Counting quickly the vectors with integer coordinates and with a given length

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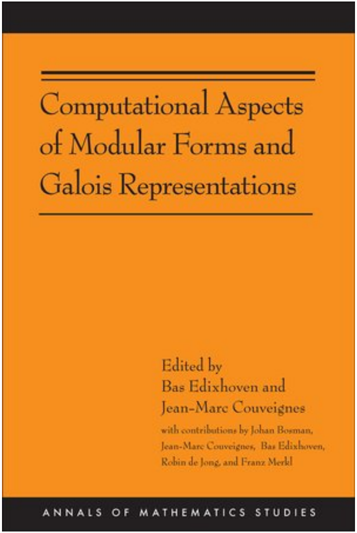
The commercial



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This book gives an algorithm for computing coefficients of modular forms of level one in polynomial time. For example, Ramanujan's tau of a prime number p can be computed in time bounded by a fixed power of the logarithm of p. . .

Back to mathematics: sums of squares

To illustrate the progress made in the book and Peter Bruin's PhD thesis, we consider the problem of computing quickly, for d and n in $\mathbb{Z}$:

$$r_d(n) := \#\{x \in \mathbb{Z}^d : x_1^2 + \cdots + x_d^2 = n\}.$$

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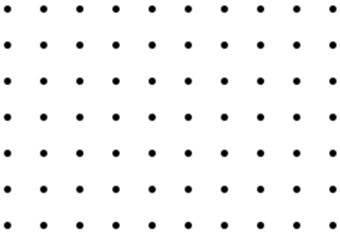
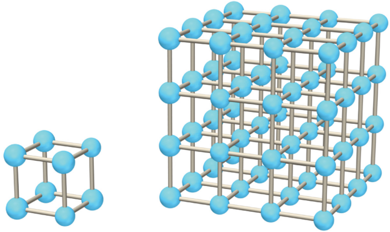
Geometric interpretation (Pythagoras): count the number of lattice points in $\mathbb{Z}^d$ at a given distance $\sqrt{n}$ from the origin.





(a) (b)

Sums of squares: some examples

$r_2(3) = 0.$

$r_2(5) = 8:$

$$5 = (\pm 2)^2 + (\pm 1)^2$$
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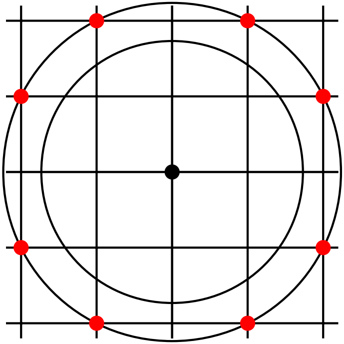

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If $n > 0$ and n is a square, then $r_1(n) = 2$, and otherwise $r_1(n) = 0$. Use the method of bisection of intervals for approximating $\sqrt{n}$, starting with $[0, n]$.

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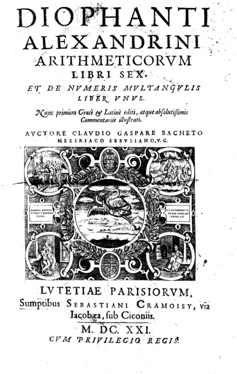
Do *not* use the factorisation of n into primes, because we do not know how to do that fast enough.

Dimension two: Diophantus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ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque abſolutiſſimis
Commentariis illuſtrati.
AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO, V.C.
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via
Iacobæa, ſub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

Diophantus of Alexandria ($\approx$ 3rd century):

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

Dimension two: Fermat

Pierre de Fermat (lawyer, Toulouse, 17th century), for $n \geq 1$: $r_2(n) \neq 0$ if and only if every prime factor of n that is 3 modulo 4, occurs an even number of times in the factorisation of n.

Dimensions 2 and 3: Legendre, Gauss

 

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For $n > 1$ squarefree, 1 or 2 mod 4, $r_3(n) = 12 \cdot h(\mathbb{Z}[\sqrt{-n}])$.

Higher even dimensions: Jacobi

Carl Gustav Jacob Jacobi (1829) proved for $n \geq 1$:

$$r_2(n) = 4 \sum_{d|n} \chi(d), \quad \text{with} \quad \chi(d) = \begin{cases} 0 \text{ if } d \text{ is even,} \\ 1 \text{ if } d = 4r+1, \\ -1 \text{ if } d = 4r+3, \end{cases}$$

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and:

$$r_4(n) = 8\sum_{2\nmid d|n} d + 16 \sum_{2\nmid d|(n/2)} d.$$

Eisenstein, Smith

 

It follows from work of Jacobi, Ferdinand Eisenstein and Henry Smith that:

$$r_6(n) = 16\sum_{d|n} \chi(n/d)d^2 - 4\sum_{d|n} \chi(d)d^2,$$

$$r_8(n) = 16\sum_{d|n} d^3 - 32\sum_{d|(n/2)} d^3 + 256\sum_{d|(n/4)} d^3.$$

Dimension 10: Liouville

For $d = 10$ Joseph Liouville (1865) found a formula in terms of the Gaussian integers $d = a + bi$ with a and b in $\mathbb{Z}$:

$$r_{10}(n) = \frac{4}{5}\sum_{d|n}\chi(d)d^4 + \frac{64}{5}\sum_{d|n}\chi(n/d)d^4 + \frac{8}{5}\sum_{d\in\mathbb{Z}[i],\,|d|^2=n} d^4.$$

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Note: unlike for $d \leq 10$, this formula does *not* lead to computation of $r_{12}(n)$ in time polynomial in $\log n$, if n is given with its factorisation into primes.

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$$r_4(n) = 8(1 + p + q + n).$$

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Conclusion. From an algorithmic perspective this classical problem is now solved for *all* even d. The question for *formulas* has a negative answer, but for *computing* that negative answer does not matter and we now have a *positive* answer.

Explanation: generating series

It is more than time to explain what is going on behind all these formulas. Generating series:

$$\theta_d := \sum_{x \in \mathbb{Z}^d} q^{x_1^2 + \cdots + x_d^2} = \sum_{n \geq 0} r_d(n) q^n \quad \text{in } \mathbb{Z}[[q]].$$

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$$\theta^d = \left(\sum_{x_1 \in \mathbb{Z}} q^{x_1^2} \right) \cdots \left(\sum_{x_d \in \mathbb{Z}} q^{x_d^2} \right) = \theta_d.$$

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Compute θ^d in $\mathbb{Z}[[q]]/(q^{n+1})$: gives $r_d(n)$ but takes time at least linear in nd.

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Key idea: $q\colon \mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\} \to \mathbb{C}, \quad z \mapsto e^{2\pi i z}$.

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Complex analytic geometry

To get further ($\mathrm{SL}_2(\mathbb{Z})$ does not suffice, we need Galois symmetry), interpret $M_k(\Gamma)$ in terms of de Rham cohomology of the quotient E^{k-2} of $\mathbb{C}^{k-2} \times \mathbb{H}$ by an action of $\mathbb{Z}^{2(k-2)} \rtimes \Gamma$:

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The coefficients $a_n(f)$ of the modular forms $f = \sum_{n\geq 0} a_n(f)q^n$ are closely related to Hecke operators T_n coming from the $\mathrm{GL}_2(\mathbb{Q})^+$-action on $\mathbb{H}$.

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- We do *not* know (yet?): $\exists \sigma \in \mathrm{Aut}(\mathbb{C}), \sigma(\pi) = e \text{ and } \sigma(e) = \pi$.

Just the case Δ

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Deligne: for every integer $m > 0$ there is $\rho_m\colon \mathrm{Aut}(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$, such that for every prime $p \nmid m$, $\tau(p) = \mathrm{trace}(\mathrm{Frob}_p)$ in $\mathbb{Z}/m\mathbb{Z}$.

Galois representations in concrete terms

For each $m > 0$ there is

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- for $p \gg 0$, the trace of $\mathrm{Frob}_p \colon x \mapsto x^p$ acting on $\mathrm{Roots}(f_m, \overline{\mathbb{F}}_p)$ is $\tau(p) \mod m$,
- where $\overline{\mathbb{Z}} \to \overline{\mathbb{F}}_p$ gives $(\mathbb{Z}/m\mathbb{Z})^2 \to \mathrm{Roots}(f_m, \mathbb{C}) \to \mathrm{Roots}(f_m, \overline{\mathbb{F}}_p)$.

The book and two theses

The *book* explains, in about 400 pages, that one can compute, for ℓ prime, such an f_ℓ in time polynomial in ℓ, and then $\tau(p)$ in time polynomial in $\log p$. More generally: for $M_k(\mathrm{SL}_2(\mathbb{Z}))$.

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Peter Bruin's PhD thesis: generalises the theory to $M_k(\Gamma_1(N))$.

An example by Johan Bosman

The polynomial:

$$
\begin{aligned}
f = x^{24} &- 2x^{23} + 115x^{22} + 23x^{21} + 1909x^{20} + 22218x^{19} \\
&+ 9223x^{18} + 121141x^{17} + 1837654x^{16} - 800032x^{15} \\
&+ 9856374x^{14} + 52362168x^{13} - 32040725x^{12} \\
&+ 279370098x^{11} + 1464085056x^{10} + 1129229689x^{9} \\
&+ 3299556862x^{8} + 14586202192x^{7} + 29414918270x^{6} \\
&+ 45332850431x^{5} - 6437110763x^{4} - 111429920358x^{3} \\
&- 12449542097x^{2} + 93960798341x - 31890957224
\end{aligned}
$$

has Galois group $\mathrm{PGL}_2(\mathbb{Z}/23\mathbb{Z})$, and (reduced) discriminant 23^{43}; it comes from étale cohomology of degree 21 of a variety of complex dimension 21.

The commercial, end

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Bounds for the required precision–in other words, bounds for the height of the rational numbers that describe the Galois representation to be computed–are obtained from Arakelov theory…

The end

Thank you for your attention!

Questions?

With: Jean-Marc Couveignes (Toulouse), Robin de Jong, Franz Merkl (München), Johan Bosman, Peter Bruin, Ila Varma.