Katz modular forms. Leiden, 2012/05/14.

§1. Over $\mathbb{C}$. Let $n \in \mathbb{Z}_{\geq 1}$. Let $k \in \mathbb{Z}$.

Def. A complex modular form of level n and weight k is a rule f that assigns to every (E, α) with E a complex ell. curve and $\alpha : (\mathbb{Z}/n\mathbb{Z})^2 \xrightarrow{\sim} E[n]$ an element $f(E,\alpha) \in \underline{\omega}_E^{\otimes k}$ $\left(\underline{\omega}_E = 0^* \Omega^1_{E/\mathbb{C}} = \mathrm{Cot}_E(0) = m/m^2 \ldots \right)$ such that:

(1) $\forall$ isomorphisms $\varphi : E_1 \xrightarrow{\sim} E_2$, $(\varphi^*)^{\otimes k} f(E_2, \alpha_2) = f(E_1, \alpha_1)$

$$\begin{array}{ccc} E_1 & \xrightarrow{\ \sim\ } & E_2 \\ {\scriptstyle \alpha_1}\nwarrow & \circlearrowleft & \nearrow{\scriptstyle \alpha_2} \\ & (\mathbb{Z}/n\mathbb{Z})^2 & \end{array}$$

(this implies the usual identity $F_f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k F_f(\tau)$ if $f(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau) = F_f(\tau)(dz)^{\otimes k}$)

(2) $\forall g \in \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$, the function $f_g : D(0,1)^* \longrightarrow \mathbb{C}$ defined $\left(D(0,1)^* = \{z \in \mathbb{C} : 0 < |z| < 1\} \right)$

by $q \longmapsto f\left(\mathbb{C}^\times/q^{n\mathbb{Z}}, (\mathbb{Z}/n\mathbb{Z})^2 \xrightarrow[g]{\sim} (\mathbb{Z}/n\mathbb{Z})^2 \xrightarrow{\sim} (\mathbb{C}^\times/q^{n\mathbb{Z}})[n]\right)$, $\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \zeta_n^a q^b$

$= f_g(q) \cdot \left(\frac{dt}{t}\right)^{\otimes k}$ $\quad$ ($t : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ inclusion)

is given by a powerseries $\sum_{m \geq 0} a_m(f,g) \cdot q^m$.

Notation. $M(n,k,\mathbb{C}) :=$ the set of such f. $\mathbb{C}$-vect. space, one can prove it is finite dimensional. $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-action:

$$(g \cdot f)(E, \alpha) = f(E, \alpha \circ g)$$

For $H \subset \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$: $M(n,H,k,\mathbb{C}) := M(n,k,\mathbb{C})^H$

Cuspforms: $S(n,H,k,\mathbb{C}) := \{ f \in M(n,H,k,\mathbb{C}) : \forall g : a_0(f,g) = 0 \}$

Lemma. $M(n,H,k,\mathbb{C})$ is the set of rules $f : (E,\alpha) \mapsto f(E,\alpha) \in \underline{\omega}_E^{\otimes k}$, with $\alpha \in \mathrm{Isom}((\mathbb{Z}/n\mathbb{Z})^2, E[n])/H$, such that (1) & (2).

Special cases. • $H = \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$: $M(n, \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}), k, \mathbb{C}) = M(1,k,\mathbb{C})$

• $d | n$, $H = \ker(\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/d\mathbb{Z}))$ $\quad = M(d,k,\mathbb{C})$

• $H = GL_2(\mathbb{Z}/n\mathbb{Z})_{\binom{1}{0}} = \{\begin{pmatrix}1&*\\0&*\end{pmatrix} \subset GL_2(\mathbb{Z}/n\mathbb{Z})\}$ $\mathbb{Z}/n\mathbb{Z} \xrightarrow{i} (\mathbb{Z}/n\mathbb{Z})^2$, $a \longmapsto a\cdot\binom{1}{0}$

$\alpha \longmapsto \alpha\circ i$

Then, $\forall E$:

$$\begin{tikzcd} \mathrm{Isom}((\mathbb{Z}/n\mathbb{Z})^2, E[n]) \arrow[r] \arrow[dr] & \mathrm{Inj}(\mathbb{Z}/n\mathbb{Z}, E[n]) \\ & \mathrm{Isom}((\mathbb{Z}/n\mathbb{Z})^2, E[n])/\{\begin{pmatrix}1&*\\0&*\end{pmatrix}\} \arrow[u, "\sim"] \end{tikzcd}$$

So: $M(n, \{\begin{pmatrix}1&*\\0&*\end{pmatrix}\}, k, \mathbb{C})$ = the set of rules $f : (E,\alpha) \to f(E,\alpha) \in \underline{\omega}_E^{\otimes k}$ with $\alpha \in \mathrm{Inj}(\mathbb{Z}/n\mathbb{Z}, E[n])$, such that (1) and (2).

More classical notation: $M(\Gamma_1(n), k, \mathbb{C})$, $S(\Gamma_1(n), k, \mathbb{C})$.

Hecke operators T_m, $m \in \mathbb{Z}_{\geq 1}$:

$$(T_m f)(E,\alpha) = \frac{1}{m}\cdot \sum_{\substack{G\subset E\\ \#G = m\\ G\cap \mathrm{Im}(\alpha) = \{0\}}} (\varphi_G^*)^{\otimes k} f(E/G, \bar{\alpha})$$

where
$$\begin{tikzcd} E \arrow[r, "\varphi_G"] & E/G \\ \mathbb{Z}/n\mathbb{Z} \arrow[u, "\alpha"] \arrow[ur, "\bar{\alpha}"'] & \end{tikzcd}$$

Diamant operators: $\forall a \in (\mathbb{Z}/n\mathbb{Z})^\times$: $(\langle a\rangle f)(E,\alpha) = f(E, \alpha\circ(a\cdot))$.

<u>Relation with geometry</u>. Assume $n \geq 3$. Then the category of $(E/S/\mathbb{Z}[1/n], \alpha)$ with $\alpha : (\mathbb{Z}/n\mathbb{Z})_S \xrightarrow{\sim} E[n]$, and morphisms
$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "\square"] & E_2 \arrow[d, "\alpha_2"] \\ S_1 \arrow[r] & S_2 \end{tikzcd}$$
has a final object
$$\begin{tikzcd} \mathbb{E}^{univ} \arrow[d] \\ Y(n) \end{tikzcd}, \alpha^{univ},$$
with $Y(n) \to \mathrm{Spec}\, \mathbb{Z}[1/n]$ a smooth affine curve, whose geometric fibers have $\varphi(n)$ connected components.

Then $Y(n)(\mathbb{C})$ is Riemann surface, with $\varphi(n)$ connected components

The map $j : Y(n)(\mathbb{C}) \longrightarrow \mathbb{C}$, $y \longmapsto j(E_y)$, is the quotient for the $GL_2(\mathbb{Z}/n\mathbb{Z})$-action:

$\forall g \in GL_2(\mathbb{Z}/n\mathbb{Z})$ $\exists!\ \tilde{\alpha}(g), \alpha(g)$:
$$\begin{tikzcd} E^{univ} \arrow[r, "\tilde{\alpha}(g)", "\sim"'] \arrow[d, "\alpha^{univ}\circ g"'] & E^{univ} \arrow[d, "\alpha^{univ}"] \\ Y(n) \arrow[r, "\sim", "\alpha(g)"'] & Y(n) \end{tikzcd}$$
(for example: $g = -1$: $a(-1) = id$, $\tilde{\alpha}(-1) = -id$)

$$\begin{tikzcd} Y(n)(\mathbb{C}) \arrow[r, "j"] \arrow[d, hook] & \mathbb{C} \arrow[d, hook] \\ X(n)(\mathbb{C}) \arrow[r] & \mathbb{P}^1(\mathbb{C}) \end{tikzcd}$$

j^{-1}(disk around ∞) = union of punctured disk, ram. index n.

Can be seen using the complex Tate curve.

Then, we have inv. $\mathcal{O}_{Y(n)}$-module $\underline{\omega}_{E^{univ}/Y(n)}$, has a natural extension over $X(n)(\mathbb{C})$, and then: $M(n,k,\mathbb{C}) = H^0(X(n)(\mathbb{C}), \underline{\omega}^{\otimes k})$.

§2. Over any $\mathbb{Z}[1/n]$-algebra A.

We need some properties of Tate curves. (See Katz–Mazur §8.8).

Let $\mathrm{Tate}(q^n)$ denote the quotient "$\mathbb{G}_m/q^{n\mathbb{Z}}$" over $\mathbb{Z}((q))$.

It is an elliptic curve, $\mathrm{Tate}(q^n)_{\mathbb{Z}[1/n,\zeta_n]((q))}[n] \xleftarrow[\alpha_n]{\sim} (\mathbb{Z}/n\mathbb{Z})^2_{\mathbb{Z}[\frac{1}{n},\zeta_n]((q))}$

$$\zeta_n^a \cdot q^b \longleftarrow\!\shortmid \binom{a}{b},$$

and $\underline{\omega}_{\mathrm{Tate}(q^n)/\mathbb{Z}((q))} = \mathbb{Z}((q)) \cdot \frac{dt}{t}$, where $\mathbb{G}_m = \mathrm{Spec}\,\mathbb{Z}[t,t^{-1}]$.

<u>Def.</u> A modular form of level n and weight k <u>over A</u> is a rule

$f: (E/S/A, \alpha) \longmapsto f(E/S/A,\alpha) \in \underline{\omega}^{\otimes k}_{E/S}(S)$ $\quad \left(\begin{array}{l}\underline{\omega}_{E/S} = 0^*\Omega^1_{E/S},\\ \text{inv. } \mathcal{O}_S\text{-module}\end{array}\right)$

$(\alpha : (\mathbb{Z}/n\mathbb{Z})^2_S \xrightarrow{\sim} E[n])$ such that:

(1) $\forall$
$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2 \\ \alpha_1 \downarrow & \square & \downarrow \alpha_2 \\ S_1 & \longrightarrow & S_2 \end{array} \qquad f(E_1/S_1,\alpha_1) = (\varphi^*)^{\otimes k} f(E_2/S_2,\alpha_2)$$

(2) $\forall g \in GL_2(\mathbb{Z}/n\mathbb{Z})$ the element $f_g \in \overset{A[x]/(\Phi_n)}{\overset{\shortparallel}{A[\zeta_n]}}((q))$ given by

$$f(\mathrm{Tate}(q^n)_{A[\zeta_n]((q))}, \alpha\circ g) = f_g \cdot \left(\frac{dt}{t}\right)^{\otimes k} \quad \text{is in } A[\zeta_n][[q]].$$

<u>Notation:</u> $M(n,H,k,A)$, $S(n,H,k,A)$ ...

<u>Geometry</u>. For $n \geq 3$, the category $[\Gamma(n)]_{\mathbb{Z}[1/n]}$ of $(E/S/\mathbb{Z}[1/n],\alpha)$ has a final object: $(E^{univ}/Y(n), \alpha^{univ})$,

with $Y(n) \longrightarrow \mathrm{Spec}\,\mathbb{Z}[1/n]$ smooth affine curve (actually $Y(n) \to \mathrm{Spec}\,\mathbb{Z}[\frac{1}{n},\zeta_n] \to \mathrm{Spec}\,\mathbb{Z}[1/n]$ because of Weil pairing $e_n(\alpha^{univ}\binom{1}{0}, \alpha^{univ}\binom{0}{1}) \in \mathcal{O}(Y(n))^*$.)

For $n \geq 4$ we have (Maarten's lectures)

$$\begin{array}{c} E_1 \\ \downarrow \\ Y_1(n) \end{array}, \quad (\mathbb{Z}/n\mathbb{Z})_{Y_1(n)} \rightarrowtail E_1[n], \quad 1 \longmapsto P$$

and $Y(n) \subset E_1[n]$ is the open and closed subscheme over which the points P, Q of $(E_1)_{E_1[n]}$ give an isomorphism $(\mathbb{Z}/n\mathbb{Z})^2 \longrightarrow (E_1)_{E_1[n]}[n]$ ..

$$P \left(\begin{array}{c} (E_1)_{E_1[n]} \\ \uparrow \downarrow Q \\ E_1[n] \end{array} \right)$$

For $n = 3$, see Katz-Mazur, § 2.2.10, or use $\lambda(X^3+Y^3+Z^3) - 3\mu XYZ$ over $\mathbb{P}^1$ − discr. locus. with the points $(0:+1:\zeta)$ with $\zeta^3 = 1$ + cyclic perm's. (Hesse pencil).

Compactification: (still $n \geq 3$)

$$\begin{tikzcd} Y(n) \arrow[loop above, "GL_2(\mathbb{Z}/n\mathbb{Z})"] \arrow[r, hook] \arrow[d, "j"'] & X(n) \arrow[d, "\text{normalise } \mathbb{P}^1_{\mathbb{Z}[1/n]} \text{ in the function field of } Y(n)."] & \text{Cusps} \arrow[l] \arrow[d, "\text{finite étale}"] \\ \mathbb{A}^1_{\mathbb{Z}[1/n]} \arrow[r, hook] & \mathbb{P}^1_{\mathbb{Z}[1/n]} & \operatorname{Spec} \mathbb{Z}[1/n] \arrow[l, "\infty"'] \end{tikzcd}$$

Deligne-Rapoport:

$$\begin{tikzcd} E^{univ} \arrow[r, hook] \arrow[d] & \overline{E^{univ}} \arrow[d] \\ Y(n) \arrow[r, hook] & X(n) \end{tikzcd}$$

$\overline{E^{univ}}$ — a "generalised elliptic curve", over $\widehat{\text{Cusps}}$ get $(\mathrm{Tate}(q^n), \alpha \cdot g)_{\mathbb{Z}[1/n, \zeta_n]((q))}$

Katz-Mazur: (§10.13): extend $\underline{\omega}$ from $Y(n)$ to $X(n)$ using the Tate curve: one decrees that $\frac{dt}{t}$ is a generator on $\widehat{\text{Cusps}}$.

Then, for $n \geq 3$, $H \subset GL_2(\mathbb{Z}/n\mathbb{Z})$, $\mathbb{Z}[1/n] \to A$:

$$M(n, H, k, A) = H^0(X(n)_A, \underline{\omega}^{\otimes k})^H.$$

$$\begin{tikzcd} \underline{\omega} & \underline{\omega} \\ X(n)_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X(n) \arrow[d] \\ \operatorname{Spec}(A) \arrow[r] & \operatorname{Spec} \mathbb{Z}[1/n] \end{tikzcd}$$

Why can this be interesting?

Because $A \otimes_{\mathbb{Z}[1/n]} H^0(X(n), \underline{\omega}^{\otimes k}) \longrightarrow H^0(X(n)_A, \underline{\omega}^{\otimes k})$ need not be surjective, and also $A \otimes_{\mathbb{Z}[1/n]} M^H \longrightarrow (A \otimes_{\mathbb{Z}[1/n]} M)^H$ need not be surjective (if $\mathbb{Z}[1/n] \to A$ is flat, then both are isomorphisms).

<u>Hasse invariant</u> Let p be prime. For $E/S/\mathbb{F}_p$ we have:

$$\begin{tikzcd} E \arrow[rr, bend left, "F_E"] \arrow[r] \arrow[dr] & E^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d] \\ & S \arrow[r, "F_S"] & S \end{tikzcd} \qquad F_{E/S}$$

$$V_{E/S} = F^t_{E/S} : E^{(p)} \longrightarrow E, \qquad E \xrightarrow{F_{E/S}} E^{(p)}, \quad E^{(p)} \xrightarrow{[p]} E^{(p)}$$

For $U \subset S$ open, $\omega \in \underline{\omega}_{E/S}(U) = \Omega^1_{E/S}(E_U)$:

$$V^*_{E/S}(\omega) \in \Omega^1_{E^{(p)}/S}(E^{(p)}_U) = \underline{\omega}_{E^{(p)}/S}(U) = (F_S^* \underline{\omega}_{E/S})(U) =$$
$$= (\underline{\omega}^{\otimes p}_{E/S})(U).$$

So, $V^*_{E/S} \in \operatorname{Hom}_{O_S}(\underline{\omega}_{E/S}, \underline{\omega}^{\otimes p}_{E/S}) = \operatorname{Hom}_{O_S}(O_S, \underline{\omega}^{\otimes(p-1)}_{E/S}) = (\underline{\omega}^{\otimes(p-1)}_{E/S})(S)$.

<u>Notation:</u> $A(E/S)$, $\quad A \in M(1, p-1, \mathbb{F}_p)$

indeed: $A(\mathrm{Tate}(q)/_{\mathbb{F}_p((q))}) = \left(\frac{dt}{t}\right)^{\otimes p-1}_{\langle q \rangle}$, the q-exp. is 1. (Katz-Mazur, Thm 12.4.2.).

$$\begin{tikzcd} (\mu_p)_{\mathbb{F}_p((q))} \arrow[r, hook] & \mathrm{Tate}(q)_{\mathbb{F}_p((q))} \arrow[r, "F"] \arrow[dr, "{[p]}"'] & \mathrm{Tate}(q^p)_{\mathbb{F}_p((q))} \arrow[d, "V"] \\ & & \mathrm{Tate}(q)_{\mathbb{F}_p((q))} \end{tikzcd}$$

$$V^* \frac{dt}{t} = \frac{dt}{t} \in \underline{\omega}_{\mathrm{Tate}(q^p)}(\ldots), \qquad \frac{dt}{t} \mapsto \left(\frac{dt}{t}\right)^{\otimes p}, \qquad \underline{\omega}_{\mathrm{Tate}(q^p)}(\ldots) = \underline{\omega}^{(\otimes p)}_{\mathrm{Tate}(q)}(\ldots)$$

Now take $p=2$, then $k=1$, $M(1,1,\mathbb{Z}) = 0$ $\quad((-1)^k = -1 \ldots)$

So, for $n \geqslant 3$ odd, get on $X(n)_{\mathbb{Z}[1/n]}$: $\underline{\omega} \xrightarrow{\cdot 2} \underline{\omega} \twoheadrightarrow i_* i^* \underline{\omega}$,

where $i : X(n)_{\mathbb{F}_2} \longrightarrow X(n)_{\mathbb{Z}[1/n]}$ ~~an obstruction somewhere~~ gives

For $n=3$: $X(3) \cong \mathbb{P}^1_{\mathbb{Z}[\frac{1}{3}, \zeta_3]}$, $\deg(\underline{\omega}) = 1$ (over $\mathbb{Z}[1/3, \zeta_3]$), so A lifts, but

$\nexists$ $GL_2(\mathbb{Z}/3\mathbb{Z})$-invariant lift:

$$M(3,1,\mathbb{Z}[1/3])^G \xrightarrow{2\cdot} M(3,1,\mathbb{Z}[1/3])^G \longrightarrow \left(\cancel{\mathbb{F}_2 \otimes} M(3,1,\mathbb{Z}[1/3]/2)\right)^G \longrightarrow H^1(G, M(3,1,\mathbb{Z}[1/3]))$$
$$M(3,1,\mathbb{Z}[1/3])^G = 0, \qquad M(3,1,\mathbb{Z}[1/3])^G = 0, \qquad A \in \left(M(3,1,\mathbb{Z}[1/3]/2)\right)^G$$

Finally: $\deg(\underline{\omega}) = 1$ on $\mathbb{P}^1_{\mathbb{Z}[1/3, \zeta_3]}$ $\rightsquigarrow$ $\forall p \neq 3$: $\displaystyle\sum_{\substack{x \in \overline{\mathbb{F}}_p \\ E_x \text{ supersingular}}} \frac{1}{\# \operatorname{Aut}(E_x)} = \frac{p-1}{24}$, mass formula

<u>Katz-Mazur: 12.4.6.</u>