(qU) \\ \mathcal{O}(U) \xleftarrow{(q\cdot)^*} \mathcal{O}(qU) \\ U \xrightarrow{q\cdot} qU \end{array}\right)$ to be defined: choose a function to multiply by. functions: $q\cdot t \leftarrow t$

For $(q\cdot)^*D = \sum_i \binom{i}{2}(q\cdot)^*D_i = \sum_i \binom{i}{2} D_{i-1}$, hence $D - (q\cdot)^*D = \sum_i \left(\binom{i}{2} - \binom{i+1}{2}\right)\cdot D_i$

$= \sum_i -i\cdot D_i = -\mathrm{div}(t) = \mathrm{div}(t^{-1})$. So $(q\cdot)^*_{\mathcal{L}}$ on $\mathcal{L}$: $f \mapsto ((q\cdot)^*f)\cdot t^{-1}$,

$-t^{-1}$?! yes

e.g. $(q\cdot)^*_{\mathcal{L}}$: $q^n\cdot t^m \mapsto q^n\cdot(qt)^m\cdot t^{-1}$.

$-t^{-1}$?

Now we want to make a $\Theta \in \mathcal{L}(\tilde{P})$ that is $\mathbb{Z}$-invariant, and has divisor $\sum_{i\in\mathbb{Z}} q^i$. For this, we must q-adically complete (power series in q) (not in t!).

Note: on $D_0 = \mathbb{P}^1_{\mathbb{Z}}$ coord. t, $\mathcal{L} = \mathcal{O}(-\infty)$, and ~~on~~ around $(D_0, 1)$, $\mathcal{L} = \mathcal{O}$. ~~So~~ and t has divisor $(q\cdot)^*D - D$, so $t \in \mathcal{L}(\tilde{P})$ and $\mathrm{div}(t) = (q\cdot)^*D$ so that means that "t decreases rapidly" and we will be able to sum.

So: $t \xmapsto{(q\cdot)^*_{\mathcal{L}}} (-1)^1\cdot q\cdot t^2 \longmapsto (-1)^2\cdot q^{1+2}\cdot t^3 \longmapsto (-1)^3\cdot q^{1+2+3}\cdot t^4 \longmapsto \cdots (-1)^{n-1}\cdot q^{\binom{n}{2}}\cdot t^n$

$\hookleftarrow (-1)^{-1}\cdot q^0\cdot t^0 \longleftarrow (-1)^{-2}\cdot q^1\cdot t^{-1} \longleftarrow \cdots$

$\varprojlim_n \mathbb{Z}[q][t,t^{-1}]/(q^n)$

$\parallel$

So: $\Theta := -\sum_{n\in\mathbb{Z}} ((q\cdot)^*_{\mathcal{L}})^n t = \sum_{n\in\mathbb{Z}} (-1)^n\cdot q^{\binom{n}{2}}\cdot t^n \in \mathbb{Z}[t,t^{-1}][[q]]$

Note: $\Theta(1) = \sum_{n\in\mathbb{Z}} (-1)^n\cdot q^{\binom{n}{2}} = 0$: $\binom{n}{2} = \binom{1-n}{2}$.

Now everything is ready for taking the quotient by $q^{\mathbb{Z}}$, over the base rings $\mathbb{Z}[q]/(q^n)$, for all $n \geq 0$ (the action is discrete enough, for the Zariski topology). That gives schemes $P_n / \mathbb{Z}[q]/(q^n)$, that make ~~an inductive~~ system

$$\begin{array}{ccc} P_n & \rightarrowtail & P_{n+1} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, \mathbb{Z}[q]/(q^n) & \rightarrow & \cdots \end{array}$$

(The limit is a formal scheme).

$\mathcal{L}^{\otimes 3}$ gives compatible embeddings in(to) $\mathbb{P}^2_{\mathbb{Z}[[q]]/(q^n)}$, and the limit of the Weierstrass equations defines a curve $\mathrm{Tate}(q)/\mathbb{Z}[[q]]$ in $\mathbb{P}^2_{\mathbb{Z}[[q]]}$.

(To do: product formula for Θ?) (log structure?)

exercise: Put $\psi = \prod_{n>0}(1-q^n t^{-1})\cdot(1-t)\cdot\prod_{n>0}(1-q^n t)$. Compute $\mathrm{div}(\psi)$, and prove that $(q\cdot)^*_{\mathcal{L}}\psi = \psi$

conclude that $\exists!\, u \in \mathbb{Z}[[q]]^\times$ s.t. $\psi = u\cdot\Theta$.

§ 4.6. The moduli problem. (p. 121)

Let $g \in \mathbb{Z}_{\geq 0}$.

$\mathcal{K}_g :=$ the fibered cat. $/\mathbb{Z}$ s.t. $\forall$ scheme B:

$\mathcal{K}_g(B) =$ the groupoid with objects:

$$(G, M_B, f:(X, M_X) \to (B, M_B), L, \Theta, \rho)$$

where: (i) M_B is a fine log str. on B ($M_B \xrightarrow{\alpha} O_B$ s.t. $\alpha^{-1} O_B^\times \xrightarrow{\sim} O_B^\times$,

(ii) f is ~~..~~ log smooth, and proper

(iii) ~~..~~ L is relatively ample inv. O_X-module on X/B.

(iv) G/B semiabelian rel. dim. g, ρ: G-action on f.

(v) $\Theta \in (f_* L)(B)$

s.t. $\forall\ \bar{s} \to B$ (geom. pt.):

(vi) $\Theta_{\bar{s}} \in L_{\bar{s}}(X_{\bar{s}})$ does not vanish on any $G_{\bar{s}}$-orbit

(vii) $(G_{\bar{s}}, M_B|_{\bar{s}}, X_{\bar{s}}, M_{X_{\bar{s}}}, L_{\bar{s}}, G_{\bar{s}}\text{-action})$ is isom. to the saturation of a standard family over $\bar{s}$.

Thm (i) $\mathcal{K}_g$ is a proper alg. stack $/\mathbb{Z}$, with finite diagonal, and containing $\mathcal{A}_g$ as dense open substack (locus where $M_{\mathcal{K}_g}$ is trivial).

(ii) ~~..~~ $(\mathcal{K}_g, M_{\mathcal{K}_g})$ is log smooth $/\mathbb{Z}$; in part., $\mathcal{K}_g$ has toroidal sing's.

(iii) $\mathcal{K}_g$ is isom. to the normalisation of the main component of $\mathcal{A}_g^{Alex}$.

must be explained. (but note: for this def.'n, it is only ~~matters~~ ~~to~~ needed with $B = \bar{s}$)

For $g=1$: ell. curve $/\bar{s}$, or $\propto$: $\mathbb{P}^1/0 \sim \infty$, $\mathbb{G}_m$. (rigidified)

First, let us look at how to embed $\mathcal{A}_g$ into it.

I think: $A \xrightarrow[\sim]{\lambda} A^t$ on B.

$$\begin{tikzcd} A \arrow[r, hook] & \mathrm{Pic}_{A^t/B} \arrow[r] & \mathrm{Hom}(A^t, A) \\ & P \arrow[u, "i"] \arrow[r] \arrow[ur, phantom, "\square"] & B \arrow[u, "\text{``}\lambda^{-1}\text{''}"'] \end{tikzcd}$$

$\mathcal{L}^{univ}$ on $A^t \times_B \mathrm{Pic}_{A^t/B}$

~~...~~ $i^* \mathcal{L}^{univ}$ on $A^t \times_B P \xrightarrow{pr_2} P \xrightarrow{p} B$

And from (A, P, L) to (A, λ):
$\lambda_L : A \to \mathrm{Pic}^0_{P/B} = A^t$ (p. 3).

~~...~~ $L' := pr_{2*} i^* \mathcal{L}^{univ}$

$L := L' \otimes_{O_P} (p^* p_* L')^{-1}$.

$\Theta :=$ the taut. ~~.....~~ gener. section of $p_* L$.

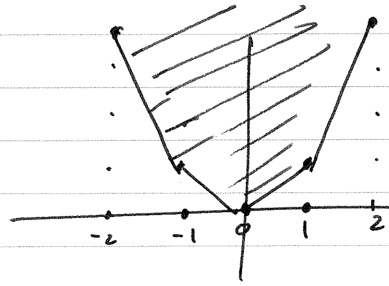
§4.1 The standard construction.

4.1.1 Show on screen; polytope (in $X_{\mathbb{R}}$): convex hull of finite subset of $X_{\mathbb{R}}$
integral: ______________________ X.

Get pairings from $a: X \to \mathbb{R}$ of degree ≤ 2, with pos. def. a_2.
$G_a := \{(x, a(x)) \in X \times \mathbb{R} : x \in \mathbb{R}\}$, hull$(G_a)$ its convex hull
$f_a: X_{\mathbb{R}} \to \mathbb{R}: \; x \mapsto \min\{y \in \mathbb{R} : (x,y) \in G_a\}$
Then $S_a :=$ set of domains of linearity of f_a, + intersections.

Suggested examples: $X = \mathbb{Z}$, $a(n) = n^2$; $X = \mathbb{Z}^2$, $a_\varepsilon(n,m) = n^2 + \varepsilon nm + m^2$
$\varepsilon \in \{-1, 0, 1\}$ isom.

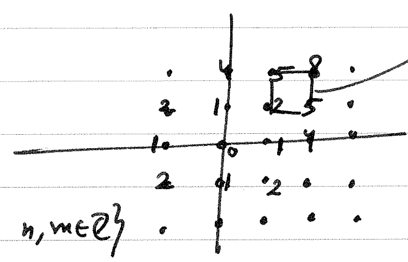
$X = \mathbb{Z}$, $a(n) = n^2$:



Then $S = \{[n, n+1] : n \in \mathbb{Z}\} \cup \{\{n\} : n \in \mathbb{Z}\} \cup \{\emptyset\}$.

$$f_a(x) = \lfloor x \rfloor^2 + \left((\lfloor x \rfloor + 1)^2 - \lfloor x \rfloor^2\right) \cdot (x - \lfloor x \rfloor) = \lfloor x \rfloor^2 + (2\lfloor x \rfloor + 1) \cdot (x - \lfloor x \rfloor)$$

$X = \mathbb{Z}^2$, $a(n,m) = n^2 + m^2$:



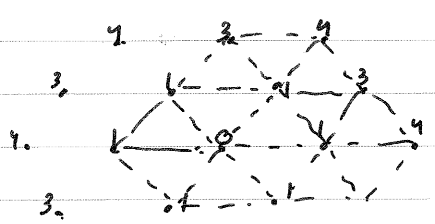
note: $5+5 = 2+8$ the 4 points $a(1,1)$, $a(2,1)$, $a(1,2)$, $a(2,2)$ lie in a plane.

Claim: $S = \{[n, n+1] \times [m, m+1] : n, m \in \mathbb{Z}\}$ + intersections.

Exercise: prove that this is correct. (for me it came down to: $\forall a \in \mathbb{Z}: \; a \leq a^2$.).

Put $f(x,y) = \lfloor x \rfloor^2 + \lfloor y \rfloor^2 + (2\lfloor x \rfloor + 1)(x - \lfloor x \rfloor) + (2\lfloor y \rfloor + 1)(y - \lfloor y \rfloor)$
and prove that $\forall (n,m) \in \mathbb{Z}^2, \forall (a,b) \in \mathbb{Z}^2$:
$$n^2 + m^2 + (2n+1) \cdot a + (2m+1) \cdot b \leq (n+a)^2 + (m+b)^2.$$

$X = \mathbb{Z}[\zeta] \subset \mathbb{C}$, $\zeta = e^{2\pi i/6}$, $a(x) = |x|^2 = N(x)$.

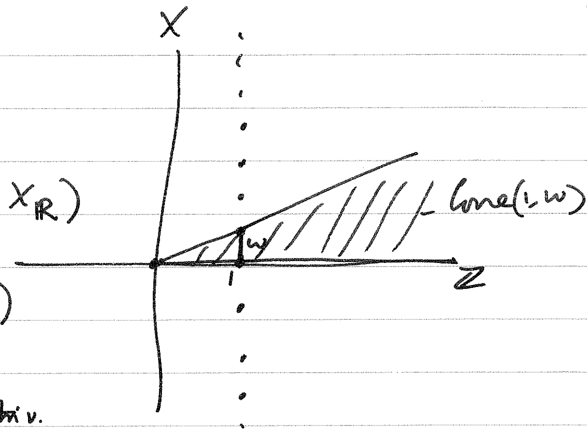


Then $S = \{\text{triangles}\} \cup \{\text{intersections}\}$.
Proof: came down to: $\forall a, b \in \mathbb{Z}: \; a^2 + ab + b^2 \geq a + b$.

Explain X-invariant S's, lemma 4.1.2. "$\forall y \in X: \; a(x+y) - a(x)$ is linear in x."

on the screen.

4.1.3. S integral regular paving of $X_{\mathbb{R}}$.



$\mathbb{X} := \mathbb{Z} \oplus X$

$\mathrm{Cone}(1,X) = \sum_{x \in X} \mathbb{R}_{\geq 0} \cdot x = \{(0,0)\} \cup (\mathbb{R}_{>0} \times X_{\mathbb{R}})$

$P := \mathbb{X} \cap \mathrm{Cone}(1,X) = \{(0,0)\} \cup (\mathbb{Z}_{>0} \times X)$

$X \circlearrowright \mathbb{X}$ linearly, by transl. on $(1,X)$, triv. on $(0,X)$. $y \in X$: $(1,0) \mapsto (1,y)$, $(d,x) \mapsto (d, x+dy)$. $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$

$\rho : P \longrightarrow \varinjlim_{w \in S} \left(\mathrm{Cone}(1,w)^{gp}\right)$ ← this needs some contemplation. the limit.

$\mathrm{Cone}(1,w)$ is a monoid, $\mathrm{Cone}(1,w)^{gp}$ the associate group, it is $\mathbb{R}\cdot(1,w)$, an $\mathbb{R}$-vect. space of dim. $\dim(w)+1$.

$w \mapsto \mathrm{Cone}(1,w)^{gp}$: system of $\mathbb{R}$-vect. spaces.

+ ~~inclusions~~ $\quad \mathrm{I}$

$\eta \mapsto \mathrm{Cone}(1,\eta)^{gp}$ $\qquad \varinjlim$ in cat. of $\mathbb{Z}$-modules or $\mathbb{R}$-vect. spaces, that's the same.

<u>Example</u> $X = \mathbb{Z}$, $a(n) = n^2$. $\mathrm{Cone}(1,[n,n+1])^{gp} = \mathbb{R}\cdot(1,n) \oplus \mathbb{R}\cdot(1,n+1)$

$\mathrm{Cone}(1,\{n\})^{gp} = \mathbb{R}\cdot(1,n)$

The system of the $\mathrm{Cone}(1,w)^{gp}$ has compatible bases! $((1, \mathrm{extr}(w)))$

So $\varinjlim_{w \in S} \mathrm{Cone}(1,w)^{gp}$ = the $\mathbb{R}$-vect. sp. with basis $\varinjlim_{w \in S} (1, \mathrm{extr}(w)) = \mathbb{R}[X] = \mathbb{R}^{(\mathbb{Z})}$

<u>In a diagram:</u>

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \{(1,1)\} & \rightarrowtail & \{(1,1),(1,2)\} & \searrow & \text{that's the limit (in } \underline{\text{Set}}\text{).} \\ & \searrow & \{(1,0),(1,1)\} & \rightarrowtail & (1,\mathbb{Z}) \\ \{(1,0)\} & \searrow & \{(1,-1),(1,0)\} & \nearrow & \\ \vdots & & \vdots & & \end{array}$$

$X = \mathbb{Z}^2$, $a(n,m) = n^2+m^2$ $\quad \mathrm{Cone}(1, \square_{(n,m)})^{gp} = \dfrac{\mathbb{R}\cdot(1,n,m) \oplus \mathbb{R}\cdot(1,n+1,m) \oplus \mathbb{R}\cdot(1,n,m+1) \oplus \mathbb{R}(1,n+1,m+1)}{(1,n,m)+(1,n+1,m+1) = (1,n+1,m)+(1,n,m+1)}$

related to $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

$X = \mathbb{Z}[\zeta]$, a = norm,

$\mathrm{Cone}(1,\Delta_x) = \mathbb{R}^{\mathrm{extr}(\Delta_x)}$

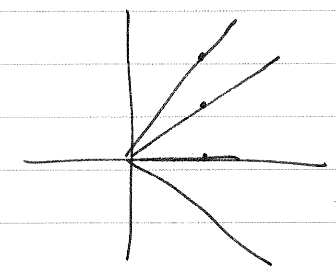
So, here: $\varinjlim_{w \in S} \mathrm{Cone}(1,w)^{gp} = \mathbb{R}^{(X)}$.

$\rho : P \longrightarrow \varinjlim_{w \in S} \mathrm{Cone}(1,w)^{gp}$ is the tautological set map.

For $p, q \in P$: $\quad p * q := \rho(p) + \rho(q) - \rho(p+q)$.

Note: if $p, q \in \mathrm{Cone}(1,w)$, then $\rho(p+q) = \rho(p) + \rho(q)$ (ρ is linear on $\mathrm{Cone}(1,w)$).

Example $X = \mathbb{Z}$, $a(n) = n^2$. $\quad \rho : \mathbb{Z}_{>0} \times X \longrightarrow \mathbb{Z}^{(X)} = \mathbb{Z}[t, t^{-1}] \subset \mathbb{R}[t, t^{-1}]$.



$$(d, n) \longmapsto (d-r) \cdot t^{q} + r \cdot t^{q+1}$$

$$\| $$

$$\underbrace{d \cdot (1, n/d)}_{\in [\lfloor n/d \rfloor, \lfloor n/d \rfloor + 1] = [q, q+1]} =$$

$n = q \cdot d + r$, $0 \leq r < n$; $\quad (d, n) = d \cdot (1, q) + r \cdot (0, 1)$

$$= d \cdot (1, q) + r \cdot ((1, q+1) - (1, q))$$

$$= (d-r) \cdot (1, q) + r \cdot (1, q+1)$$

$\widetilde{H}_S \subset \varinjlim_{w \in S} \mathrm{Cone}(1,w)^{gp}$ the monoid generated by the $p * q$, ($p, q \in P$).

$\widetilde{H}_S \twoheadrightarrow H_S$: co-invariants for X-action

(§5.12 example of higher dimensional Tate curve

Book, lemma 5.12.8. if $X = \mathbb{Z}^g$, $a(x) = x_1^2 + \cdots + x_g^2$, then

$$\begin{array}{ccc} \mathbb{N}^g & \xrightarrow{\sim} & H_S \\ \cap & & \cap \\ \mathbb{Z}^g & \xrightarrow{\sim} & H_S^{gp} \\ e_i & \longmapsto & s(e_i \otimes e_i) \end{array}$$

$s : X \times X \longrightarrow H_S^{gp} : (x, y) \mapsto (1, x+y) * (1, 0) - (1, x) * (1, y)$.

show on screen

4.1.5. $P \rtimes H_S$:= the monoid with set $P \times H_S$,

in ~~example~~ lemma 5.12.10 $(X = \mathbb{Z}^g,\ x_1^2 + \cdots + x_g^2)$ $P \rtimes H_S = \left(P^{(1)} \rtimes H_S^{(1)} \xrightarrow{\deg} \mathbb{N}\right)^g$

(g-fold fibre product)

and $P^{(1)} \rtimes \underbrace{H_S^{(1)}}_{= \mathbb{N} \cdot q}$ generators: q, x_n $(n \in \mathbb{Z})$

$\{(0,0)\} \cup \mathbb{Z}_{>0} \times \mathbb{Z}$ $\quad \mathbb{Z}[q]$ relations: $x_{n+2} + x_n = q + 2x_{n+1}$ $(n \in \mathbb{Z})$.

Now scroll to 4.1.10. The geometry starts.

B any scheme, X, a, S as before.

(i) A/B ab. scheme, (ii) $\mathcal{M}$ gives $\lambda : A \xrightarrow{\sim} A^t$, $a \mapsto [(tr_a^* \mathcal{M}) \otimes \mathcal{M}^{-1}] \in A^t$.

(iv) $\beta : H_S \longrightarrow \mathcal{O}_B(B)$: gives a log str. M_B on B.

(v) $c : X \to A^t(B)$, gives $T \rightarrowtail G \twoheadrightarrow A$, T split torus over B char. gp. X $\quad = X^\vee \otimes \mathbb{G}_m$.

$X_B \xrightarrow{c} A^t \xrightarrow{\sim} A$: $c^* \mathcal{M}^{-1}$ on X_B.

(vi) $\psi \in (c^* \mathcal{M}^{-1})(X_B)$ a trivialization, inducing a triv $\tau \in ((c \times c)^* \mathcal{B}^{-1})((X \times X)_B)$.

note: $\mathcal{B}$ on $A \times_B A$: $\mathcal{B}_{(x,y)} = \mathcal{M}_{x+y} \otimes \mathcal{M}_x^{-1} \otimes \mathcal{M}_y^{-1} \otimes \mathcal{M}_0$, (autom.) compatible ? with symm. biext. structure.

$$\mathcal{R} := \bigoplus_{(d,x) \in P} \underbrace{\mathcal{M}^d \otimes \mathcal{L}_x}_{\text{inv. } \mathcal{O}_A\text{-module}} \qquad \text{where } \mathcal{L}_x \sim c(x) \in A^t(B)$$

quasi-coherent $\mathcal{O}_A$-algebra.

$\boxed{\text{VERY IMPORTANT: } \varphi : P \to \lim_{w \in S} \mathrm{Gme}(\mathrm{Lw})^{gr} \text{ occurs in multiplication}}$ β and by $\beta : H_S \to \mathcal{O}_B$.

Locally on A: $\mathcal{R} \cong \mathbb{Z}[P \rtimes H_S] \otimes_{\mathbb{Z}[H_S]} \mathcal{O}_A$ $\qquad \mathbb{Z}[H_S] \longrightarrow \mathcal{O}_B \to \mathcal{O}_A$

~~as~~ not as algebra

$\tilde{\mathcal{P}} := \mathrm{Proj}_A(\mathcal{R})$, $\quad (\tilde{\mathcal{P}}, M_{\tilde{\mathcal{P}}}) \longrightarrow (B, M_B)$.

$\uparrow$ log smooth, integral

$\uparrow$ fine

G $\circlearrowleft$

$\nearrow X$ $\quad G(\tilde{\mathcal{P}}, M_{\tilde{\mathcal{P}}}, \mathcal{O}(1))$ $\qquad$ Tate curve

commute.

this action, in the case $A = 0$, $X = \mathbb{Z}$, determines the ~~ample~~ line bundle on the quotient.

(This seems too automatic to me....)

Scroll to 4.1.22

If $H_S \to \mathcal{O}_B(B)$ has image in $\sqrt{0}$, then we get a quotient. These are the standard families.

Also can saturate them...

~~Back to the Tate curve $X = \mathbb{Z}$, ... $\mathbb{Z}$, Σ = ...~~

Consider the case $A = 0$, $B = \mathbb{Z}[H_S]$. Then $G = X^\vee \otimes \mathbb{G}_{m,B} = T$.

There is no more data than this.

H_S — 0 on $H_S - \{0\}$

We want to look at the most degenerate piece of $\tilde{\mathcal{P}}$;

$$\begin{tikzcd} \mathbb{Z}[H_S] \arrow[r, two heads] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Z}[P \rtimes H_S] \arrow[r, two heads] & \mathbb{Z}[P] \end{tikzcd}$$

with algebra structure involving ρ.

Example of Tate curve.