

Katz modular forms. Leiden, 2012/05/14.

§1. Over \mathbb{C} . let $n \in \mathbb{Z}_{\neq 1}$. let $k \in \mathbb{Z}$.

Def. A complex modular form of level n and weight k is a rule f that assigns to every (E, α) with E a complex ell. curve and $\alpha: (\mathbb{Z}/n\mathbb{Z})^2 \xrightarrow{\sim} E[n]$ an element $f(E, \alpha) \in \underline{w}_E^{\otimes k}$ such that:

(1) \forall isomorphisms $\varphi: E_1 \xrightarrow{\sim} E_2$, $(\varphi^*)^{\otimes k} f(E_2, \alpha_2) = f(E_1, \alpha_1)$

(this implies the usual identity $F_E \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (cd)^k F_E \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ if $f(\mathbb{C}/\mathbb{Z} + z\mathbb{Z}) = F_E(z)$)

(2) $\forall g \in GL_2(\mathbb{Z}/n\mathbb{Z})$, the function $f_g: D(\mathbb{C}, 1)^* \rightarrow \mathbb{C}$ defined $\{z \in \mathbb{C}: 0 < |z| < 1\}$

by $g \mapsto f(\mathbb{C}^x/q^n\mathbb{Z}, (\mathbb{Z}/n\mathbb{Z})^2 \xrightarrow{g} (\mathbb{Z}/n\mathbb{Z})^2 \xrightarrow{\sim} (\mathbb{C}^x/q^n\mathbb{Z})[n])$

\parallel $\begin{pmatrix} a & \\ & b \end{pmatrix} \mapsto \zeta_n^a q^{\frac{1}{2}}$

$f_g(q) \cdot \left(\frac{dq}{q}\right)^{\otimes k}$ ($t: \mathbb{C}^x \hookrightarrow \mathbb{C}$ inclusion)

is given by a power series $\sum_{m \geq 0} a_m(f, g) \cdot q^m$.

Notation $M(n, k, \mathbb{C}) :=$ the set of such f . \mathbb{C} -vect. space, one can prove it is finite dimensional. $GL_2(\mathbb{Z}/n\mathbb{Z})$ -action:

$(g \cdot f)(E, \alpha) = f(E, \alpha \circ g)$

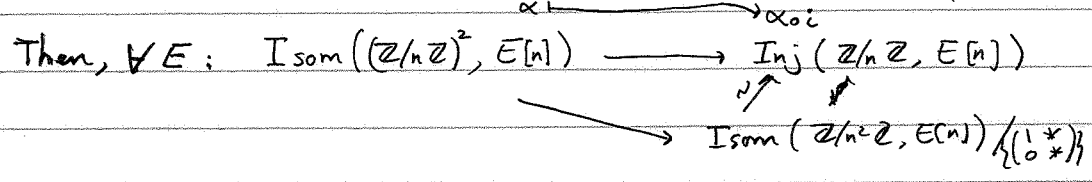
For $H \subset GL_2(\mathbb{Z}/n\mathbb{Z})$: $M(n, H, k, \mathbb{C}) := M(n, k, \mathbb{C})^H$

Cusprforms: $S(n, H, k, \mathbb{C}) := \{f \in M(n, H, k, \mathbb{C}) : \forall g: a_0(f, g) = 0\}$

Lemma. $M(n, H, k, \mathbb{C})$ is the set of rules $f: (E, \alpha) \mapsto f(E, \alpha) \in \underline{w}_E^{\otimes k}$, with $\alpha \in \text{Isom}((\mathbb{Z}/n\mathbb{Z})^2, E[n]) / H$, such that (1) & (2).

Special cases. • $H = GL_2(\mathbb{Z}/n\mathbb{Z})$: $M(n, GL_2(\mathbb{Z}/n\mathbb{Z}), k, \mathbb{C}) = M(1, k, \mathbb{C})$
 • $d|n$, $H = \ker(GL_2(\mathbb{Z}/n\mathbb{Z}) \rightarrow GL_2(\mathbb{Z}/d\mathbb{Z})) = M(d, k, \mathbb{C})$

• $H = GL_2(\mathbb{Z}/n\mathbb{Z})_{\neq 1} = \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL_2(\mathbb{Z}/n\mathbb{Z}) \right\}$ $\mathbb{Z}/n\mathbb{Z} \xrightarrow{i} (\mathbb{Z}/n\mathbb{Z})^2$



So: $M(n, \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\}, k, \mathbb{C}) =$ the set of rules $f: (E, \alpha) \rightarrow f(E, \alpha) \in \underline{\omega}_E^{\otimes k}$
with $\alpha \in \text{Inj}(\mathbb{Z}/n\mathbb{Z}, E[n])$, such that (v) and (e).

More classical notation: $M(\Gamma, (n), k, \mathbb{C}), S(\Gamma, (n), k, \mathbb{C})$.

Hecke operators $T_m, m \in \mathbb{Z}_{>1}$:

$(T_m f)(E, \alpha) = \frac{1}{m} \cdot \sum_{\substack{G \subseteq E \\ \#G=m \\ G \cap \text{Im}(\alpha) = \{0\}}} \left(\varphi_G^* \right)^{\otimes k} f(E/G, \bar{\alpha})$ where $E \xrightarrow{\varphi_G} E/G$

$\uparrow \quad \nearrow$

$\mathbb{Z}/n\mathbb{Z} \quad \bar{\alpha}$

Diamant operators: $\forall a \in (\mathbb{Z}/n\mathbb{Z})^* : (\langle a \rangle f)(E, \alpha) = f(E, \alpha \circ (a \cdot))$.

Relation with geometry. Assume $n \geq 3$. Then the category of $(E/S/\mathbb{Z}[1/n], \alpha)$ with $\alpha: (\mathbb{Z}/n\mathbb{Z})_S \xrightarrow{\sim} E[n]$, and morphisms $E_1 \rightarrow E_2$

$\alpha_1 \downarrow \square \downarrow \alpha_2$

$S_1 \rightarrow S_2$

has a final object $E^{\text{univ}} \downarrow, \alpha^{\text{univ}}$

$Y(n)$ with $Y(n) \rightarrow \text{Spec } \mathbb{Z}[1/n]$ a smooth affine curve, whose geometric fibers have $\varphi(n)$ connected components.

Then $Y(n)(\mathbb{C})$ is Riemann surface, with $\varphi(n)$ connected components

The map $j: Y(n)(\mathbb{C}) \rightarrow \mathbb{C}, \gamma \mapsto j(E_\gamma)$, is the quotient for the $GL_2(\mathbb{Z}/n\mathbb{Z})$ -action:

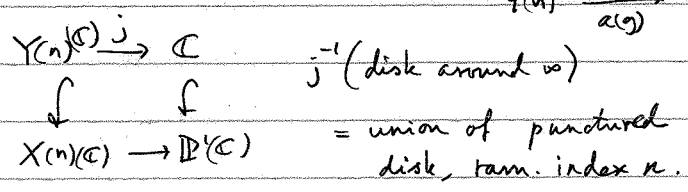
$\forall g \in GL_2(\mathbb{Z}/n\mathbb{Z}) \exists! \tilde{\alpha}(g), \alpha(g):$

$E^{\text{univ}} \xrightarrow{\tilde{\alpha}(g)} E^{\text{univ}}$

$\alpha^{\text{univ}} \downarrow \quad \downarrow \alpha^{\text{univ}}$

$Y(n) \xrightarrow{\alpha(g)} Y(n)$

(for example: $g = -1:$
 $\alpha(-1) = \text{id}$
 $\tilde{\alpha}(-1) = -\text{id}$)



Can be seen using the complex Tate curve.

Then, we have inv. $\mathcal{O}_{Y(n)}$ -module $\underline{\omega}_{E^{\text{univ}}/Y(n)}$, has a natural extension over $X(n)(\mathbb{C})$, and then: $M(n, k, \mathbb{C}) = H^0(X(n)(\mathbb{C}), \underline{\omega}^{\otimes k})$.

§2. Over any $\mathbb{Z}[1/n]$ -algebra A .

We need some properties of Tate curves. (See Katz-Mazur §8.8).

Let $\text{Tate}(q^n)$ denote the quotient " $G_m/q^n\mathbb{Z}$ " over $\mathbb{Z}((q))$.

It is an elliptic curve,
$$\begin{array}{ccc} \text{Tate}(q^n) & [n] & \xleftarrow{\alpha} (\mathbb{Z}/n\mathbb{Z})^2 \\ \mathbb{Z}[1/n, \zeta_n]((q)) & & \mathbb{Z}[1/n, \zeta_n]((q)) \\ \sum_n a \cdot q^b & & \longleftarrow \begin{pmatrix} a \\ b \end{pmatrix} \end{array}$$

and $\omega_{\text{Tate}(q^n)/\mathbb{Z}((q))} = \mathbb{Z}((q)) \cdot \frac{dt}{t}$, where $G_m = \text{Spec } \mathbb{Z}[\epsilon, \epsilon^{-1}]$.

Def. A modular form of level n and weight k over A is a rule

$$f : (E/S/A, \alpha) \mapsto f(E/S/A, \alpha) \in \omega_{E/S}^{\otimes k}(S) \quad \left(\omega_{E/S} = \mathcal{O}^* \Omega^1_{E/S}, \text{ inv. } \mathcal{O}_S\text{-module} \right)$$

 $(\alpha: (\mathbb{Z}/n\mathbb{Z})^2_S \xrightarrow{\sim} E[n])$ such that:

(1)
$$\begin{array}{ccc} E_1 & \xrightarrow{\psi} & E_2 \\ \alpha_1 \downarrow & \square & \downarrow \alpha_2 \\ S_1 & \longrightarrow & S_2 \end{array} \quad f(E_1/S_1, \alpha_1) = (\psi^*)^{\otimes k} f(E_2/S_2, \alpha_2)$$

$$\begin{array}{c} A[x]/(\Phi_n) \\ \parallel \end{array}$$

(2) $\forall g \in G_2(\mathbb{Z}/n\mathbb{Z})$ the element $f_g \in \widehat{A}[\zeta_n]((q))$ given by

$$f(\text{Tate}(q^n)_{A[\zeta_n]((q))}, \alpha \circ g) = f_g \cdot \left(\frac{dt}{t}\right)^{\otimes k}$$
 is in $A[\zeta_n][[q, \mathbb{I}]$.

Notation: $M(n, H, k, A), S(n, H, k, A) \dots$

Geometry. For $n \geq 3$, the category $[\Gamma(n)]_{\mathbb{Z}[1/n]}$ of $(E/S/\mathbb{Z}[1/n], \alpha)$ has a final

object: $(E^{\text{univ}}/Y(n), \alpha^{\text{univ}})$,

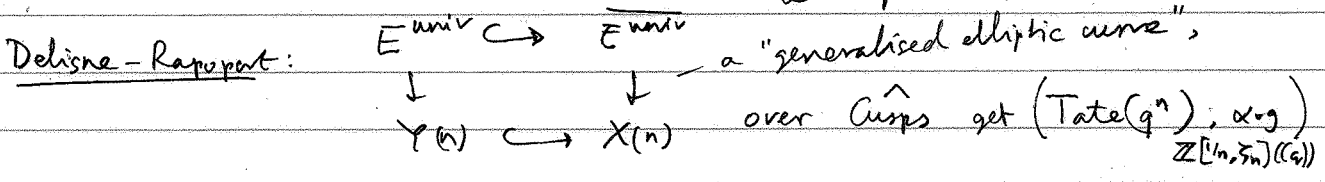
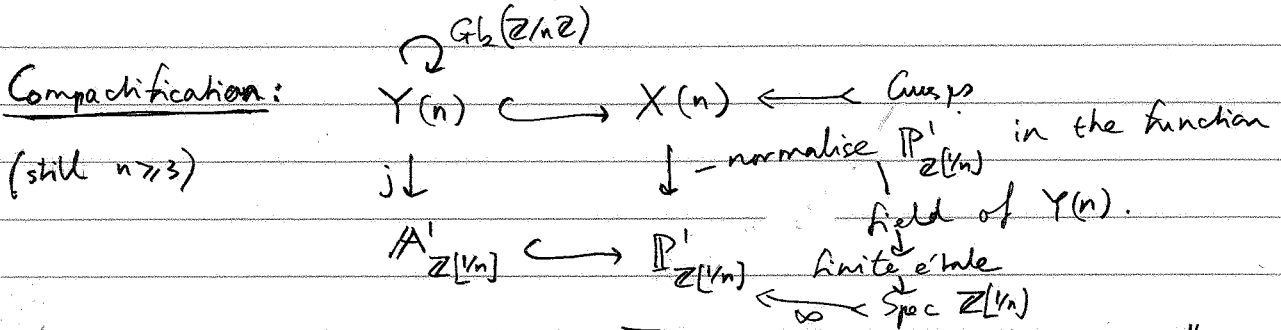
with $Y(n) \rightarrow \text{Spec } \mathbb{Z}[1/n]$ smooth affine curve (actually $Y(n) \rightarrow \text{Spec } \mathbb{Z}[1/n, \zeta_n]$ because of Weil pairing $e_n(\alpha^{\text{univ}}(i), \alpha^{\text{univ}}(j)) \in \mathcal{O}(Y(n))^*$)

For $n \geq 4$ we have (Maarten's lectures)
$$\begin{array}{ccc} E_1 & (\mathbb{Z}/n\mathbb{Z}) & \longrightarrow E_1[n] \\ \downarrow & \downarrow \gamma(n) & \\ Y_1(n) & \longrightarrow & P \end{array}$$
 and $Y(n) \subset E_1[n]$ is the open and closed subscheme over which the

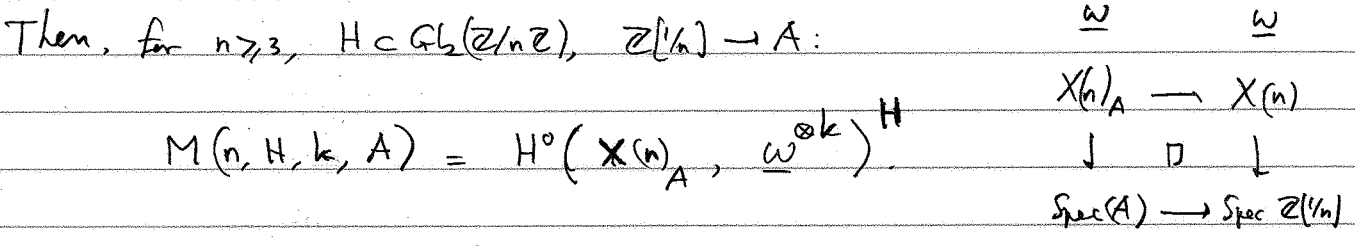
points P, Q of $(E_1)_{E_1[n]}$ give an isomorphism $(\mathbb{Z}/n\mathbb{Z})^2 \xrightarrow{\sim} (E_1)_{E_1[n]}[n]$

$$P \begin{pmatrix} \uparrow \\ Q \\ \downarrow \end{pmatrix} \begin{array}{c} \\ E_1[n] \end{array}$$

For $n=3$, see Katz-Mazur, § 2.2.10, or use $d(X^3 + Y^3 + Z^3) - 3XYZ$ over \mathbb{P}^1 -discr. with the points $(0:1:\zeta)$ with $\zeta^3=1$ + cyclic perm's. (Hesse pencil).
 (Hesse pencil) braun.



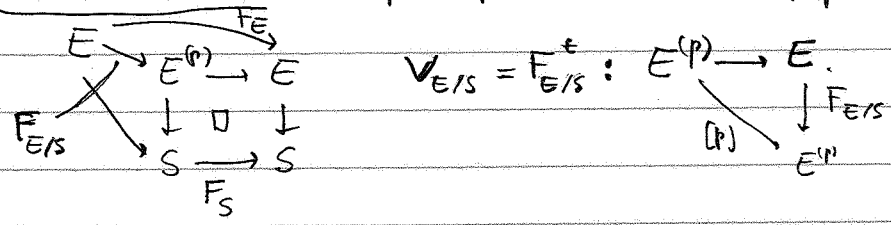
Katz-Mazur: (§10.13) : extend ω from $Y(n)$ to $X(n)$ using the Tate curve:
 one checks that $\frac{dt}{t}$ is a generator on $\widehat{\text{Cusps}}$.



Why can this be interesting?

Because $A \otimes_{\mathbb{Z}[1/n]} H^0(X(n), \omega^{\otimes k}) \rightarrow H^0(X(n)_A, \underline{\omega}^{\otimes k})$ need not be surjective, and also $(A \otimes_{\mathbb{Z}[1/n]} M)^H \rightarrow (A \otimes_{\mathbb{Z}[1/n]} M^H)^H$ need not be surjective (if $\mathbb{Z}[1/n] \rightarrow A$ is flat, then both are isomorphisms).

Hasse invariant Let p be prime. For $E/S/\mathbb{F}_p$ we have:



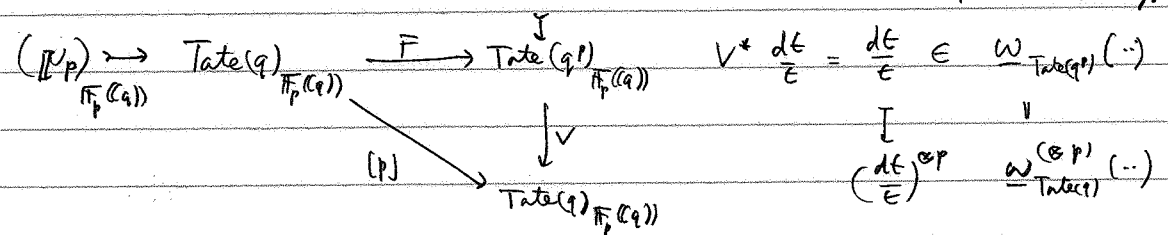
For $u \in S$ gen, $w \in \underline{\omega}_{E/S}(u) = \Omega'_{E/S}(E_u)$.

$$\begin{aligned}
 V_{E/S}^*(w) &\in \Omega'_{E^{(p)}/S}(E^{(p)}_u) = \underline{\omega}_{E^{(p)}/S}(u) = (F_S^* \underline{\omega}_{E/S})(u) = \\
 &= (\underline{\omega}_{E/S}^{\otimes p})(u).
 \end{aligned}$$

So, $V_{E/S}^* \in \text{Hom}_{O_S}(\underline{\omega}_{E/S}, \underline{\omega}_{E/S}^{\otimes p}) = \text{Hom}_{O_S}(O_S, \underline{\omega}_{E/S}^{\otimes(p-1)}) = (\underline{\omega}_{E/S}^{\otimes(p-1)})/S$.

Notation: $A(E/S), A \in M(1, p-1, \mathbb{F}_p)$

indeed: $A(\text{Tate}(q)/\mathbb{F}_p((q))) = \left(\frac{dt}{t}\right)^{\otimes p-1}_{\langle q \rangle}$, the q -exp. is 1. (Katz-Mazur, Thm 12.4.2.)



Now take $p=2$, then $k=1$, $M(1, 1, \mathbb{Z}) = 0$ $(-1)^t = -1 \dots$

So, for $n \geq 3$ odd, get on $X^{(n)}_{\mathbb{Z}[1/2]}$ $\underline{\omega} \xrightarrow{2} \underline{\omega} \rightarrow i_* i^* \underline{\omega}$,
 where $i: X^{(n)}_{\mathbb{F}_2} \rightarrow X^{(n)}_{\mathbb{Z}[1/2]}$ gives an obstruction somewhere.

For $n=3$: $X(3) \cong \mathbb{P}^1_{\mathbb{Z}[1/3, 5/3]}$, $\deg(\underline{\omega}) = 1$ (over $\mathbb{Z}[1/3, 5/3]$), so A lifts, but $\nexists G_2(\mathbb{Z}/3\mathbb{Z})$ -invariant lift:

$$\begin{array}{ccccc}
 M(3, 1, \mathbb{Z}[1/3]) & \xrightarrow{G} & M(3, 1, \mathbb{Z}[1/3]) & \xrightarrow{G} & M(3, 1, \mathbb{F}_3) & \xrightarrow{G} & H^1(G, M(3, 1, \mathbb{Z}[1/3])) \\
 \parallel & & \parallel & & \downarrow & & \\
 0 & & 0 & & A & &
 \end{array}$$

Finally: $\deg(\underline{\omega}) = 1$ on $\mathbb{P}^1_{\mathbb{Z}[1/3, 5/3]} \rightsquigarrow \forall p \neq 3: \sum_{x \in \overline{\mathbb{F}_p}} \frac{1}{\# \text{Aut}(E_x)} = \frac{p-1}{24}$, mass formula
 E_x supersingular Katz-Mazur: 12.4.6.