Counting quickly the vectors with integer coordinates and with a given length

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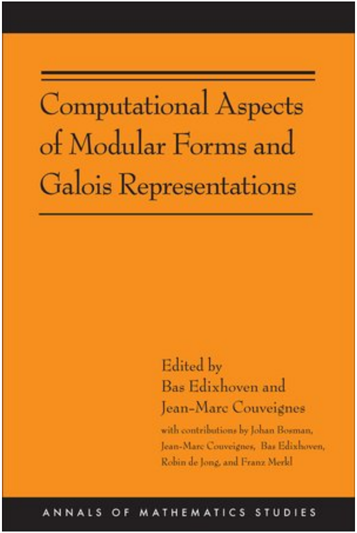
The commercial



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This book gives an algorithm for computing coefficients of modular forms of level one in polynomial time. For example, Ramanujan's tau of a prime number p can be computed in time bounded by a fixed power of the logarithm of p...

Back to mathematics: sums of squares

To illustrate the progress made in the book and Peter Bruin's PhD thesis, we consider the problem of computing quickly, for d and n in $\mathbb{Z}$:

$$r_d(n) := \#\{x \in \mathbb{Z}^d : x_1^2 + \cdots + x_d^2 = n\}.$$

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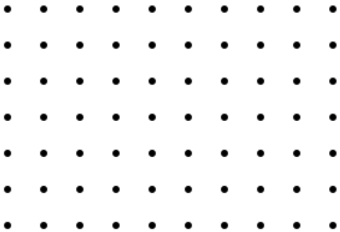
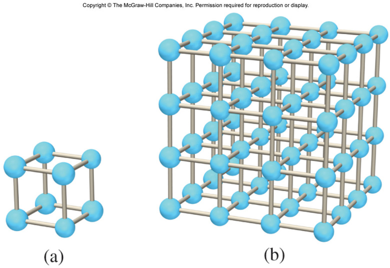
Geometric interpretation (Pythagoras): count the number of lattice points in $\mathbb{Z}^d$ at a given distance $\sqrt{n}$ from the origin.





Sums of squares: some examples

$r_2(3) = 0.$

$r_2(5) = 8:$

$$5 = (\pm 2)^2 + (\pm 1)^2$$
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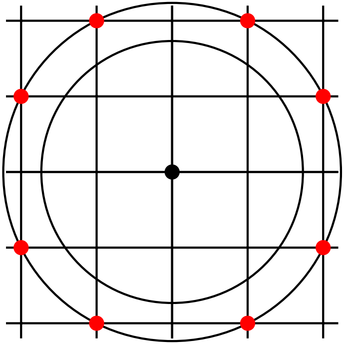

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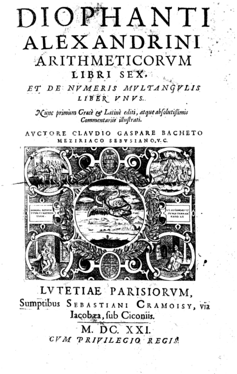
Do *not* use the factorisation of n into primes, because we do not know how to to that fast enough.

Dimension two: Diophantus



Diophantus of Alexandria ($\approx$ 3rd century):

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

Dimension two: Fermat

Pierre de Fermat (lawyer, Toulouse, 17th century), for $n \geq 1$: $r_2(n) \neq 0$ if and only if every prime factor of n that is 3 modulo 4, occurs an even number of times in the factorisation of n.

Dimensions 2 and 3: Legendre, Gauss

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Carl Friedrich Gauss (1801) gave a general formula for $r_2(n)$, and a formula for $r_3(n)$ that shows that the $r_d(n)$ for odd d are more complicated (involve class numbers).

Higher even dimensions: Jacobi

Carl Gustav Jacob Jacobi (1829) proved for $n > 1$:

$$r_2(n) = 4\sum_{d|n} \chi(d), \quad \text{with} \quad \chi(d) = \begin{cases} 0 \text{ if } d \text{ is even,} \\ 1 \text{ if } d = 4r+1, \\ -1 \text{ if } d = 4r+3, \end{cases}$$

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and:

$$r_4(n) = 8\sum_{2\nmid d|n} d + 16\sum_{2\nmid d|(n/2)} d.$$

Eisenstein, Smith

It follows from work of Jacobi, Ferdinand Eisenstein and Henry Smith that:

$$r_6(n) = 16\sum_{d|n} \chi(n/d)d^2 - 4\sum_{d|n} \chi(d)d^2,$$

$$r_8(n) = 16\sum_{d|n} d^3 - 32\sum_{d|(n/2)} d^3 + 256\sum_{d|(n/4)} d^3.$$

Dimension 10: Liouville

For $d = 10$ Joseph Liouville (1865) found a formula in terms of the Gaussian integers $d = a + bi$ with a and b in $\mathbb{Z}$:

$$r_{10}(n) = \frac{4}{5}\sum_{d|n}\chi(d)d^4 + \frac{64}{5}\sum_{d|n}\chi(n/d)d^4 + \frac{8}{5}\sum_{d\in\mathbb{Z}[i],\,|d|^2=n} d^4.$$

Dimension 12: Glaisher, Ramanujan

James Whitbread Lee Glaisher, reinterpreted by Srinivasa Ramanujan in 1916, proved that:

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Note: unlike for $d \leq 10$, this formula does *not* lead to computation of $r_{12}(n)$ in time polynomial in $\log n$, if n is given with its factorisation into primes.

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Note: for $n = pq$ with p and q distinct odd primes:

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Conclusion. From an algorithmic perspective this classical problem is now solved for *all* even d. The question for *formulas* has a negative answer, but for *computing* that negative answer does not matter and we now have a *positive* answer.

Explanation: generating series

It is more than time to explain what is going on behind all these formulas. Generating series:

$$\theta_d := \sum_{x \in \mathbb{Z}^d} q^{x_1^2 + \cdots + x_d^2} = \sum_{n \geq 0} r_d(n) q^n \quad \text{in } \mathbb{Z}[[q]].$$

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Let $\theta := \theta_1$ (Jacobi theta function at $z = 0$). Then:

$$\theta^d = \left(\sum_{x_1\in\mathbb{Z}} q^{x_1^2}\right)\cdots\left(\sum_{x_d\in\mathbb{Z}} q^{x_d^2}\right) = \theta_d.$$

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Compute θ^d in $\mathbb{Z}[[q]]/(q^{n+1})$: gives $r_d(n)$ but takes time at least linear in nd.

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This implies: θ_d is in the $\mathbb{C}$-vector space $M_{d/2}(\Gamma_1(4))$ of modular forms of weight $d/2$ on the subgroup $\Gamma_1(4)$ of $\mathrm{SL}_2(\mathbb{Z})$. Assume from now on that d is even. Then $k = d/2$ is in $\mathbb{Z}$.

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Complex analytic geometry

To get further ($\mathrm{SL}_2(\mathbb{Z})$ does not suffice, we need Galois symmetry), interpret $M_k(\Gamma)$ in terms of de Rham cohomology of the quotient E^{k-2} of $\mathbb{C}^{k-2} \times \mathbb{H}$ by an action of $\mathbb{Z}^{2(k-2)} \rtimes \Gamma$:

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The coefficients $a_n(f)$ of the modular forms $f = \sum_{n \geq 0} a_n(f) q^n$ are closely related to Hecke operators T_n coming from the $\mathrm{GL}_2(\mathbb{Q})^+$-action on $\mathbb{H}$.

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- We do *not* know (yet?): $\exists \sigma \in \mathrm{Aut}(\mathbb{C}), \sigma(\pi) = e$ *and* $\sigma(e) = \pi$.

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To simplify, consider $M_{12}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}{\cdot}E_{12} \oplus \mathbb{C}{\cdot}\Delta$, where

$$E_{12} = 1 + \frac{65520}{691} \sum_{n \geq 1} \left(\sum_{d|n} d^{11} \right) q^n,$$

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n.$$

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For p prime, the action of T_p on $H^{k-1}(E^{k-2}, \mathbb{Z}/m\mathbb{Z})$ can be computed from the Galois action, as the trace of a Frobenius element at p.

To simplify, consider $M_{12}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}{\cdot}E_{12} \oplus \mathbb{C}{\cdot}\Delta$, where

$$E_{12} = 1 + \frac{65520}{691} \sum_{n \geq 1} \left(\sum_{d|n} d^{11} \right) q^n,$$

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n.$$

Deligne: for every integer $m > 0$ there is $\rho_m \colon \mathrm{Aut}(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$, such that for every prime $p \nmid m$, $\tau(p) = \mathrm{trace}(\mathrm{Frob}_p)$ in $\mathbb{Z}/m\mathbb{Z}$.

The book and two theses

The *book* explains, in about 400 pages, that one can compute, for ℓ prime, ρ_ℓ in time polynomial in ℓ, and then $\tau(p)$ in time polynomial in $\log p$. More generally: for $M_k(\mathrm{SL}_2(\mathbb{Z}))$.

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Peter Bruin's PhD thesis: generalises the theory to $M_k(\Gamma_1(N))$.

Galois groups

Let $f = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be in $\mathbb{Q}[x]$.

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For σ in $\mathrm{Aut}(\mathbb{C})$ and z in $\mathrm{Roots}(f)$:

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$\mathrm{Gal}(f)$ is the group of permutations of $\mathrm{Roots}(f)$ given by elements of $\mathrm{Aut}(\mathbb{C})$.

Example: roots of unity

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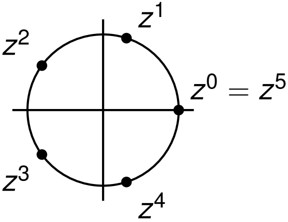

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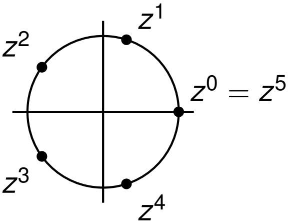

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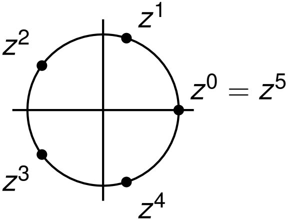

$\mathrm{Gal}(x^n - 1) = \{a \mapsto ka \mod n : k \in (\mathbb{Z}/n\mathbb{Z})^\times\}$

Conclusion: in terms of the labelling Gal(f) is given by elements of $\mathrm{GL}_1(\mathbb{Z}/n\mathbb{Z})$.

Two-dimensional Galois representations

A 2-dimensional Galois representation mod n is a polynomial $f = x^{n^2} + \cdots + a_1 x + a_0$ in $\mathbb{Q}[x]$ of degree n^2, with a bijection $\mathbb{Z}/n\mathbb{Z}^2 \to \mathrm{Roots}(f)$, such that each element of $\mathrm{Gal}(f)$ acts as multiplication by an element of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$.

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40 years ago the Langlands program started, relating Galois representations and automorphic forms.

Question: can one efficiently compute the Galois representations whose existence is guaranteed by the Langlands program?

It looks as if the answer will be 'yes'.

An example by Johan Bosman

The polynomial:

$$\begin{aligned} f = x^{24} &- 2x^{23} + 115x^{22} + 23x^{21} + 1909x^{20} + 22218x^{19} \\ &+ 9223x^{18} + 121141x^{17} + 1837654x^{16} - 800032x^{15} \\ &+ 9856374x^{14} + 52362168x^{13} - 32040725x^{12} \\ &+ 279370098x^{11} + 1464085056x^{10} + 1129229689x^{9} \\ &+ 3299556862x^{8} + 14586202192x^{7} + 29414918270x^{6} \\ &+ 45332850431x^{5} - 6437110763x^{4} - 111429920358x^{3} \\ &- 12449542097x^{2} + 93960798341x - 31890957224 \end{aligned}$$

has Galois group $\mathrm{PGL}_2(\mathbb{Z}/23\mathbb{Z})$, and (reduced) discriminant 23^{43}; it comes from étale cohomology of degree 21 of a variety of complex dimension 21.

The commercial, end

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This is avoided by numerical approximations with a precision that suffices to derive exact results from them.

Bounds for the required precision–in other words, bounds for the height of the rational numbers that describe the Galois representation to be computed–are obtained from Arakelov theory. . .

The end

Thank you for your attention!

Questions?

Nederlandse Organisatie voor Wetenschappelijk Onderzoek

With: Jean-Marc Couveignes (Toulouse), Robin de Jong, Franz Merkl (München), Johan Bosman, Peter Bruin, Ila Varma.