# Shimura varieties,
# notes for some introductory lectures

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These notes were written, in the Spring of 2012, to be used just by myself for lectures to be given at the Summer School "Around the Zilber-Pink conjectures," Paris June 2012. I was unfortunately unable to give these lectures, and was replaced at very short notice by Christophe Cornut and Bruno Klingler (many thanks to them).

At many places these notes are not as complete as I would like. But now Ariyan Javanpeykar asked me if he can use them for a "reading course", so I make them available. Comments, suggestions, corrections etc. are welcome (`edix@math.leidenuniv.nl`).

## Contents

# 1 Introduction

Shimura varieties form an intimidating subject. The subject was started by Shimura, and then reshaped by Deligne.

It is not easy to give a good introduction into this subject, because of its many prerequisites: Hodge theory, algebraic geometry, algebraic groups, number theory. One can say that it took a large part of the 20th century to develop the necessary tools for the theory of Shimura varieties (algebraic geometry, Hodge theory, algebraic and arithmetic groups, class field theory, for example).

**References**: Deligne's papers Travaux de Griffiths, Théorie de Hodge, Travaux de Shimura, Variétés de Shimura, Milne's notes "Introduction to Shimura varieties", Ullmo's notes "Autour de la conjecture d'André-Oort", Yafaev's notes "Introduction to André-Oort and Zilber-Pink", Voisin's book "Hodge theory and complex algebraic geometry I". Moonen's notes "An introduction to Mumford-Tate groups", Moonen's article "Models of Shimura varieties in mixed characteristics", Pink's thesis (compactifactions), etc....

I will do my best to make it into more than just a series of statements of facts with no proofs... My goal is to present this theory, but for details you will really have to read these references. For motivation, let me start with an example of a Shimura variety.

## 1.1 Moduli of complex elliptic curves

Let $\mathbb{H} = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$. We have $\mathrm{SL}_2(\mathbb{Z})$ acting on it:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau = \frac{a\tau + b}{c\tau + d}.$$

In this case, everything is quite explicit. We have the standard fundamental domain. The quotient can be interpreted as the set of complex elliptic curves up to isomorphism:

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \{E/\mathbb{C}\}/\cong, \quad \tau \mapsto [E_\tau := \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})]$$

As elliptic curves up to isomorphism are classified by their $j$-invariant, the quotient map is the holomorphic map

$$\mathbb{H} \to \mathbb{C}, \quad \tau \mapsto j(E_\tau) = q^{-1} + 744 + 196884q + \cdots, \quad q = e^{2\pi i\tau}.$$

The quotient $\mathbb{C}$ is the moduli space of complex elliptic curves; it is an algebraic variety, defined over $\mathbb{Q}$ (even over $\mathbb{Z}$) because we have the notion of elliptic curves over arbitrary fields and rings (and schemes).

For $\tau \in \mathbb{H}$,

$$\mathrm{End}(E_\tau) = \{z \in \mathbb{C} : z\cdot(\mathbb{Z}\tau + \mathbb{Z}) \subset \mathbb{Z}\tau + \mathbb{Z}\}.$$

Hence $\mathrm{End}(E_\tau) = \mathbb{Z}$ if $\dim_\mathbb{Q} \mathbb{Q}(\tau) \neq 2$, and $\mathrm{End}(E_\tau)$ is an order in $\mathbb{Q}(\tau)$ if $\dim_\mathbb{Q} \mathbb{Q}(\tau) = 2$, and then $E_\tau$ is called a CM-elliptic curve, and $\tau \in \mathbb{H}$ and $j(E_\tau) \in \mathbb{C}$ are called CM-points and also special points.

CM-theory for elliptic curves says:

$$\text{if } \tau \text{ special, then } j(\tau) \in \mathbb{Q}(\tau)^{\mathrm{ab}}, \text{ algebraic integer,}$$

and it gives a description of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\tau))$-action in terms of class field theory.

A theorem of Theodor Schneider says:

$$\text{if } \tau \text{ is algebraic but not special, then } j(\tau) \text{ is transcendental.}$$

So the special $\tau$ are the only $\tau$ with $\tau$ and $j(\tau)$ algebraic.

## 1.2 The André-Oort conjecture for $\mathbb{C}^2$

Now we consider $\mathbb{C}^2$ as the moduli space of pairs of elliptic curves:

$$\mathbb{C}^2 = (\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}))\backslash(\mathbb{H} \times \mathbb{H}).$$

A point $x = (x_1, x_2) \in \mathbb{C}^2$ is called *special* if both $x_1$ and $x_2$ are special.

**1.2.1 Conjecture. (André-Oort)** *Let $\Sigma \subset \mathbb{C}^2$ be a set of special points, and let $Z$ be an irreducible component of $\Sigma^{\mathrm{Zar}}$. Then $Z$ is one of the following:*

1. *a special point,*

2. *$\{x_1\} \times \mathbb{C}$ with $x_1$ special,*

3. *$\mathbb{C} \times \{x_2\}$ with $x_2$ special,*

4. *the image $T_n$ (a Hecke correspondence) of*

$$j \colon \mathbb{H} \to \mathbb{H} \times \mathbb{H} \to \mathbb{C}^2, \quad \tau \mapsto (\tau, n\tau) \mapsto (j(\tau), j(n\tau))$$

   *for some $n \in \mathbb{Z}_{\geq 1}$,*

5. *$\mathbb{C}^2$ itself.*

In general, the André-Oort conjecture says that the irreducible components of the Zariski closure of a set of special points are special subvarieties: images of "Shimura-Hecke morphisms" of Shimura varieties. In the case of $\mathbb{C}^2$ one gets the list that is given. The Hecke correspondences $T_n$ are precisely the images $T_{g_1,g_2}$ of

$$\mathbb{H} \to \mathbb{H} \times \mathbb{H} \to \mathbb{C} \times \mathbb{C}, \quad \tau \mapsto (g_1\tau, g_2\tau) \mapsto (j(g_1\tau), j(g_2\tau))$$

with $g_1$ and $g_2$ in $\mathrm{GL}_2(\mathbb{Q})^+$ (positive determinant).

## 1.3 A little bit of history of the A-O and Z-P conjecture.

A very recent reference: Umberto Zannier's 2012 book "Some problems of unlikely intersections in arithmetic and geometry".

André stated the case for curves in Shimura varieties in 1989, and Oort stated the case of subvarieties in $A_g$ (moduli space of principally polarised abelian varieties) in 1995. The foundations were laid by Ben Moonen in his thesis (1995). He also obtained the first results under an assumption on the reduction of $\Sigma$ at a suitable prime number $p$. The conjecture is an analog of the Manin-Mumford conjecture: the ambient variety is a complex abelian variety, special points are torsion points, special subvarieties are translates of abelian subvarieties by torsion points. The Manin-Mumford conjecture for proved first by Raynaud (1983) in the case where $Z$ is a curve, and then later by many peoply using many different tools. The culmination (for me) is an equidistribution result by Shouwu Zhang: if $A$ is an abelian variety over a number field $K \subset \mathbb{C}$, and $(x_n)_n$ is a *small* and *strict* sequence ($h(x_n) \to 0$, and no subsequence in a proper *special* subvariety) in $A(\overline{\mathbb{Q}})$, then the Galois orbits $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x_n$ are equidistributed in $A(\mathbb{C})$ for the translation invariant probability measure. Therefore, such sequences are *generic*: no subsequence in a proper closed subvariety. In the André-Oort case, there is such an equidistribution conjecture, but we seem to be still very far from such a general result. See work of Zhang. Of course, abelian varieties can be replaced with other algebraic groups: tori, semi-abelian varieties, $\mathbb{C}$ can be replaced with other fields. Here too, much has been done, but I know too little about it.... See Chambert-Loir's Bourbaki lecture from 2011. Let me just mention work of Breuer and Hubschmid on the case of Drinfeld modules.

Conjecture 1.2.1 above was proved by myself in 1995, assuming GRH for number fields, and by Yves André, unconditionally, in 1997(?). We both used the Galois action on special points, I used Hecke correspondences and André used a result of Masser on the $j$-function at rational values.

The method I used, Galois action, Hecke correspondences and intersections, has been generalised, first by myself and Andrei Yafaev to the case of curves in Shimura varieties, and later by Klingler, Ullmo and Yafaev to the general case, but still assuming GRH. Also equidistribution techniques have been added. Results by Clozel, Ullmo, Yafaev only concern sequences of positive dimensional special subvarieties. The method I used does its work in the quotient, that is, in the algebraic variety, and therefore uses techniques mostly from algebraic geometry and number theory.

A recent development is the addition, mainly by Jonathan Pila, of tools (o-minimal structure, Pila-Wilkie counting theorem, of course, Thomas Scanlon reports on that!) that can be used in $\mathbb{H}^n$ and its analog in the general case. For example, building on work of himself with Bombieri, and with Wilkie, Pila has proved André-Oort for $\mathbb{C}^n$ unconditionally.

**1.4 Theorem. (Pila)** *The André-Oort conjecture for $\mathbb{C}^n$ is true.*

Galois orbits of special points keep playing the same role as in the older approach. It seems to me that one big issue now is to extend Pila's method to the general case.

Since about 2000 Bombieri, Masser and Zannier, and then others, studied intersections of subvarieties of algebraic groups (tori, abelian varieties, semi-abelian varieties) with closed subgroups. This has led to the Zilber-Pink conjecture; the role of special points is then played by "unlikely intersections". Richard Pink has formulated the most general form in the context of *mixed Shimura varieties*, in his preprint "A Common Generalization of the Conjectures of André-Oort, Manin-Mumford, and Mordell-Lang" (2005):

**1.5 Conjecture.** *Let $S$ be a connected component of a mixed Shimura variety, over $\mathbb{C}$, $Z \subset S$ an irreducible closed subvariety that is not contained in any special $S' \subsetneq S$. Then*

$$\bigcup\{Z \cap S' : S' \subset S \text{ special}, \operatorname{codim}_S(S') > \dim(Z)\}$$

*is not Zariski dense in $Z$.*

Zilber's conjecture is indeed a special case: if $Y$ is an irreducible component of the Zariski closure of that union, then Pink's conjecture implies that $Y$ is contained in some $S' \subsetneq S$.

I mention Pink's conjecture here, because it means that the correct notion of special subvariety, in this full generality, comes from that context, and hence from Hodge theoretical conditions. (I am thinkking in particular of the case of relative Manin-Mumford for semi-abelian schemes.)

This should motivate us to go into the theory of Shimura varieties. The most important cases here are the moduli spaces $A_g$ of principally polarised abelian varieties of dimension $g$. However, in order to keep the complexity under control when dealing with all special subvarieties, it seems preferable to use Deligne's general definition, which is technically demanding, but which gives flexiblity and a convenient description of the Galois action on special points:

$$\operatorname{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K).$$

Here, $G$ is a reductive algebraic group over $\mathbb{Q}$, and $X$ is a $G(\mathbb{R})$-orbit in $\operatorname{Hom}(\mathbb{S}, G_{\mathbb{R}})$, where $\mathbb{S}$ is the Deligne torus whose representations are the $\mathbb{R}$-Hodge structures, $\mathbb{A}$ is the adèle ring of $\mathbb{Q}$..., there is a lot to be explained.

# 2 Some Hodge theory

Let us begin the story with Hodge structures. These are really at the start of the theory of Shimura varieties.

Let $V$ be a finite dimensional $\mathbb{R}$-vector space, and $V_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V$ its complexification. Then complex conjugation acts on $V_{\mathbb{C}}$ by $\lambda \otimes v \mapsto \overline{\lambda} \otimes v$. If $e = (e_1, \ldots, e_d)$ is an $\mathbb{R}$-basis of $V$ then $e$ (or rather $1 \otimes e$) is also a $\mathbb{C}$-basis of $V_{\mathbb{C}}$ and $\overline{\sum_j \lambda_j e_j} = \sum_j \overline{\lambda_j} e_j$.

A *Hodge decomposition* of $V$ is a direct sum decomposition of $V_{\mathbb{C}}$ into $\mathbb{C}$-subspaces $V^{p,q}$ indexed by $\mathbb{Z}^2$:

$$V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}, \quad \text{such that} \quad \overline{V^{p,q}} = V^{q,p}.$$

Note that not only the subspaces are important, but also their labelling with pairs $(p, q)$ is part of the data.

An $\mathbb{R}$-*Hodge structure* is a finite dimensional $\mathbb{R}$-vector space $V$ with a Hodge decomposition.

The *type* of this Hodge structure is the set of $(p, q)$ such that $V^{p,q} \neq 0$.

The functor $\mathbb{C} \otimes_{\mathbb{R}} \cdot$ from $\mathbb{R}$-vector spaces to $\mathbb{C}$-vector spaces with a semi-linear involution, sending $V$ to $(V, \bar{\cdot})$ is an equivalence, with inverse $(W, \sigma) \mapsto W^{\sigma}$.

For $n$ in $\mathbb{Z}$, the subspace

$$\bigoplus_{p+q=n} V^{p,q}$$

of $V_{\mathbb{C}}$ is stable under complex conjugation, and therefore the complexification of the subspace $V_n$ of $V$ consisting of elements in $\oplus_{p+q=n} V^{p,q}$ that are invariant under complex conjugation. The decomposition $V = \oplus_{n \in \mathbb{Z}} V_n$ is called the *weight decomposition* of $V$.

A $\mathbb{Q}$-*Hodge structure* is a finite dimensional $\mathbb{Q}$-vector space $V$ together with an $\mathbb{R}$-Hodge structure on $V_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Q}} V$, and a $\mathbb{Z}$-*Hodge structure* is a free $\mathbb{Z}$-module of finite rank $V$ together with an $\mathbb{R}$-Hodge structure on $V_{\mathbb{R}}$.

**2.1 Example. (Complex tori)** The simplest way in which $\mathbb{Z}$-Hodge structures arise is from complex tori, quotients of $\mathbb{C}$-vector spaces by lattices (for example complex elliptic curves). Indeed, let $V$ be a finite dimensional $\mathbb{C}$-vector space, and let $L \subset V$ be a lattice, that is, the free $\mathbb{Z}$-module generated by an $\mathbb{R}$-basis of $V$. Then $L_{\mathbb{R}} = V$ is a $\mathbb{C}$-vector space, and let us denote by $J$ the multiplication by $i$ on $L_{\mathbb{R}}$, note that $J^2 + \mathrm{id} = 0$ on $L_{\mathbb{R}}$; we view $L_{\mathbb{R}}$ as an $\mathbb{R}$-vector space. Then on $L_{\mathbb{C}}$ we have $J^2 + \mathrm{id} = (J - i)(J + i)$, and therefore $L_{\mathbb{C}}$ decomposes into two conjugate $\mathbb{C}$-subspaces: $V^{-1,0}$ on which $J$ acts as $i$ and $V^{0,-1}$ on which $J$ acts as $-i$ (the reason why we label them with $(-1, 0)$ and $(0, -1)$ will become clear in a moment). More generally, the endomorphism $a + bJ$ of $L_{\mathbb{R}}$ (with $a$ and $b$ in $\mathbb{R}$) induces $a + bi$ on $V^{-1,0}$ and $a - bi$ on $V^{0,-1}$. We see that the action of $\mathbb{C}^{\times}$ on $L_{\mathbb{R}} = V$ coming from the $\mathbb{C}$-vector space structure of $V$ induces, for $z \in \mathbb{C}^{\times}$, multiplication by $z = z^1 \overline{z}^0$ on $V^{-1,0}$ and by $\overline{z} = z^0 \overline{z}^1$ on $V^{0,-1}$. The reason to give $L$ weight $-1$ is that $L$ is the first *homology* group of $V/L$ and that we reserve positive weights for *cohomology*. We will now see that all $\mathbb{R}$-Hodge structures are given by an action of $\mathbb{C}^{\times}$.

## 2.2 The Deligne torus

Let $V$ be an $\mathbb{R}$-Hodge structure. Then we let $\mathbb{C}^\times$ act on $V_\mathbb{C}$ by letting $z$ act on $V^{p,q}$ as multiplication by $z^{-p}\overline{z}^{-q}$ (there is also the other convention $z^p\overline{z}^q$, but the one we use seems to be standard now for Shimura varieties). Then the multiplication by $z$ commutes with complex conjugation on $V_\mathbb{C}$ (for $v$ in $V^{p,q}$ we have $\overline{z{\cdot}v} = \overline{z^{-p}\overline{z}^{-q}v} = \overline{z}^{-p}z^{-q}\overline{v} = z{\cdot}\overline{v}$ because $\overline{v} \in V^{q,p}$), and therefore this $\mathbb{C}^\times$-action comes from an $\mathbb{R}$-linear $\mathbb{C}^\times$-action on $V$, $h\colon \mathbb{C}^\times \to \mathrm{GL}(V)$, and from it one recovers the decomposition of $V_\mathbb{C}$. This action is *algebraic* in the sense that, with respect to any $\mathbb{R}$-basis of $V$, $a+bi$ acts as a matrix whose coefficients are polynomials in $a$, $b$ and $1/(a^2+b^2)$. This condition is usually stated in terms of the action of an algebraic group $\mathbb{S}$ over $\mathbb{R}$, the Deligne torus. My favourite way of working with algebraic groups is with their functors of points; this seems the most natural way. The Deligne torus $\mathbb{S}$ is defined as the Weil restriction from $\mathbb{C}$ to $\mathbb{R}$ of the multiplicative group over $\mathbb{C}$:

$$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m}\mathbb{C}}$$

This means that for all $\mathbb{R} \to A$:

$$\mathbb{S}(A) = \mathbb{G}_{\mathrm{m}\mathbb{C}}(A \otimes_\mathbb{R} \mathbb{C}) = (A \otimes_\mathbb{R} \mathbb{C})^\times = (A[x]/(x^2+1))^\times = \{a + bx : a, b \in A, a^2+b^2 \in A^\times\}.$$

One can also consider the action of $\mathbb{C}^\times$ on itself, and embed $\mathbb{S}$ into $\mathrm{GL}_{2,\mathbb{R}}$:

$$\mathbb{S}(A) = \{\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) : a, b \text{ in } A \text{ with } a^2+b^2 \in A^\times\}.$$

From this we see that $\mathbb{S}$ is represented by $\mathbb{R}[a, b, 1/(a^2+b^2)]$, that is, this $\mathbb{R}$-algebra is the coordinate ring of $\mathbb{S}$. We also see that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$. One can show that the $\mathbb{C}^\times$-action above corresponds to a unique representation of $\mathbb{S}$ on $V$, that is, the action is given by a morphism of group schemes $h\colon \mathbb{S} \to \mathbf{GL}(V)$, where for every $\mathbb{R} \to A$, $\mathbf{GL}(V)(A) = \mathrm{Aut}_{A\text{-Mod}}(V_A)$, with $V_A = A \otimes_\mathbb{R} V$. And vice versa, every representation of $\mathbb{S}$ is an $\mathbb{R}$-Hodge structure.

Another way to see this is by looking at the character group $\mathrm{Hom}(\mathbb{S}_\mathbb{C}, \mathbb{G}_{\mathrm{m}\mathbb{C}})$ of the torus $\mathbb{S}$, with its action by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Let us compute it. For $A$ a $\mathbb{C}$-algebra, we have:

$$A \otimes_\mathbb{R} \mathbb{C} \xrightarrow{\sim} A \times A, \quad a \otimes z \mapsto (az, a\overline{z}),$$

hence

$$\mathbb{S}(A) = (A \times A)^\times = A^\times \times A^\times.$$

This gives, by Yoneda and the defining property of the pullback $\mathbb{S}_\mathbb{C}$, an isomorphism

$$(z, \overline{z})\colon \mathbb{S}_\mathbb{C} \xrightarrow{\sim} \mathbb{G}_{\mathrm{m}\mathbb{C}}^2, \quad \text{inducing } \mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \to \mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times,\ z \mapsto (z, \overline{z}).$$

This means that $X := \mathrm{Hom}(\mathbb{S}_\mathbb{C}, \mathbb{G}_{\mathrm{m}\mathbb{C}})$ is a free $\mathbb{Z}$-module with basis two characters that we can write $z$

and $\overline{z}$ and that are exchanged by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Indeed, let us compute the complex conjugation on $\mathbb{S}(\mathbb{C})$:

$$\begin{tikzcd}
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \arrow[r] & \mathbb{C} \times \mathbb{C} & \\
a \otimes z \arrow[r, mapsto] \arrow[d, mapsto] & (az, a\overline{z}) \arrow[r, equal] \arrow[d, mapsto] & (x, y) \arrow[d, mapsto] \\
\overline{a} \otimes z \arrow[r, mapsto] & (\overline{a}z, \overline{a}\,\overline{z}) \arrow[r, equal] & (\overline{y}, \overline{x})
\end{tikzcd}$$

which is the ordinary complex conjugation on $\mathbb{C} \times \mathbb{C}$, followed by the swap, that is, the exchange of coordinates. For later use we also introduce the *cocharacter* $\mu$ of $\mathbb{S}_{\mathbb{C}}$:

$$\begin{tikzcd}
\mu\colon \mathbb{G}_{\mathrm{m}\mathbb{C}} \arrow[r] & \mathbb{G}_{\mathrm{m}\mathbb{C}} \times \mathbb{G}_{\mathrm{m}\mathbb{C}} \arrow[r, "(z,\overline{z})^{-1}"] & \mathbb{S}_{\mathbb{C}} \\
a \arrow[r, mapsto] & (a, 1) &
\end{tikzcd}$$

By its definition, $\mu$ is characterised by:

$$z \circ \mu = 1 \quad \overline{z} \circ \mu = 0, \quad \text{in } X(\mathbb{G}_{\mathrm{m}\mathbb{C}}).$$

A representation of $\mathbb{S}$ on a finite dimensional $\mathbb{R}$-vector space $V$ is then a grading $V_{\mathbb{C}} = \oplus_x V_x$, compatible with Galois action: $V_{\sigma x} = \overline{V_x}$. We conclude again that $\mathbb{R}$-Hodge structure on $V$ is the same thing as a representation $h\colon \mathbb{S} \to \mathbf{GL}(V)$. A $\mathbb{Q}$-Hodge structure is a $\mathbb{Q}$-vector space with $h\colon \mathbb{S} \to \mathbf{GL}(V_{\mathbb{R}})$. Etc.

Using this interpretation as representations, we define morphisms of Hodge structures, tensor products, duals. We have $w\colon \mathbb{G}_{\mathrm{m}\mathbb{R}} \to \mathbb{S}$, corresponding on $\mathbb{R}$-points to $\mathbb{R}^\times \to \mathbb{C}^\times$, $x \mapsto x^{-1}$. This gives the weight decomposition of $V$: $V = \oplus V_n$, $x$ acts as $x^n$ on $V_n$.

## 2.3 Tate twists

For $n \in \mathbb{Z}$ we let $\mathbb{Q}(n)$ be the $\mathbb{Q}$-Hodge structure $V = \mathbb{Q}$, where $\mathbb{S}$ acts on $V_{\mathbb{R}} = \mathbb{R}$ by the character $(z\overline{z})^n$, hence $V_{\mathbb{C}} = V_{\mathbb{C}}^{-n,-n}$. Etc. for $\mathbb{Z}(n)$ and $\mathbb{R}(n)$. Often one lets the underlying $\mathbb{Z}$-module of $\mathbb{Z}(n)$ be $(2\pi i)^n\mathbb{Z}$ in $\mathbb{C}$. The reason for this is the factors $2\pi i$ that occur in comparison between singular and algebraic or holomorphic de Rham cohomology (natural generators on both sides differ by such factors). For example, the exponential map $\mathbb{C} \to \mathbb{C}^\times$ has kernel $2\pi i\mathbb{Z}$, and therefore it identifies $2\pi i\mathbb{Z}$ with the image of $\mathrm{H}_1(\mathbb{C}^\times, \mathbb{Z})$ in $\mathbb{C}$. A bit more seriously:

$$\mathrm{H}_1(\mathbb{C}^\times, \mathbb{Z}) = \mathbb{Z}\cdot[\gamma], \quad \gamma\colon [0,1] \to \mathbb{C}^\times, \quad t \mapsto e^{2\pi i t},$$

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{C}^\times) = \mathbb{C}\cdot \left[\frac{dz}{z}\right],$$

and note that $(dz)/z$ is the natural generator "over $\mathbb{Z}$" (more on this later, where we consider $\mathrm{H}^2(\mathbb{P}^1(\mathbb{C}), \mathbb{Z})$ and $\mathrm{H}^2_{\mathrm{dR}}(\mathbb{P}^1/\mathbb{Z})$), and under the natural pairing we get

$$\int_\gamma \frac{dz}{z} = 2\pi i.$$

## 2.4 Weil operator

For an $\mathbb{R}$-Hodge structure $(V, h)$ we have the *Weil operator* $C := h(i)$, acting as $i^{q-p}$ on $V^{p,q}$, and $C^2 = h(-1)$ acts as $(-1)^n$ on $V_n$.

## 2.5 The Hodge filtration

For $V$ an $\mathbb{R}$-Hodge structure of some weight $n$ we let $F^\bullet$ be the (descending) filtration on $V_\mathbb{C}$ given by:

$$F^i(V_\mathbb{C}) = \bigoplus_{p \geq i} V^{p,q} = \bigoplus_{p \geq i} V^{p,n-p}.$$

Note that for $i + j = n$ we have:

$$F^i(V_\mathbb{C}) \cap \overline{F^j(V_\mathbb{C})} = \bigoplus_{p \geq i} V^{p,n-p} \cap \bigoplus_{q \geq j} \overline{V^{q,n-q}} = \bigoplus_{p \geq i} V^{p,n-p} \cap \bigoplus_{q \geq j} V^{n-q,q} = V^{i,j}.$$

We let $h_\mathbb{C} \colon \mathbb{S}_\mathbb{C} \to \mathbf{GL}(V_\mathbb{C})$. We define $\mu_h := h_\mathbb{C} \circ \mu \colon \mathbb{G}_{\mathrm{m}\mathbb{C}} \to \mathbf{GL}(V_\mathbb{C})$. Then

$$V^{p,n-p} = \{v \in V_\mathbb{C} : \text{for all } z \text{ in } \mathbb{C}^\times,\ \mu_h(z)v = z^{-p}v\},$$

and $F^i(V_\mathbb{C})$ consists of sums of $v$ in $V_\mathbb{C}$ such that $\mu_h(z)v = z^{-p}v$ for some $p \geq i$.

## 2.6 Polarisations

Let $V$ be a $\mathbb{Q}$-Hodge structure of some weight $n$. A polarisation on $V$ is a morphism of Hodge structures:

$$\psi \colon V \otimes V \to \mathbb{Q}(-n)$$

such that

$$\psi_C \colon V_\mathbb{R} \times V_\mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto \psi(x, h(i)y)$$

is symmetric and positive definite (recall that $C = h(i)$ is called the Weil operator, and that $\mathbb{Q}(-n) = \mathbb{Q}$). That $\psi$ is a morphism means that $\psi$ is $\mathbb{Q}$-linear (corresponds to a $\mathbb{Q}$-bilinear map $V \times V \to \mathbb{Q}$) such that for all $z \in \mathbb{C}^\times$, $x$ and $y$ in $V_\mathbb{R}$, we have

$$\psi_\mathbb{R}(h(z)x, h(z)y) = (z\overline{z})^{-n}\psi_\mathbb{R}(x, y).$$

The symmetry condition means then that for all $x$ and $y$ in $V_\mathbb{R}$:

$$\psi(x, y) = \psi(h(i)x, h(i)y) = \psi(y, h(i)^2x) = \psi(y, h(-1)x) = \psi(y, (-1)^n x) = (-1)^n\psi(y, x).$$

Hence $\psi$ is alternating if $n$ is odd and symmetric if $n$ is even. Of course, similar for $\mathbb{Z}$ and $\mathbb{R}$-Hodge structures.

Let us also look at $\psi_\mathbb{C} \colon V_\mathbb{C} \times V_\mathbb{C} \to \mathbb{C}$. For $x$ in $V_\mathbb{C}^{p,q}$ and $y$ in $V_\mathbb{C}^{p',q'}$ we have

$$\begin{aligned}\psi_\mathbb{C}(h(z)x, h(z)y) &= \psi_\mathbb{C}(z^{-p}\overline{z}^{-q})x, z^{-p'}\overline{z}^{-q'})y) = z^{-p-p'}\overline{z}^{-q-q'}\psi_\mathbb{C}(x, y)\\ &= (z\overline{z})^{-n}\psi_\mathbb{C}(x, y).\end{aligned}$$

This means that $\psi_{\mathbb{C}}$ gives a perfect pairing between $V^{p,q}$ and $V^{q,p}$ and zero if $(p',q') \neq (q,p)$.

The motivation for the definition of $\psi_C$ comes from cohomology, as we will see soon. The condition that $\psi_C$ is symmetric and positive definite is equivalent to the sesquilinear form

$$\psi'_C \colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}, \quad (x,y) \mapsto \psi_{\mathbb{C}}(x, h(i)\overline{y})$$

being hermitian ($\psi'_C(y,x) = \overline{\psi'_C(x,y)}$) and positive definite. The decomposition $V_{\mathbb{C}} = \oplus_{p,q} V^{p,q}$ is then orthogonal. If $W \subset V$ is a sub $\mathbb{Q}$-Hodge structure, and $\psi$ a polarisation on $V$, then $\psi$ induces a polarisation on $W$ because of the positive definiteness of $\psi_C$ and $\psi'_C$, and then $V$ splits as $V = W \oplus W^{\perp}$ with $W^{\perp}$ a sub $\mathbb{Q}$-Hodge structure. A polarisation on a $\mathbb{Q}$-Hodge structure $V = \oplus_n V^n$ (weight decomposition) is a system $(\psi^n)_n$ of polarisations on the $V^n$. A $\mathbb{Q}$-Hodge structure $V$ is called polarisable if there exists a polarisation on it. Direct sums, tensor products and duals of polarised $\mathbb{Q}$-Hodge structures are naturally polarised. Polarisable $\mathbb{Q}$-Hodge structures form a semi-simple $\mathbb{Q}$-linear abelian category.

**2.7 Example.** Let $V$ a $\mathbb{Z}$-Hodge structure of type $\{(-1,0),(0,-1)\}$ as in the example above. Such Hodge structures correspond to $\mathbb{C}$-structures on $V_{\mathbb{R}}$, via $V_{\mathbb{R}} \hookrightarrow V_{\mathbb{C}} \twoheadrightarrow V^{-1,0}$, or, equivalently, via $h \colon \mathbb{C}^{\times} \to \mathrm{GL}(V_{\mathbb{R}})$ which extends to an $\mathbb{R}$-algebra morphism $\mathbb{C} \to \mathrm{End}(V_{\mathbb{R}})$. In this case, a polarisation is a symplectic form $\psi \colon V \times V \to \mathbb{Z}$ such that for all $x$ and $y$ in $V_{\mathbb{R}}$ one has $\psi(ix,iy) = \psi(x,y)$ and such that $(x,y) \mapsto \psi(x,iy)$ is symmetric and positive definite on $V_{\mathbb{R}}$. The $\mathbb{Z}$-valued form $-\psi$, or maybe its opposite, is called the Riemann form. In this case, $H \colon V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{C}$ given by $(x,y) \mapsto (\psi(x,iy) - i\psi(x,y))$ is hermitian and positive definite. Note that $-\psi$ is the imaginary part of $H$. Such a Riemann form is used to construct a holomorphic line bundle on $V/L$ by which it can be embedded in a projective space. Only the polarisable complex tori are algebraic.

## 2.8 Variations of Hodge structures

Let $S$ be a complex (analytic) manifold, and $V$ a finite dimensional $\mathbb{R}$-vector space. Suppose that for each $s$ in $S$ we have a Hodge structure $h_s$ on $V$, of weight $n$ (independent of $s$). Then we have $V_{\mathbb{C}} = \oplus_{p+q=n} V_s^{p,q}$, and we have the filtration $F_s^{\bullet}(V_{\mathbb{C}})$. Then we say that $h_s$ or $F_s^{\bullet}(V_{\mathbb{C}})$ varies continuously with $s$ if the dimensions of the $V_s^{p,q}$ are constant, and if the subspaces $V_s^{p,q}$ vary continuously with $s$. To state this, consider the Grassmannians $G_d(V_{\mathbb{C}})$ that parametrise the $d$-dimensional sub-$\mathbb{C}$-vector spaces of $V_{\mathbb{C}}$. Then we ask that for each $(p,q)$ the map $S \to G_{d(p,q)}(V_{\mathbb{C}})$, $s \to V_s^{p,q}$ is continuous. How to see $G_d(V_{\mathbb{C}})$ as a manifold? Well, at a point $W$ in it, choose a complementary subspace $U$: $V_{\mathbb{C}} = W \oplus U$ and let $G_d(V_{\mathbb{C}})_U$ be the set of $W'$ with $W' \cap U = \{0\}$. Then each such $W'$ is the graph of a unique linear map $\phi \colon W \to U$. Now note that $U \to V_{\mathbb{C}}/W$ is an isomorphism, so we have $\mathrm{Hom}(W, V_{\mathbb{C}}/W)$ in bijection with $G_d(V_{\mathbb{C}})_U$. These are charts. Even, this gives an isomorphism from $\mathrm{Hom}(W, V_{\mathbb{C}}/W)$ to $T_{G_d(V_{\mathbb{C}})}(W)$ that is independent of the choice of $U$; see Voisin, or say this. The algebraic group $\mathbf{GL}(V)$ acts transitively on $G_d(V)$. Let $\mathbf{GL}(V)_W$ be the stabiliser of $W$. Then $T_{G_d(V)}(W)$ is the quotient $T_{\mathbf{GL}(V)}(1)/T_{\mathbf{GL}(V)_W}(1)$, that is, it is the quotient of $\mathrm{End}(V)$ by its subspace of $f$ with $fW \subset W$. Consider $\mathrm{End}(V) \to \mathrm{Hom}(W, V/W)$. This is surjective and its kernel is precisely the set of $f$ with $fW \subset W$.

We also have the Plücker embedding: $G_d(V_{\mathbb{C}}) \to \mathbb{P}(\bigwedge^d(V_{\mathbb{C}}))$, sending $W$ to the line $\bigwedge^d(W)$ in $\bigwedge^d(V_{\mathbb{C}})$.

We say that the $h_s$ vary *holomorphically* with $s$ if the filtration $F_s^\bullet(V_{\mathbb{C}})$ varies holomorphically with $s$ in the sense that the maps from $S$ to the appropriate Grassmannians of $V_{\mathbb{C}}$ are holomorphic. The tangent maps are then $\mathbb{C}$-linear. Note that $\overline{F_s^\bullet(V_{\mathbb{C}})}$ then varies anti-holomorphically.

Finally, in order for the $h_s$ to be called a *variation of Hodge structures* the *Griffiths transversality condition* must be verified: for each $s$ in $S$ and each $p$, the image of $T_S(s)$ in the tangent space of $G_d(C_{\mathbb{C}})$ must be in the subspace $\mathrm{Hom}(F_s^p, F_s^{p-1}/F_s^p)$ of $\mathrm{Hom}(F_s^p, V_{\mathbb{C}}/F_s^p)$. All these definitions are motivated by the geometric origin of HS's.

If $F_s^\bullet(V_{\mathbb{C}})$ varies holomorphically with $s$, then the $\mathcal{O}_S$-module $\mathcal{V} := \mathcal{O}_S \otimes_{\mathbb{R}} V$ has the filtration $F^\bullet\mathcal{V}$ by locally free submodules given by $F_s^\bullet$ at each $s$, such that the $F^p\mathcal{V}/F^{p+1}\mathcal{V}$ are locally free. For $v$ in $V$ and $p$ in $\mathbb{Z}$, the set

$$\{s \in S : v \in F_s^p V_C C\} \subset S$$

is an analytic subset, because it is the set of zeros of $v$ in $\mathcal{V}/F^p\mathcal{V}$.

**2.8.1 Definition.** Let $S$ be a non-singular complex algebraic variety, and let $S^{\mathrm{an}}$ be its associated complex analytic manifold. A polarised variation of $\mathbb{Z}$-HS on $S$ of weight $n$ is:

1. a locally free $\mathcal{O}_S$-module $\mathcal{V}$ of finite rank,
2. a filtration $F^\bullet\mathcal{V}$ such that the $F^i\mathcal{V}/F^{i+1}\mathcal{V}$ are locally free,
3. a locally constant $\mathbb{Z}_{S^{\mathrm{an}}}$-module $V$ and an isomorphism $\mathcal{O}_{S^{\mathrm{an}}} \otimes V \xrightarrow{\sim} \mathcal{V}^{\mathrm{an}}$
4. a $\psi\colon V \otimes V \to \mathbb{Z}(-n)_{S^{\mathrm{an}}}$

satisfying Griffiths transversality and such that for all $s$ in $S^{\mathrm{an}}$, $(V_s, F_s^\bullet, \psi_s)$ is a polarised $\mathbb{Z}$-HS of weight $n$.

## 2.9 Hodge classes, Hodge loci

**2.9.1 Definition.** Let $V$ be a $\mathbb{Q}$-HS, of weight $0$. Then the $\mathbb{Q}$-vector space of Hodge classes in $V$ is $V \cap (V_{\mathbb{C}})^{(0,0)}$.

The importance of this will become clear in the next section (Hodge conjecture). The following theorem gives a strong result on the nature of the loci that are given by the condition that a given class is a Hodge class in a variation of $\mathbb{Z}$-HS. It is of interest for us because these are our "special subvarieties" in the context of Hodge structures.

**2.9.2 Theorem. (Cattani-Deligne-Kaplan, 1995)** *Let $S$ be a connected non-singular complex algebraic variety, and $(\mathcal{V}, F^\bullet, V)$ a polarisable variation of $\mathbb{Z}$-HS of weight $0$. Let $s \in S^{\mathrm{an}}$, and $v_s \in V_s$ a Hodge class. Let $\widetilde{S^{\mathrm{an}}} \to S^{\mathrm{an}}$ be the universal covering and $v \in V(\widetilde{S^{\mathrm{an}}})$ the continuation of $v_s$. Then the image in $S$ of the set $\{t \in \widetilde{S^{\mathrm{an}}} : v_t$ is a Hodge class$\}$ is an* algebraic *subvariety of $S$.*

# 3 Geometric origin of Hodge structures.

First we look at it complex analytically. Let $X$ be a complex manifold, meaning that it is a topological space, Haussdorf and second countable in order to have partitions of unity, with for each open $U \subset X$ the $\mathbb{C}$-algebra $\mathcal{O}_X(U)$ of holomorphic functions $U \to \mathbb{C}$; $\mathcal{O}_X$ is a sheaf of $\mathbb{C}$-valued functions, and locally $(X, \mathcal{O}_X)$ is isomorphic to an open subset of $\mathbb{C}^n$ for some $n$, with its sheaf of holomorphic functions. More explicitly: each $x$ in $X$ has an open neighborhood $U$, an integer $n$ and functions $z_1, \ldots, z_n$ in $\mathcal{O}(U)$ that induce an isomorphism $U \to V \subset \mathbb{C}^n$ open. We say that $z_1, \ldots, z_n$ are local coordinates at $x$ if for all $j$, $z_j(x) = 0$. The stalk $\mathcal{O}_{X,x}$ is then the $\mathbb{C}$-algebra of convergent power series in the $z_j$.

## 3.1 The real de Rham complex

We can view $X$ as a smooth ($C^\infty$) real manifold by defining the sheaf $C_{X,\mathbb{R}}$ of smooth real functions, we can use $x_j := \Re(z_j), y_j := \Im(z_j)$, $1 \leq j \leq n$, as local coordinates. The real tangent space at $x$, $T_{X,\mathbb{R}}(x) = \mathrm{Der}_\mathbb{R}(C_{X,\mathbb{R},x}, \mathbb{R})$, has $\mathbb{R}$-basis $f \mapsto (\partial f/\partial x_1)x$, $f \mapsto (\partial f/\partial y_1)x$, $\ldots$, $f \mapsto (\partial f/\partial x_n)x$, $f \mapsto (\partial f/\partial y_n)x$. The tangent spaces at all $x$ in $X$ together make up the real tangent bundle of $X$, whose sheaf of sections we denote by $T_{X,\mathbb{R}}$. For $U \subset X$ open we have $T_{X,\mathbb{R}}(U) = \mathrm{Der}_\mathbb{R}(C_{X,\mathbb{R}}(U))$, the set of smooth vector fields on $U$, and in local coordinates each derivation can be written uniquely as $\sum_j (a_j \partial/\partial x_j + b_j \partial/\partial y_j)$, with $a_j$ and $b_j$ smooth functions. In other words, $T_{X,\mathbb{R}}$ is a sheaf of $C_{X,\mathbb{R}}$-modules, locally free, with basis the $(\partial/\partial x_j)_j$ and the $(\partial/\partial y_j)_j$.

We let $\Omega^1_{X,\mathbb{R}}$ be the dual of $T_{X,\mathbb{R}}$ as $C_{X,\mathbb{R}}$-module, and, locally, we let $(dx_j)_j, (dy_j)_j$ be the basis dual to $(\partial/\partial x_j)_j, (\partial/\partial y_j)_j$. Then we have the total derivative $d\colon C_{X,\mathbb{R}} \to \Omega^1_{X,\mathbb{R}}$ given locally by

$$df = \sum_j \left( \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \right).$$

The exterior algebra

$$\Omega^\bullet_{X,\mathbb{R}} := \bigwedge_{C_{X,\mathbb{R}}} \Omega^1_{X,\mathbb{R}}$$

is graded commutative: $\omega_i \omega_j = (-1)^{ij} \omega_j \omega_i$ for $\omega_i$ and $\omega_j$ of degrees $i$ and $j$, and has a unique $d\colon \Omega^i_{X,\mathbb{R}} \to \Omega^{i+1}_{X,\mathbb{R}}$ such that $d \circ d = 0$ and for all $U \subset C$, all $\omega \in \Omega^i_{X,\mathbb{R}}(U)$ and $\eta \in \Omega^j_{X,\mathbb{R}}(U)$: $d(\omega\eta) = (d\omega)\eta + (-1)^i \omega d(\eta)$. In local coordinates, an $i$-form $\omega$ can be written uniquely as $\omega = \sum_{I,J} f_{I,J} dx_I dy_J$, with $I$ and $J$ subsets of $\{1, \ldots, n\}$ and $\#I + \#J = i$.

## 3.2 Decomposition of the complex de Rham complex

We can also consider $\mathbb{C}$-valued functions. We let $C_{X,\mathbb{C}}$ be the sheaf given by

$$C_{X,\mathbb{C}}(U) = \{f\colon U \to \mathbb{C} : \Re(f) \text{ and } \Im(f) \text{ are in } C_{X,\mathbb{R}}(U)\}.$$

Then we have the complex tangent space $T_{X,\mathbb{C}}(x) = \mathrm{Der}_\mathbb{C}(C_{X,\mathbb{C},x}, \mathbb{C})$, the sheaf of smooth complex vector fields $T_{X,\mathbb{C}}$, and the complex smooth differentials $\Omega^\bullet_{X,\mathbb{C}}$ with differential $d$, all with the same bases as in the real case, but now over $\mathbb{C}$ or $C_{X,\mathbb{C}}$.

The sheaf $\mathcal{O}_X$ then gives us the $C_{X,\mathbb{C}}$-submodule $\Omega_X^{1,0}$ of $\Omega^1_{X,\mathbb{C}}$ that is generated, locally by the $df$ where $f$ is holomorphic. In local coordinates, $\Omega_X^{1,0}$ has basis $(dz_1, \ldots, dz_n)$. And we have the submodule $\Omega_X^{0,1}$ that is generated by the $df$ for $f$ antiholomorphic (that is, $f$ such that $\overline{f}$ is holomorphic). In local coordinates, $\Omega_X^{0,1}$ has basis $(d\overline{z_1}, \ldots, d\overline{z_n})$. Note that $dz_j = d(x_j + iy_j) = dx_j + idy_j$ and that $d\overline{z_j} = dx_j - idy_j$. We see that $\Omega^1_{X,\mathbb{C}} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$. It follows that

$$\Omega^\bullet_{X,\mathbb{C}} = \bigwedge \Omega_X^{1,0} \otimes \bigwedge \Omega_X^{0,1}, \quad \Omega^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega_X^{p,q}, \quad \Omega_X^{p,q} = \bigwedge^p \Omega_X^{1,0} \otimes \bigwedge^q \Omega_X^{0,1}$$

and in local coordinates $\omega$ of type $(p,q)$ is uniquely written

$$\omega = \sum_{I,J} f_{I,J} dz_I d\overline{z_J}, \quad I, J \subset \{1, \ldots, n\}, \quad \#I = p, \quad \#J = q, \quad f_{I,J} \in C_{X,\mathbb{C}}(U).$$

For $\omega \in \Omega_X^{p,q}(U)$ we write $d\omega = \partial\omega + \overline{\partial}\omega$, with $\partial\omega \in \Omega_X^{p+1,q}(U)$ and $\overline{\partial}\omega \in \Omega_X^{p,q+1}(U)$. Then we have $d = \partial + \overline{\partial}$, with $\partial$ of degree $(1,0)$ and $\overline{\partial}$ of degree $(0,1)$. We have $0 = d^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2$, which means that $\partial^2 = 0$, $\overline{\partial}^2 = 0$, $\partial\overline{\partial} + \overline{\partial}\partial = 0$. In other words, the de Rham complex $(\Omega^\bullet_{X,\mathbb{C}}, d)$ is the total complex of the double complex $(\Omega_X^{\bullet,\bullet}, \partial, \overline{\partial})$.

As $T_{X,\mathbb{C}}$ is the $C_{X,\mathbb{C}}$-dual of $\Omega^1_{X,\mathbb{C}}$, it decomposes as

$$T_{X,\mathbb{C}} = T_X^{-1,0} \oplus T_X^{0,-1}.$$

In local coordinates, the basis $(dz_j)_j$, $(d\overline{z_j})_j$ of $\Omega^1_{X,\mathbb{C}} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$ has dual basis $(\partial/\partial z_j)_j$, $(\partial/\partial\overline{z_j})_j$ of $T_{X,\mathbb{C}} = T_X^{-1,0} \oplus T_X^{0,-1}$, hence with

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

According to the conventions on Hodge structures that we are using, this decomposition means that $T_{X,\mathbb{R}}$ should have a natural complex structure. This is indeed so: each local coordinate system $z = (z_j)_j$ gives an isomorphism of $\mathbb{R}$-vector spaces $T_{X,\mathbb{R}}(x) \to \mathbb{C}^n$. The $\mathbb{C}$-vector space structure induced by this on $T_{X,\mathbb{R}}(x)$ does not depend on the choice of coordinate system.

The de Rham complex (of sheaves)

$$(\Omega^\bullet_{X,\mathbb{R}}, d)$$

is a resolution of the constant sheaf $\mathbb{R}_X$ by fine sheaves (all $C_{X,\mathbb{R}}$-modules are fine, comes from partition of unity in $C_{X,\mathbb{R}}$) and hence are soft (restrictions to closed subsets are surjective) and $\Gamma(X,-)$-acyclic. Therefore we have:

$$\mathrm{H}^k(X, \mathbb{R}_X) = \mathrm{H}^k(\Omega^\bullet_{X,\mathbb{R}}(X), d).$$

Likewise:

$$\mathbb{C} \otimes_\mathbb{R} \mathrm{H}^k(X, \mathbb{R}_X) = \mathrm{H}^k(X, \mathbb{C}_X) = \mathrm{H}^k(\Omega^\bullet_{X,\mathbb{C}}(X), d).$$

For $k \in \mathbb{Z}$ and $p + q = k$ we let $\mathrm{H}^{p,q}(X)$ denote the sub-$\mathbb{C}$-vector space of $\mathrm{H}^k(X, \mathbb{C}_X)$ consisting of classes of closed forms $\omega \in \Omega_X^{p,q}(X)$, that is, with $\partial\omega = 0$ and $\overline{\partial}\omega = 0$. If our complex manifold $X$ is a projective, that is, a closed submanifold of some complex projective space $\mathbb{P}^N(\mathbb{C})$ (with the archimedian topology), or more generally if $X$ is a proper complex algebraic variety, or if $X$ is compact and has a Kähler metric on $T_{X,\mathbb{R}}$, then:

$$\mathbb{C} \otimes_{\mathbb{R}} \mathrm{H}^k(X, \mathbb{Q}_X) = \mathrm{H}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathrm{H}^{p,q}(X), \quad \text{and} \quad \overline{\mathrm{H}_X^{p,q}(X)} = \mathrm{H}_X^{q,p}(X).$$

Note that the type of $\mathrm{H}^k(X, \mathbb{Q}_X)$ is contained in

$$\{(p,q) : p+q = k, 0 \leq p, q \leq \dim_{\mathbb{C}}(X)\}.$$

This is a famous and difficult result of Hodge and others, obtained using a lot of analysis (elliptic complexes, Sobolev spaces,...). Later it was proved, in the algebraic case, by algebraic tools (Faltings, Deligne, Illusie). Hence the rational Betti cohomology groups of smooth projective complex varieties are naturally $\mathbb{Q}$-Hodge structures.

The proof of the Hodge decomposition above gives at the same time isomorphisms

$$\mathrm{H}^{p,q}(X) \xrightarrow{\sim} \mathrm{H}^q(X, \Omega_X^p),$$

with $\Omega_X^p$ the $\mathcal{O}_X$-module of holomorphic $p$-forms, that in terms of local coordinates has basis $dz_{i_1} \cdots dz_{i_p}$, with $i_1 < \cdots < i_p$. These isomorphisms are independent of the choice of Kähler metric.

## 3.3 Poincaré, Lefschetz and polarisations

These Hodge structures are polarised as follows. Assume for simplicity that $X$ is connected and of (complex) dimension $n$, say. Then $\mathrm{H}^{2n}(X, \mathbb{Z})$ is canonically isomorphic to $\mathbb{Z}$ ($X$ is compact, connected, of real dimension $2n$ and oriented by its complex structure). As a Hodge structure it is $\mathbb{Z}(-n)$, of type $(n,n)$. The product structure on cohomology then induces perfect pairings (Poincaré duality)

$$\mathrm{H}^i(X, \mathbb{Q}) \times \mathrm{H}^{2n-i}(X, \mathbb{Q}) \to \mathrm{H}^{2n}(X, \mathbb{Q}) = \mathbb{Q}(-n).$$

In terms of real de Rham cohomology Poincaré duality is given by integration:

$$([\omega], [\eta]) \mapsto \int_X \omega\eta$$

Let now $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, that is, the sheaf of sections of a holomorphic line bundle on $X$. Then, via the *exponential sequence*, $\mathcal{L}$ gives an element $[\omega]$ (class of a 2-form) of $\mathrm{H}^2(X, \mathbb{Z})$:

$$0 \longrightarrow \mathbb{Z}_X \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 1.$$

This induces:

$$0 \to \mathrm{H}^1(X, \mathbb{Z}) \to \mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \to \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathcal{O}_X).$$

This shows a lot: $\mathrm{Pic}^0(X)$ is $\mathrm{H}^{0,1}(X)/\mathrm{H}^1(X, \mathbb{Z}) = \mathrm{Gr}^0(\mathrm{H}^1(X, \mathbb{C}))/\mathrm{H}^1(X, \mathbb{Z})$ (notation: $\mathrm{Gr}^p = F^p/F^{p+1}$), the tangent space at 0 of $\mathrm{Pic}(X)$ is $\mathrm{H}^1(X, \mathcal{O}_X)$, and that $[\omega]$ is in $\mathrm{H}^2(X, \mathbb{Z}) \cap \mathrm{H}^{1,1}(X)$

(because it goes to zero in $\mathrm{H}^2(X, \mathcal{O}_X) = \mathrm{H}^{0,2}(X)$ hence is in $F^1\mathrm{H}^2(X, \mathbb{C})$ and it is real hence invariant under complex conjugation, so also in $\overline{F^1\mathrm{H}^2(X, \mathbb{C})}$).

If we equip $\mathcal{L}$ with a smooth hermitian metric (using partitions of unity), then the image of $\mathcal{L}$ in $\mathrm{H}^2(X, \mathbb{C})$ is $[\omega]$, where, locally on $X$ for $s$ any generator of $\mathcal{L}$ we have:

$$\omega = \frac{1}{2\pi i}\partial\overline{\partial}\log(\|s\|^2).$$

This also shows that $[\omega]$ is in $\mathrm{H}^{1,1}(X)$.

Let us now come back to polarisations on $\mathrm{H}^k(X, \mathbb{Q})$. We assume $X$ projective, and $\mathcal{L}$ ample on $X$ (for example the restriction to $X$ of the line bundle $\mathcal{O}(1)$ of a projective space in which $X$ is embedded). Then we have the multiplication by $[\omega]$ on $\mathrm{H}^\bullet(X, \mathbb{Q})$ (the *Lefschetz operator*, often denoted $L$). For $k \leq n$ it gives an isomorphism

$$\mathrm{H}^k(X, \mathbb{Q}) \xrightarrow{\omega^{n-k}} \mathrm{H}^{2n-k}(X, \mathbb{Q})(n-k) = \mathrm{H}^{2n-k}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(n-k)$$

and the Lefschetz decomposition of $\mathrm{H}^k(X, \mathbb{Q})$: the kernel $\mathrm{H}^k(X, \mathbb{Q})_{\mathrm{prim}}$ of $\omega^{n-k+1}$ on $\mathrm{H}^k(X, \mathbb{Q})$ is the *primitive part*, and the decomposition is:

$$\mathrm{H}^k(X, \mathbb{Q}) = \bigoplus_{2r \leq k} \omega^r \mathrm{H}^{k-2r}(X, \mathbb{Q})_{\mathrm{prim}}$$

See Voisin's book (Section 7.1.2) for the result that

$$\psi\colon \mathrm{H}^k(X, \mathbb{Q}) \times \mathrm{H}^k(X, \mathbb{Q}) \to \mathrm{H}^k(X, \mathbb{Q}) \times \mathrm{H}^{2n-k}(X, \mathbb{Q})(n-k) \to \mathbb{Q}(-n+n-k) = \mathbb{Q}(-k),$$

sending $(x, y)$ to $xy[\omega]^{n-k}$, induces a polarisation on $\mathrm{H}^k(X, \mathbb{Q})_{\mathrm{prim}}$, up to a sign depending on $k$.

## 3.4 Holomorphic and algebraic de Rham cohomology

Let now $X$ be a smooth projective algebraic variety over a field $k$, and $n = \dim(X)$. Then we have the algebraic de Rham complex

$$(\Omega^\bullet_{X/k}, d) = [\mathcal{O}_X = \Omega^0_{X/k} \to \Omega^1_{X/k} \to \cdots \to \Omega^n_{X/k} \to 0]$$

on $X$. Its *hypercohomology* gives the *algebraic de Rham cohomology*. A simple description is as follows. Let $X = \cup_{i=0}^n U_i$ be an affine open cover of $X$. This gives the Cech-de Rham double complex (of $k$-vector spaces):

$$\mathcal{C}^p\Omega^q = \bigoplus_{i_0 < \cdots < i_p} \Omega^q(U_{i_0} \cap \cdots \cap U_{i_p}),$$

with differentials

$$d\colon \mathcal{C}^p\Omega^q \to \mathcal{C}^p\Omega^{q+1} \quad \text{induced by } d\colon \Omega^q \to \Omega^{q+1}$$

and

$$\delta\colon \mathcal{C}^p\Omega^q \to \mathcal{C}^{p+1}\Omega^q, \quad (\delta\omega)_{i_0,\ldots,i_p} = (-1)^q \sum_j (-1)^{j+1} \omega_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}}.$$

With these choices of signs we have $d\delta + \delta d = 0$ and we can define the *total complex*:

$$\mathrm{Tot}(\mathcal{C}^\bullet\Omega^\bullet)^i = \bigoplus_{p+q=i} \mathcal{C}^p\Omega^q, \quad d = d + \delta.$$

Then the algebraic de Rham cohomology is defined as:

$$\mathrm{H}^i_{\mathrm{dR}}(X) = \mathrm{H}^i(\mathrm{Tot}(\mathcal{C}^\bullet\Omega^\bullet)) = \mathbb{H}^i(X, \Omega^\bullet),$$

where the last notation $\mathbb{H}^i$ means *hypercohomology*.

It is a theorem of Grothendieck that for $k \to \mathbb{C}$ one has

$$\mathbb{C} \otimes_k \mathbb{H}^i(X, \Omega^\bullet) = \mathbb{H}^i(X_\mathbb{C}, \Omega^\bullet) = \mathrm{H}^i(X(\mathbb{C}), \mathbb{C}_X) = \mathrm{H}^i(X(\mathbb{C}), \mathbb{C}).$$

The first equality is that the construction is compatible with field extensions. The second equality is GAGA (comparison complex algebraic and complex analytic de Rham cohomology, plus the fact that the analytic de Rham complex is a resolution of $\mathbb{C}_X$).

The Hodge filtration can be defined algebraically. For $i$ in $\mathbb{Z}$, we let $\Omega^{\geq i}_{X/k}$ be the subcomplex of $\Omega^\bullet_{X/k}$ consisting of the $\Omega^j$ with $j \geq i$. The morphism of complexes $\Omega^{\geq i} \to \Omega^\bullet$ induces a morphism of double complexes $\mathcal{C}^\bullet\Omega^{\geq i} \to \mathcal{C}^\bullet\Omega^\bullet$, then a morphism of total complexes and finally morphisms $\mathbb{H}^j(X, \Omega^{\geq i}) \to \mathbb{H}^j(X, \Omega^\bullet)$. The image of this one is $F^i(\mathbb{H}^j(X, \Omega^\bullet)) = F^i\mathrm{H}^j_{\mathrm{dR}}(X/k)$. For $k \to \mathbb{C}$ it induces the Hodge filtration.

Assume now $k = \mathbb{C}$. We can interpret the Hodge decomposition $\mathrm{H}^i(X, \mathbb{C}) = \oplus_{p+q=i}\mathrm{H}^{p,q}(X)$ and the isomorphisms $\mathrm{H}^{p,q}(X) = \mathrm{H}^q(X, \Omega^p)$ as follows in terms of hypercohomology. For each $p$, we have a short exact sequence of complexes

$$0 \to \Omega^{>p} \to \Omega^{\geq p} \to \Omega^p[-p] \to 0$$

where $\Omega^p[-p]$ is the sheaf $\Omega^p$, placed in degree $p$ (general notation: $(C^\bullet[i])^j = C^{i+j}$). For each $i$ this gives an exact sequence

$$\mathbb{H}^i(X, \Omega^{>p}) \to \mathbb{H}^i(X, \Omega^{\geq p}) \to \mathrm{H}^{i-p}(X, \Omega^p)$$

The first two terms map surjectively to $F^{p+1}\mathrm{H}^i_{\mathrm{dR}}(X)$ and $F^p\mathrm{H}^i_{\mathrm{dR}}(X)$ and the Hodge decomposition says that $\mathrm{Gr}^p\mathrm{H}^i_{\mathrm{dR}}(X)$ is isomorphic to $\mathrm{H}^{i-p}(X, \Omega^p)$. This means that $\mathbb{H}^i(X, \Omega^{\geq p}) \to \mathrm{H}^{i-p}(X, \Omega^p)$ is surjective, and hence that $\mathbb{H}^i(X, \Omega^{>p}) \to \mathbb{H}^i(X, \Omega^{\geq p})$ is injective. We conclude that all $\mathbb{H}^i(X, \Omega^{\geq p}) \to F^p\mathrm{H}^i_{\mathrm{dR}}(X)$ are isomorphisms, and that the

$$0 \to \mathbb{H}^i(X, \Omega^{>p}) \to \mathbb{H}^i(X, \Omega^{\geq p}) \to \mathrm{H}^{i-p}(X, \Omega^p) \to 0$$

are all exact.

### 3.5 Griffiths transversality

The fact that $\mathrm{H}^i(X, \mathbb{C})$ can be defined algebraically implies (theorems by Deligne, probably), by doing the same construction for a smooth projective family $f\colon X \to S$ over $\mathbb{C}$, that the Hodge filtration varies algebraically over $S$. One then has locally free $\mathcal{O}_S$-modules $\mathrm{H}^i_{\mathrm{dR}}(X/S)$ with a filtration $F^\bullet$ giving fibre-wise the de Rham cohomology with Hodge filtration. The fact that, analytically, on $S(\mathbb{C})$ we have $\mathrm{H}^i_{\mathrm{dR}}(X/S) = \mathcal{O}_{S(\mathbb{C})} \otimes_{\mathbb{C}_{S(\mathbb{C})}} \mathrm{R}^i f_*(\mathbb{C}_X)$, that is, the relative holomorphic de Rham cohomology is obtained from the locally constant sheaf $\mathrm{R}^i f_*(\mathbb{C}_X)$ by tensoring with $\mathcal{O}_{S(\mathbb{C})}$, is reflected in the algebraically defined *Gauss-Manin connection*:

$$\nabla\colon \mathrm{H}^i_{\mathrm{dR}}(X/S) \longrightarrow \Omega^1_{S/\mathbb{C}} \otimes_{\mathcal{O}_S} \mathrm{H}^i_{\mathrm{dR}}(X/S).$$

A fundamental fact is that this connection has the Griffiths transversality property

$$\nabla\colon F^p\mathrm{H}^i_{\mathrm{dR}}(X/S) \longrightarrow \Omega^1_{S/\mathbb{C}} \otimes_{\mathcal{O}_S} F^{p-1}\mathrm{H}^i_{\mathrm{dR}}(X/S).$$

To get the Gauss-Manin connection, one lets $A := \bigwedge^\bullet \Omega^1_X$, $B := \bigwedge^\bullet \Omega^1_{X/S}$ (sheaves of graded-commutative differential algebras) and $I \subset A$ the differential ideal generated by $f^*\Omega^1_S$. Then $B = A/I$, and one considers the short exact sequence

$$0 \to I/I^2 \to A/I^2 \to B \to 0$$

of complexes of $\mathcal{O}_X$-modules, and gets a map from $\mathbb{R}^i f_* B$ to $(\mathbb{R}^{i+1} f_*) I/I^2$.

**3.6 Exercise.** In order to understand why often $\mathrm{H}^2(\mathbb{P}^1(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}(-1) = (2\pi i)^{-1}\mathbb{Z} \subset \mathbb{C}$, let us compute the morphism

$$\mathrm{H}^2(\mathbb{P}^1(\mathbb{C}), \mathbb{Z}) \to \mathrm{H}^2(\mathbb{P}^1(\mathbb{C}), \mathbb{C}) = \mathrm{H}^2_{\mathrm{dR}}(\mathbb{P}^1(\mathbb{C})/\mathbb{C}).$$

Use the resolutions (writing $X = \mathbb{P}^1(\mathbb{C})$)

$$\begin{tikzcd}
\mathbb{Z}_X \arrow[r, tail, "2\pi i"] \arrow[d, "\mathrm{inj}"'] & \mathcal{O}_X \arrow[r, two heads, "\exp"] \arrow[d, "(2\pi i)^{-1}"'] & \mathcal{O}_X^\times \arrow[d, "(2\pi i)^{-1} d\log"] \\
\mathbb{C}_X \arrow[r, hook] & \mathcal{O}_X \arrow[r, two heads, "d"] & \Omega^1_X
\end{tikzcd}$$

Then

$$\mathrm{H}^2(X, \mathbb{Z}) = \mathbb{H}^2(\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times) = \mathrm{H}^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}(X) = \mathbb{Z}$$

and

$$\mathrm{H}^2(X, \mathbb{C}_X) = \mathbb{H}^2(\mathcal{O}_X \xrightarrow{d} \Omega^1_X) = \mathrm{H}^1(X, \Omega^1_X) = \mathbb{C}$$

where the identification with $\mathbb{C}$ is via residues. In fact, let $X = X_0 \cup X_\infty$ be the cover by the complements of $\infty$ and $0$, respectively. Then the class in $\mathrm{H}^1(X, \Omega^1)$ of $(dz)/z \in \Omega^1_X(X_{0,\infty})$ is sent to its residue at $0$, or $\infty$ (depending on conventions), hence to $\pm 1$. Also, $z \in \mathcal{O}_X(X_{0,\infty})^\times$ represents the positive generator of $\mathrm{Pic}(X)$ and it is mapped to $(2\pi i)^{-1}(dz)/z$ in $\Omega^1_X(X_{0,\infty})$, which is $(2\pi i)^{-1}$ times the generator $(dz)/z$ of $\mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{C})$.

Of course, one can also consider the integration pairing with singular homology to get the factor $(2\pi i)^{-1}$ (by the Mayer-Vietoris sequence for $X = X_0 \cup X_\infty$, $\mathrm{H}^1(\mathbb{C}^\times) = \mathrm{H}^2(X)$). But it is interesting to see it in terms of algebraic de Rham cohomology.

**3.7 Definition. (cohomology class)** Let $X$ be a connected proper non-singular complex algebraic variety, and $Z$ an irreducible closed subvariety, of dimensions $d_X$ and $d_Z$, hence of codimension $k = d_X - d_Z$. Then we let $\mathrm{cl}(Z) \in \mathrm{H}^{2k}(X, \mathbb{Q})(k)$ be the cohomology class of $Z$. It is a Hodge class.

If $Z$ is non-singular, it is obtained as follows. The inclusion $i\colon Z \to X$ induces

$$i^*\colon \mathrm{H}^{2d_Z}(X, \mathbb{Q}) \to \mathrm{H}^{2d_Z}(Z, \mathbb{Q}) = \mathbb{Q}(-d_Z).$$

Dualising and twisting gives:

$$\mathbb{Q} \to \left(\mathrm{H}^{2d_Z}(X, \mathbb{Q})(d_Z)\right)^\vee = \mathrm{H}^{2k}(X, \mathbb{Q})(k), \quad 1 \mapsto \mathrm{cl}(Z),$$

where for the last equality we used the perfect pairing

$$\mathrm{H}^{2d_Z}(X, \mathbb{Q})(d_Z) \times \mathrm{H}^{2k}(X, \mathbb{Q})(k) \to \mathrm{H}^{2d_X}(X, \mathbb{Q})(d_X) = \mathbb{Q}.$$

In the general case, one can invoke resolution of singularities: $\widetilde{Z} \to Z \to X$ and proceed in the same way.

A topologist would triangulate $Z$, get $[Z]$ in $\mathrm{H}_{2d_Z}(X, \mathbb{Q}) = (\mathrm{H}^{2d_Z}(X, \mathbb{Q}))^\vee = \mathrm{H}^{2k}(X, \mathbb{Q})$. See also §11 of Voisin's book.

**3.8 Conjecture. (Hodge conjecture)** *Let $X$ be a connected proper non-singular complex algebraic variety, and $\xi \in \mathrm{H}^{2k}(X, \mathbb{Q})(k)$ a Hodge class. Then $\xi$ is a $\mathbb{Q}$-linear combination $\sum_i a_i \mathrm{cl}(Z_i)$ with $Z_i$ irreducible closed subvarieties of codimension $k$.*

# 4 Background on linear algebraic groups

We need some background on linear algebraic groups. References are: T. Springer's book "Linear algebraic groups", Borel's book (same title), Waterhouse's "Introducion to affine group schemes", SGA3 (more categorical approach, more technical), Platonov-Rapinchuk's "Algebraic groups and number theory", the Corvallis (1979) articles by Springer and Tits. Also: Conrad's lectures in the Luminy Summer School of 2011.

## 4.1 Linear algebraic groups as representable functors

For $k$ a field, a *linear algebraic group over $k$* is an affine group scheme $G$ over $k$, of finite type. This means that $G = \mathrm{Spec}(H)$ with $H = \mathcal{O}(G)$ a finitely generated $k$-algebra, and that for every $k$-algebra $A$ the set $G(A)$ of $A$-points of $G$ has been given the structure of a group, functorially in $A$: for $A \to A'$, the map $G(A) \to G(A')$ is a morphism of groups. Equivalently: $G$ is a covariant functor from the category of $k$-algebras to that of groups, whose underlying functor to the category of sets is represented by an affine $k$-scheme of finite type. Morphisms $G_1 \to G_2$ of linear algebraic groups over $k$ are morphisms of functors. By Yoneda's lemma they are the morphisms of $k$-schemes $G_1 \to G_2$ that are compatible with the group laws. Also by Yoneda, the group structure of the $G(A)$ comes from a morphism of $k$-schemes $G \times_k G \to G$. The $k$-algebra $H$ has the structure of a Hopf algebra. Note that we do not demand that $G$ is reduced as $k$-scheme; this is automatically so if $k \supset \mathbb{Q}$, and then $G$ is smooth over $k$, that is, $G_{\overline{k}}$ is a non-singular variety.

A few examples. First of all we have the $\mathrm{GL}_{n,k}$, sending $A$ to the group $\mathrm{GL}_n(A)$ of automorphisms of the free $A$-module $A^n$, which we can also see as the group of invertible $n$ by $n$ matrices with coefficients in $A$. For $n = 1$ we have $\mathrm{GL}_{n,k} = \mathbb{G}_{\mathrm{m}k}$. Let us show the representability. To give an element of $\mathrm{GL}_n(A)$ is to give two elements $a$ and $b$ in $\mathrm{M}_n(A)$ with $ab = 1$, hence the $k$-algebra $k[x, y]/(xy - 1)$ with $2n^2$ variables and $n^2$ equations does the job. More generally we have, for $V$ a finite dimensional $k$-vector space, $\mathbf{GL}(V)$, sending $A$ to $\mathrm{Aut}_A(A \otimes_k V)$ (automorphisms of the $A$-module). Actually this functorial approach makes works for arbitrary $k$-vector spaces, even if $\mathbf{GL}(V)$ is then not representable.

Our first aim is to show that every $G$ can be embedded as a closed subgroup scheme of some $\mathrm{GL}_{n,k}$. This we can do by showing that $G$ has a finite dimensional faithful representation.

## 4.2 Representations are locally finite

A *representation* of $G$ is a $k$-vector space $V$ together with an action by $G$, that is, for each $k$-algebra $A$, an action of $G(A)$ on $A \otimes_k V$, functorial in $A$. (If $V$ is finite dimensional then this corresponds to a morphism $G \to \mathbf{GL}(V)$.) Such a $V$ is then automatically *locally finite*: each finite subset of $V$ is contained in a finite dimensional subspace $W$ of $V$ that is stable under the $G$-action. Let us prove that. For each $A$, each $g$ in $G(A) = \mathrm{Hom}_{k-\mathrm{Alg}}(H, A)$ arises by $g\colon H \to A$ from the universal point $\mathrm{id}_G\colon G \to G$, hence the action of $G$ on $V$ is determined by the automorphism of the $H$-module $H \otimes_k V$ attached to $\mathrm{id}_G$, which is determined by the $k$-linear map

$$\alpha\colon V \to H \otimes_k V \to H \otimes_k V.$$

Explicitly, for $g \in G(A)$, let $g^*\colon H \to A$ be the corresponding morphism of $k$-algebras and let $v$ be in $V$, then $g{\cdot}(1 \otimes v) = (g^* \otimes \mathrm{id}_V)(\alpha v)$ in $A \otimes_k V$. We see that a subvector space $W$ of $V$ is $G$-stable if and only if $\alpha(W) \subset H \otimes_k W$.

Let now $v$ be in $V$. We produce a finite dimensional $W \subset V$ that is $G$-stable and contains $W$. Let $(h_i)_{i \in I}$ be a $k$-basis of $H$. Then there are unique $v_i$ in $V$, almost all zero, such that $\alpha(v) = \sum_i h_i \otimes v_i$. Then $W := \sum_i k{\cdot}v_i$ has the required properties. Here is the computation. We let $g_1$ and $g_2$ in $G(H \otimes_k H)$ be given by $g_1^*\colon h \mapsto h \otimes 1$ and $g_2^*\colon h \mapsto 1 \otimes h$. Then we have $g_2 g_1$ in $G(H \otimes_k H)$. We write $\mu^* h_i = \sum_{j,l} r_{i,j,l} h_j \otimes h_l$ with the $r_{i,j,l}$ in $k$, where $\mu\colon G \times_k G \to G$ is the multiplication map. Then we have:

$$\begin{aligned}
g_1 \cdot (1 \otimes 1 \otimes v) &= \sum_i h_i \otimes 1 \otimes v_i \\
g_2 \cdot (g_1 \cdot (1 \otimes 1 \otimes v)) &= \sum_i h_i \otimes (g_2 \cdot (1 \otimes v_i)) \\
(g_2 g_1) \cdot (1 \otimes 1 \otimes v) &= \sum_i ((g_2 g_1)^* h_i) \otimes v_i = \sum_{i,j,l} r_{i,j,l} \, h_j \otimes h_l \otimes v_i \\
&= \sum_{i,j,l} r_{j,i,l} \, h_i \otimes h_l \otimes v_j = \sum_i h_i \otimes \left( \sum_{j,l} r_{j,i,l} \, h_l \otimes v_j \right)
\end{aligned}$$

hence we conclude that for each $i$ we have

$$g_2 \cdot (1 \otimes v_i) = \sum_{j,l} r_{j,i,l} \, h_l \otimes v_j .$$

Hence indeed $W$ is $G$-stable. (It contains $v$ tautologically, but let us write it, because of $e \in G(k)$, hence $v = e \cdot v = \sum_i (e^* h_i) \, v_i$.)

## 4.3 Embedding in $\mathrm{GL}_{n,k}$

Let $G$ be a linear algebraic group over a field $k$. Let $H := \mathcal{O}(G)$. By definition $H$ is a finitely generated $k$-algebra. It has an action by $G \times_k G$ by translations on both sides. Let us consider the action of $G$ by right translations. For $k \to A$, $g \in G(A)$ gives an automorphism of $A$-schemes $G_A \to G_A$, $g' \mapsto g' \cdot g$ ($g'$ in $G(A')$ with $A \to A'$, say), and therefore an automorphism $(\cdot g)^*$ of $A$-algebras $H_A \to H_A$ (we denote now $H_A = A \otimes_k H$). This is a faithful action: if $g$ in $G(A)$ acts trivially, then $g$ is the identity element. Let $V \subset H$ be a finite dimensional $G$-stable subspace that contains a set of generators for $H$ as $k$-algebra (exists by local finiteness of the representation). Then we have a injective morphism of $k$-group schemes $G \to \mathbf{GL}(V)$, sending $g$ in $G(A)$ to $(\cdot g)^*|_V$. We show that it is a closed immersion (this results from a generality: *the image of a morphism of linear algebraic groups is closed*, but we still show it). Let $(v_i)_{i \in I}$ be a basis for $V$. Write $\alpha(v_i) = \sum_j h_{i,j} \otimes v_j$. Then $G \to \mathbf{GL}(V) = \mathrm{GL}_{n,k}$ corresponds to $k[x, 1/\det(x)] \to H$, $x_{i,j} \mapsto h_{i,j}$. Hence the $h_{i,j}$ are in the image. One checks that $\alpha \colon V \to H \otimes_k V$ is the inclusion $V \to H$ followed by $\mu^* \colon H \to H \times H$ maybe switched). Hence we have $v_i = \sum_j h_{i,j} \, e^* v_j$, hence the $v_i$ are in the image of $k[x, 1/\det(x)] \to H$, hence $k[x, 1/\det(x)] \to H$ is surjective.

## 4.4 Closed subgroups are stabilisers of lines

Let $G$ be a linear algebraic group over a field $k \supset \mathbb{Q}$. Let $H$ be a closed subgroup scheme of $G$, and $V$ a finite dimensional faithful representation of $G$. Then there exists a line $L$ in some finite dimensional representation of $G$ of the form $T^\nu = \oplus_i (V^{\otimes n_i} \otimes (V^\vee)^{\otimes m_i})$, such that $H$ is the stabilizer of $L$ ($\nu$ denotes the collection $(n_i, m_i)_i$).

**Proof.** First of all, we may and do suppose that $G = \mathbf{GL}(V)$. The idea is now the following: let $G$ act on itself by right translation; then $G$ acts on $\mathcal{O}(G)$, and $H$ is the stabilizer of the ideal $I_H$; then use that $I_H$ is finitely generated, and that $\mathcal{O}(G)$ is locally finite. Let us first write down what $\mathcal{O}(G)$ is, as a $G$-module via right translation on $G$. Well, $G = \mathbf{GL}(V) \subset \mathbf{End}(V)$, hence

$$\mathcal{O}(G) = \mathcal{O}(\mathbf{GL}(V)) = \mathcal{O}(\mathbf{End}(V))[1/\det] = \mathrm{Sym}_k(\mathrm{End}(V)^\vee)[1/\det] = \mathrm{Sym}_k(V^d)[1/\det],$$

where the last equality comes from the fact that $\mathrm{End}(V)^\vee$, as $G$-module given by right translation on $\mathrm{End}(V)$, is simply $V^d$, where $d$ is of course the dimension of $V$ (note that the $G$-action on $\mathrm{End}(V)^\vee$ extends to an $\mathrm{End}(V)$-action and makes $\mathrm{End}(V)^\vee$ into an $\mathrm{End}(V)$-module, and each such module is a direct sum of a number of copies of $V$). Also, note that $\det$ is in $\mathrm{Sym}^d(\mathrm{End}(V)^\vee)$, and that we have $g{\cdot}\det = \det(g)\det$. The scalar subgroup $\mathbb{G}_\mathrm{m}$ of $G$ induces a $\mathbb{Z}$-grading on $\mathcal{O}(G)$. We have

$$\mathcal{O}(G) = \mathcal{O}(\mathbf{End}(V))[1/\det] = \mathcal{O}(\mathbf{End}(V))[z]/(z{\cdot}\det - 1),$$

hence:

$$\mathcal{O}(G)_i = \bigcup_j \mathcal{O}(\mathbf{End}(V))_{i+dj}\det^{-j},$$

and (note that we use $k \supset \mathbb{Q}$ to see $\mathrm{Sym}^i$ as a *submodule* of $\otimes^i$ in stead of a quotient):

$$\begin{aligned}
&\mathcal{O}(\mathbf{End}(V))_i = \mathrm{Sym}^i(\mathrm{End}(V)^\vee) = \mathrm{Sym}^i(V^d) \subset (V^d)^{\otimes i} = (V^{\otimes i})^{d^i},\\
&k{\cdot}\det = \bigwedge^d V \quad \text{is a quotient of } V^{\otimes d},\\
&k{\cdot}\det^{-1} = (\bigwedge^d V)^\vee \subset (V^\vee)^{\otimes d}.
\end{aligned}$$

This describes $\mathcal{O}(G)$ as $G$-module. Let $f_1, \ldots, f_r$ be a finite set of generators of the ideal $I_H$ of $\mathcal{O}(G)$. Let $W \subset \mathcal{O}(G)$ be a finite dimensional sub-$G$-module containing the $f_i$. Then $H$ is the stabilizer of the subspace $I_H \cap W$ of $W$, hence of the line $\bigwedge^n(I_H \cap W) \subset \bigwedge^n(W)$, with $n = \dim(I_H \cap W)$. Now note that $W$ is a subrepresentation of a representation of the form $\oplus_i(V^{\otimes n_i} \otimes (V^\vee)^{\otimes m_i})$. Hence $\bigwedge^n W$, being a submodule of $W^{\otimes n}$ is of that form too. $\square$

**4.4.1 Remark.** A subgroup $H$ such that $G/H$ is not quasi-affine cannot be the stabiliser of an *element* in a representation of $G$, because the orbits of $G$ in representations of $G$ are quasi-affine. This happens for example if $G/H$ is projective and of positive dimension.

**4.4.2 Remark.** If we allow subquotients of the $\oplus_i(V^{\otimes n_i} \otimes (V^\vee)^{\otimes m_i})$, then we can drop the hypothesis that $k$ is of characteristic zero.

**4.4.3 Remark.** If $H$ contains the scalars in $G = \mathrm{GL}(V)$, then one can take $L$ to be in some representation of the form $(V^{\otimes n})^m$. To prove this, consider the Zariski closure $\overline{H}$ of $H$ in $\mathrm{End}(V)$, and use that it is a cone. The ideal $I$ of $\overline{H}$ is then positively graded, and for all $i$ sufficiently large we have that $\overline{H} = Z(I_i)$ (standard argument in projective spaces, vanishing of $\mathrm{H}^1(\mathbb{P}^{d^2-1}, I_i)$).

**4.4.4 Example.** Just for fun, let us look at some examples in $G := \mathrm{GL}_2$. The Borel subgroup $B := \{(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix})\}$ is the stabilizer of the line generated by $(1,0)$ in $V := k^2$. The subgroup $\{(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix})\}$ is the stabilizer of $k(1,(1,0))$ in $k \oplus V$. The subgroup $\{(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix})\}$ is the stabilizer of $k(1,(0,1)^*)$ in $k \oplus V^*$. The subgroup $\{(\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix})\}$ is the stabilizer of $k((1,0),(0,1))$ in $V \oplus V$. The subgroup $\{(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix})\}$ is the stabilizer of the two-dimensional subspace of the $((x,0),(0,y))$ in $V \oplus V$; note that the proof above gives the same result. Finally, the trivial subgroup $\{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})\}$ is the stabilizer of $(1,(1,0),(0,1))$ in $k \oplus V \oplus V$.

## 4.5 Embedding in sums of the regular representation

Let $G$ be a linear algebraic group over a field $k$, and let $V$ be a finite dimensional representation of $G$. For $l\colon V \to k$ in $V^\vee$ and $v$ in $V$, let $f_{l,v} \in \mathcal{O}(G) = \mathrm{Hom}_k(G, \mathbb{A}^1_k)$ be given by: for all $k \to A$ for all $g$ in $G(A)$, $f_{l,v}(g) = l(g{\cdot}v)$ in $A$. Of course, $f_{l,v}$ is determined by its value on the universal $g = \mathrm{id}\colon \mathcal{O}(G) \to \mathcal{O}(G)$, and we have

$$V \xrightarrow{\alpha} \mathcal{O}(G) \otimes_k V \xrightarrow{\mathrm{id}\otimes l} \mathcal{O}(G), \quad v \mapsto f_{l,v}.$$

Each such a composition $(\mathrm{id} \otimes l) \circ \alpha$ is a morphism of representations of $G$, if we let $G$ act on $\mathcal{O}(G)$ by right multiplication on $G$. For all $k \to A$ and all $g_1$ and $g_2$ in $G(A)$ we have

$$f_{l,g_2 v}(g_1) = l(g_1(g_2 v)) = l((g_1 g_2)v) = f_{l,v}(g_1 g_2).$$

By taking $l_1, \ldots, l_d$ a basis for $V^\vee$, we get an embedding of $V$ in $\mathcal{O}(G)^d$ (note that $f_{l,v}(1) = l(v)$, hence the map is injective).

We could also note that we have just shown that $\mathrm{Hom}_G(V, \mathcal{O}(G))$ is the same as $\mathrm{Hom}_k(V,k)$ (the map in the other direction is $\phi \mapsto (v \mapsto (\phi v)(1))$), hence $\mathcal{O}(G)$ is induced from the trivial representation of the trivial subgroup.

## 4.6 All representations from one

Let $G$ be a linear algebraic group over a field $k$, and $V$ a finite dimensional faithful representation. We have seen that $\mathcal{O}(G)$ is a union of subrepresentations that are subquotients of $\oplus_i (V^{\otimes n_i} \otimes (V^\vee)^{\otimes m_i})$. It follows from this and the existence of embeddings in sums of copies of $\mathcal{O}(G)$ that every finite dimensional representation of $G$ is obtained from $V$ by the following operations: direct sums, tensor products, duals, subrepresentations, quotients. Also: it is a subquotient of some $\oplus_i (V^{\otimes n_i} \otimes (V^\vee)^{\otimes m_i})$, because the collection of subquotients of such $T^\nu$'s is stable under $\oplus$, $\otimes$, $(\cdot)^\vee$ and subquotients.

Here is why. In any abelian category a subquotient of a subquotient is again a subquotient (the square is cartesian):

$$\begin{tikzcd}
M & M_1 \arrow[l, tail] \arrow[d, two heads] & M_5 \arrow[l, tail] \arrow[d, two heads] \\
 & M_2 & M_3 \arrow[l, tail] \arrow[d, two heads] \\
 & & M_4
\end{tikzcd}$$

A subquotient is a "quotientsub": consider for $M$, $M_1$ and $M_2$ the push-out, that is, the quotient of $M$ by $\ker(M_1 \to M_2)$. A subquotient of a $\oplus$ or $\otimes$ of $T^\nu$'s or of a dual of a $T^\nu$ is again one such.

## 4.7 Quotients $G/H$

Let $G$ be a linear algebraic group over a field $k$, and $H$ a closed subgroup scheme. Let $V$ be a finite dimensional representation of $G$ and $L \subset V$ a line such that $H$ is the stabiliser of $L$ (for all $k \to A$, $H(A) = G(A)_{L_A}$; for $A = k[\varepsilon]$ this implies that $\mathrm{Lie}(H) = \mathrm{Lie}(G)_L$). Let $X$ be the orbit of $[L]$ in $\mathbb{P}(V)$, the projective space of lines in $V$. It is a locally closed subscheme, with an action of $G$. Assume now that $X$ is reduced (certainly the case if $G$ is reduced). Then $X$ realises the quotient $G/H$ (that one could imagine as a sheaf on the fppf-site schemes over $k$, in particular, $G \to X$ is faithfully flat, and an $H$-torsor). If $H$ is a normal subgroup, then $G/H$ is again a linear algebraic group over $k$. If $H$ is smooth over $k$ and $G$ is reduced, then $G \to X$ is smooth and we have $T_X([L]) = \mathrm{Lie}(G)/\mathrm{Lie}(H)$.

## 4.8 Jordan decomposition

Let $k$ be a perfect field, $G$ a linear algebraic $k$-group, and $g \in G(k)$. Then there are unique $g_s$ and $g_u$ in $G(k)$ such that $g = g_s g_u = g_u g_s$, $g_s$ is semisimple (diagonalisable in any representation of $G$, or equivalently, on $\mathcal{O}(G)$ or on one faithful representation) and $g_u$ is unipotent (in any representation of $G$ its only eigenvalue is $1$, or equivalently, $g_u - 1$ is nilpotent in all finite dimensional representations). To prove this: let $G$ act on $\mathcal{O}(G)$ by right translations. Decompose $\rho(g) = su$, with $s$ semi-simple and $u$ unipotent and $su = us$. (In terms of $\rho(g) = d + n$, $s = d$, $u = 1 + d^{-1}n$.) Then $s$ and $u$ commute with the action of $G$ on $\mathcal{O}(G)$ by left translations, and are $k$-algebra morphisms, hence *are* right translations of uniquely determined $g_s$ and $g_u$ in $G(k)$ (the only morphisms of varieties $G \to G$ commuting with the left-action by $G$ are the right translations from $G(k)$).

## 4.9 Representations and Tannaka

Let $G$ be a linear algebraic group over a field $k$. Let $\mathrm{Rep}(G)$ denote the category of representations of $G$ on finite dimensional $k$-vector spaces. The Hom-sets in this category are finite dimensional $k$-vector spaces, and $\mathrm{Rep}(G)$ has direct sum, tensor product, dual, and it is abelian. We have the forgetful functor $F$ from $\mathrm{Rep}(G)$ to $k$-vector spaces. A theorem named after Tannaka says that every automorphism of $F$ that is compatible with the tensor product is given by a unique element of $G(k)$. Let us make this more explicit. Let $\alpha$ be such an automorphism. This means that for every $V$ in $\mathrm{Rep}(G)$ we have $\alpha(V)$ in $\mathrm{GL}(V)$, such that for every morphism $f \colon V \to W$ in $\mathrm{Rep}(G)$ we have $\alpha(W) \circ f = f \circ \alpha(V)$. Compatibility with tensor product means that for $V$ and $W$ in $\mathrm{Rep}(G)$ we have $\alpha(V \otimes_k W) = \alpha(V) \otimes \alpha(W)$. For the trivial representation $k$ this implies that $\alpha(k) = 1$ ($k \otimes_k k$ is isomorphic to $k$ and therefore $\alpha(k)^2 = \alpha(k)$). For $V$ in $\mathrm{Rep}(G)$ it implies that $\alpha(V^\vee) = \alpha(V)^{\vee,-1}$ (exercise). Let us now prove that there is a unique element $g$ in $G(k)$ such that for all $V$ we have $\alpha(V) = (g\cdot)$. Let $V$ be in $\mathrm{Rep}(G)$. Let $W$ be a finite dimensional faithful representation of $G$, then $V \oplus W$ is also one, and $\alpha(V \oplus W) = \alpha(V) \oplus \alpha(W)$ (consider the injections of $V$ and $W$ into $V \oplus W$). Let $L$ be a line in some representation $T$ of $\mathbf{GL}(V \oplus W)$ obtained by tensor product, direct sum, dual, subrepresentation, quotient, such that $G$ is the stabiliser of $L$. Then the inclusion $L \to T$ is a morphism in $\mathrm{Rep}(G)$ and therefore $\alpha(T)$ stabilises $L$, and therefore $\alpha(V \oplus W)$ is in $G(k)$. In fact, one can recover the Hopf algebra $\mathcal{O}(G)$ from $\mathrm{Rep}(G)$, and therefore $G$ itself (see §2.5 in Springer's book). As we have seen that for all $V$ in $\mathrm{Rep}(G)$ we have $\mathrm{Hom}_G(V, \mathcal{O}(G)) = \mathrm{Hom}_k(FV, k)$, it is not surprising that $\mathcal{O}(G)$ can be reconstructed from $F$. But it also works like this: for every $k \to A$, one has $A \otimes_k \mathrm{Rep}(G)$ (keep the objects, tensor the Hom's with $A$), and $F_A \colon A \otimes_k \mathrm{Rep}(G) \to A \otimes_k k{-}\mathrm{Vect}$ (the category of free $A$-modules), one will have $G(A) = \mathrm{Aut}(F_A)$, I guess.

## 4.10 Representations of tori

Let us first consider $\mathrm{Rep}(\mathbb{G}_{\mathrm{m}k})$. It is a very nice exercise to show that to give an action of $\mathbb{G}_{\mathrm{m}k}$ on a $k$-vector space $V$ is the same as giving a $\mathbb{Z}$-grading of $V$: $V = \oplus_{i \in \mathbb{Z}} V_i$, such that $\mathbb{G}_{\mathrm{m}k}$ acts on $V_i$ via the $i$th power map.

More generally, for $T$ a split torus, that is, $T = \mathbf{Hom}(X, \mathbb{G}_{\mathrm{m}k})$ with $X$ a free $\mathbb{Z}$-module of finite rank, to give an action of $T$ on a $k$-vector space $V$ is to give $V$ an $X$-grading: $V = \sum_{\chi \in X} V_\chi$. Note that we have $\mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}k}) = X$.

A $k$-torus is a linear algebraic group $T$ such that $T_{\overline{k}}$ is a split torus. One puts $X^*(T) := \mathrm{Hom}_{\overline{k}}(T_{\overline{k}}, \mathbb{G}_{\mathrm{m}\overline{k}})$, and $X_*(T) := \mathrm{Hom}_{\overline{k}}(\mathbb{G}_{\mathrm{m}\overline{k}}, T_{\overline{k}})$. These are free $\mathbb{Z}$-modules of rank $\dim(T)$, with action by $\mathrm{Gal}(\overline{k}/k)$, called character and co-character group, canonically duals of each other. The action of $\mathrm{Gal}(\overline{k}/k)$ induces injections $\mathrm{Gal}(k'/k)$ into $\mathrm{GL}(X^*(T))$ and $\mathrm{GL}(X_*(T))$, with $k \to k'$ finite separable, called the splitting field of $T$. To give an action of $T$ on a $k$-vector space $V$ is to give an $X^*(T)$-grading $V_{k'} = \oplus_{\chi \in X^*(T)} V_\chi$ such that for all $\chi$ in $X^*(T)$ and all $\sigma$ in $\mathrm{Gal}(k'/k)$ we have $\sigma V_\chi = V_{\sigma \cdot \chi}$.

## 4.11 Reductive groups in char zero

Let $k$ be a field of characteristic zero, and let $G$ be a connected linear algebraic group over $k$. Then $G$ is smooth over $k$. The $k$-group $G$ is called *reductive* if $\mathrm{Rep}(G)$ is semi-simple: every $V$ in $\mathrm{Rep}(G)$ is a direct sum of simple ones; this is a property of $V$ as a $\mathrm{Lie}(G)$-module, and it is invariant under field extensions $k \to k'$ (indeed, the formation of $\mathrm{End}_G(V) = \mathrm{End}_{\mathrm{Lie}(G)}(V)$ commutes with $k \to k'$, and a finite dimensional $k$-algebra $A$ is semi-simple if and only if $A_{\overline{k}}$ is; we use here that $k \supset \mathbb{Q}$). This is equivalent to the condition that $G$ contains no nontrivial normal subgroup isomorphic to $\mathbb{G}_{\mathrm{a}k}^{\,n}$, where $\mathbb{G}_{\mathrm{a}k}$ is the *additive group*, for all $k \to A$, $\mathbb{G}_{\mathrm{a}k}(A) = (A, +)$; it is represented by $\mathbb{A}^1_k$. Or in other words: the *unipotent radical* of $G$ is trivial (this plus smoothness is the definition over algebraically closed fields of arbitrary characteristic).

Examples. $\mathbb{G}_{\mathrm{a}k}$ is *not* reductive, and neither is the tautological semi-direct product of $\mathbb{G}_{\mathrm{m}k}$ by $\mathbb{G}_{\mathrm{a}k}$: $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$. Some positive examples are $\mathrm{GL}_n$, special orthogonal groups, symplectic groups,...

## 4.12 Compact forms

Let $G$ be a connected linear algebraic group over $\mathbb{C}$. Then $G$ is reductive if and only if there exists an $\mathbb{R}$-*model* $G_{\mathbb{R}}$ of $G$ over $\mathbb{R}$ (that is, $G$ is isomorphic to $(G_{\mathbb{R}})_{\mathbb{C}}$), with $G_{\mathbb{R}}(\mathbb{R})$ compact (for the Archimedean topology). Let us prove one direction: for $G$ over $\mathbb{R}$, linear algebraic and connected, with $G(\mathbb{R})$ compact, $G$ is reductive (and we already know that $G$ is reductive iff $G_{\mathbb{C}}$ is). Let $V$ be in $\mathrm{Rep}(G)$. Let $\langle\cdot,\cdot\rangle$ be an inner product on $V$. Then averaging over $G(\mathbb{R})$ gives a $G(\mathbb{R})$-invariant inner product on $V$. As $G$ is connected, $G(\mathbb{R})$ is Zariski dense in $G$, and the inner product is $G$-invariant. Then for $W \subset V$ a subrepresentation of $V$, the orthogonal $W^{\perp}$ is also a subrepresentation, and therefore $V = W \oplus W^{\perp}$ as $G$-representation.

## 4.13 Twisting

Let $k \to l$ be a finite Galois extension, and let $\Gamma := \mathrm{Gal}(l/k)$. Let $X$ and $Y$ be $k$-schemes, and suppose that $f\colon X_l \to Y_l$ is an isomorphism of $l$-schemes. Then, for every $\sigma$ in $\Gamma$, we have $\sigma(f)\colon X_l \to Y_l$, the pullback of $f$ under $\mathrm{Spec}(\sigma)\colon \mathrm{Spec}(l) \to \mathrm{Spec}(l)$. Then $f$ comes from $k$ if and only if for all $\sigma \in \Gamma$ we have $\sigma(f) = f$. Anyway, we get two right-actions of $\Gamma$ on $X_l \to \mathrm{Spec}(l)$, covering the tautological action of $\Gamma$ on $\mathrm{Spec}(l)$: one is where $\sigma$ acts as $\mathrm{id}_X \times \mathrm{Spec}(\sigma)$, and the other is by transporting the action by $\mathrm{id}_Y \times \mathrm{Spec}(\sigma)$ on $Y_l$ to $X_l$ via $f$: $f^{-1} \circ (\mathrm{id}_Y \times \mathrm{Spec}(\sigma)) \circ f$. Clearly, $f$ is an isomorphism from $X_l \to \mathrm{Spec}(l)$ with the second action to $Y_l \to \mathrm{Spec}(l)$ with its own action by $\mathrm{id}_Y \times \mathrm{Spec}(\sigma)$. Therefore, the quotient of $X_l$ by the second action exists, and is $X_l \to Y_l \to Y$. Assume now that $X$ is quasi-projective over $k$. Then for every right-action of $\Gamma$ on $X_l \to \mathrm{Spec}(l)$ the quotient $X_l \to X_l/\Gamma$ exists, and gives what we call the twist of $X$ by that action. In this way, one gets all $k$-schemes $Y$ such that $Y_l$ is isomorphic to $X_l$ as $l$-schemes. One can make a bijection between such actions and the set of 1-cocycles of $\Gamma$ with coefficients in $\mathrm{Aut}_l(X_l)$, the group of automorphisms of $X_l$ as $l$-scheme (Galois cohomology). A simple case is where $\Gamma$ acts from the right on $X$ as $k$-scheme. We give an example of that.

Let $\tau$ be the automorphism of the $\mathbb{R}$-scheme $\mathrm{GL}_{n,\mathbb{R}}$ given by $g \mapsto g^{t,-1}$, the inverse of the transpose; then $\tau^2 = \mathrm{id}$. Then we let $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ act (from the right) on $\mathrm{GL}_{n,\mathbb{C}} \to \mathrm{Spec}(\mathbb{C})$, by sending the complex conjugation $\iota$ to $\tau \times \mathrm{Spec}(\iota)$. Then, for all $\mathbb{R} \to A$, the $A$-points of the quotient are the $g$ in $\mathrm{GL}_n(\mathbb{C} \otimes_{\mathbb{R}} A)$ such that

$$g = \overline{g}^{t,-1}.$$

That is, the quotient is the $\mathbb{R}-$group $\mathrm{U}_{n,\mathbb{R}}$ of unitary $n$ by $n$ matrices, stabilising the standard hermitian inner product on $\mathbb{C}^n$. As $\mathrm{U}_{n,\mathbb{R}}(\mathbb{R})$ is compact, $\mathrm{U}_{n,\mathbb{R}}$ is reductive and therefore $\mathrm{GL}_{n,\mathbb{C}} = (\mathrm{U}_{n,\mathbb{R}})_{\mathbb{C}}$ is reductive, and therefore $\mathrm{GL}_{n,\mathbb{Q}}$ is reductive.

## 4.14 Reductive groups and root data

Reductive groups can be described in terms of *root data* (and this works over any separably closed field (one needs a split maximal torus)). Let us show how this works in the simplest example: $G = \mathrm{GL}_{n,k}$. We let $T = \mathbb{G}_{\mathrm{m}}{}_k^n$ be the diagonal torus in $G$; $t = (t_1, \ldots, t_n)$ corresponds to the diagonal matrix $\mathrm{diag}(t)$

with $t_{i,i} = t_i$. We let $X^*(T)$ be the character group of $T$. It has $\mathbb{Z}$-basis the $e_i\colon T \to \mathbb{G}_{\mathrm{m}k}$, projections on the factors. The action of $G$ on itself by conjugation restricts to an action by $T$ on $G$, hence on $\mathrm{Lie}(G) = \mathrm{M}_n(k)$, which decomposes as $\mathrm{Lie}(G) = \sum_{\chi \in X^*(T)} \mathrm{Lie}(G)_\chi$. The computation

$$\left(\mathrm{diag}(t)\cdot a\cdot \mathrm{diag}(t)^{-1}\right)_{i,j} = t_i a_{i,j} t_j^{-1}$$

shows that $\mathrm{Lie}(G)_0 = \mathrm{Lie}(T)$, and

$$\mathrm{Lie}(G) = \mathrm{Lie}(T) \oplus \bigoplus_{\alpha \in R} \mathrm{Lie}(G)_\alpha, \quad R = \{e_i - e_j : i \neq j\}, \quad \mathrm{Lie}(G)_{e_i - e_j} = k\cdot E_{i,j}.$$

In particular, if $\alpha \neq 0$ and $\mathrm{Lie}(G)_\alpha \neq 0$, then $\mathrm{Lie}(G)_\alpha$ is of dimension one. The subset $R$ of $X^*(T)$ is called the set of roots of $G$ with respect to $T$. Note that it is symmetric: for each $\alpha \in R$, we have $-\alpha \in R$. Also, for $\alpha$ and $\beta$ in $R$ with $\alpha \neq \pm\beta$, $\alpha$ and $\beta$ are linearly independent (reducedness of the root datum). For each root $\alpha$, there is an injective morphism of group schemes $u_\alpha\colon \mathbb{G}_{\mathrm{a}k} \to G$, unique up to $k^\times$, such that for any $k \to A$, any $a \in A$ and $t \in T(A)$,

$$t\cdot u_\alpha(a)\cdot t^{-1} = u_\alpha(\alpha(t)\cdot a) \quad \text{in } G(A).$$

For $\alpha \in R$, the subgroup scheme $\langle u_\alpha, u_{-\alpha}\rangle$ generated by $u_\alpha$ and $u_{-\alpha}$ is isomorphic to $\mathrm{SL}_{2,k}$ (in general it can also be isomorphic to $\mathrm{PGL}_{2,k}$) and $T \cap \langle u_\alpha, u_{-\alpha}\rangle$ is isomorphic to $\mathbb{G}_{\mathrm{m}k}$, uniquely up to sign, and there is a unique cocharacter denoted $\alpha^\vee$ in $X_*(T)$ factoring through $\langle u_\alpha, u_{-\alpha}\rangle$ such that $\alpha \circ \alpha^\vee = 2$ in $\mathrm{End}(\mathbb{G}_{\mathrm{m}k}) = \mathbb{Z}$. For $\alpha \in R$, let

$$s_\alpha\colon X^*(T) \to X^*(T), \quad \chi \mapsto \chi - \langle \alpha^\vee, \chi\rangle\cdot\alpha$$

denote the symmetry given by $\alpha$ and $\alpha^\vee$: it is the identity on $(\alpha^\vee)^\perp$, and sends $\alpha$ to $-\alpha$. Then one can check that the $s_\alpha$ preserves $R$, $R^\vee$, and $\Psi$, and hence

$$(X^*(T), R, X_*(T), R^\vee, \Psi),$$

where $R^\vee = \{\alpha^\vee : \alpha \in R\}$, and $\Psi\colon R \to R^\vee$ is the bijection $\alpha \mapsto \alpha^\vee$, is a *root datum*. For any such datum there exists a reductive group scheme over $\mathbb{Z}$, called a Chevalley group scheme, that gives by the above recipe, the root datum in question. It can be built up "by generators and relations" (first Borel group, then by Bruhat decomposition $G$ itself). The subgroup of $\mathrm{Aut}(X^*(T))$ generated by the $s_\alpha$ is called the Weyl group of $(G, T)$, or of the root datum. Note that the notion of root datum is symmetric: if $(X, R, X^\vee, R^\vee, \Psi)$ is a root datum, then so is $(X^\vee, R^\vee, X, R, \Psi^{-1})$.

If $G$ and $G'$ are reductive, over $k$, with split maximal tori $T$ and $T'$, and if $\phi$ is an isomorphism from $(X^*(T), R, X_*(T), R^\vee, \Psi)$ to $(X^*(T'), R', X_*(T'), R'^\vee, \Psi')$, then there is an isomorphism $\tilde{\phi}$ from $G$ to $G'$ sending $T$ to $T'$ and inducing $\phi$. Such a $\tilde{\phi}$ can be made unique by imposing "épinglages" (pinnings) on the root data and on the groups. See Cor. 5.1 in Exposé XXIII of SGA3. For completeness, a pinning of $(G, T, B)$ is a collection of $u_\alpha$, for each $\alpha \in \Delta$, the set of simple roots specified by $B$. Multiplication in $G$ induces dualities between $\mathrm{Lie}(G)_\alpha$ and $\mathrm{Lie}(G)_{-\alpha}$, hence from the $u_\alpha$ one gets $u_{-\alpha}$.

## 4.15 Semi-simple, adjoint, derived, simply connected

Let us say something about the structure of reductive group schemes. Let $G$ be a reductive group scheme over a field $k$, and let $Z_G$ be its center. If $T$ is a split maximal torus of $G$ (of course, this does not necessarily exist), and $R$ is the set of roots, then $Z_G = \cap_{\alpha \in R} \ker(\alpha)$. The maximal torus $C_G$ of $Z_G$ is the intersection of the $\ker(\beta)$, $\beta$ in $X^*(T) \cap \sum_{\alpha \in R} \mathbb{Q}\cdot\alpha$. Note that $Z_G^0$ is not necessarily a torus, for example for $G = \mathrm{SL}_n$, $Z_G = \mu_n$, which is not reduced if $k \supset \mathbb{F}_p$ with $p|n$. The reductive group $G$ is called *semi-simple* if $Z_G$ is finite. This is the case if and only if $\mathbb{Z}\cdot R \subset X^*(T)$ is of finite index.

There are two ways to get a semi-simple group from a reductive group. The first is to divide out the center: $G^{\mathrm{ad}} := G/Z_G$; $G^{\mathrm{ad}}$ is called the adjoint group of $G$. If $k \supset \mathbb{Q}$ then $G^{\mathrm{ad}}$ is the image of $G$ acting on $\mathrm{Lie}(G)$. The group scheme $G^{\mathrm{ad}}$ is again reductive, its root datum of $G^{\mathrm{ad}}$ (w.r.t. a split torus) is obtained from that of $G$ by replacing $X^*(T)$ by the submodule spanned by $R$, and the rest accordingly. The second construction is to take the *derived subgroup*: $G^{\mathrm{der}}$ is the subgroup scheme generated by commutators. It is generated by the $u_\alpha$, and its torus is generated by the $\alpha^\vee$. If $G$ is semi-simple, then its *simply connected cover* $\widetilde{G}$ is obtained by replacing $X_*(T)$ by $\mathbb{Z}\cdot R^\vee$ and adapting the rest accordingly.

For $G$ reductive, the morphism $C_G \times G^{\mathrm{der}} \to G$ is an isogeny (surjective, finite kernel), and one says that $G$ is almost a product of $C_G$ and $G^{\mathrm{der}}$. Similarly, $\widetilde{G^{\mathrm{der}}} \to G^{\mathrm{ad}}$ are the extremes in the isogeny class of $G^{\mathrm{der}}$ (well, I only consider isogenies with kernel in the center). If $G \to G'$ is a central isogeny, $T$ and $T'$ maximal tori of $G$ and $G'$, with $T \to T'$, we have $X^*(T') \to X^*(T)$, injective, and so we consider the $X^*(T)$ as lattices in one $\mathbb{Q}$-vector space, and similarly for the $X_*(T)$.

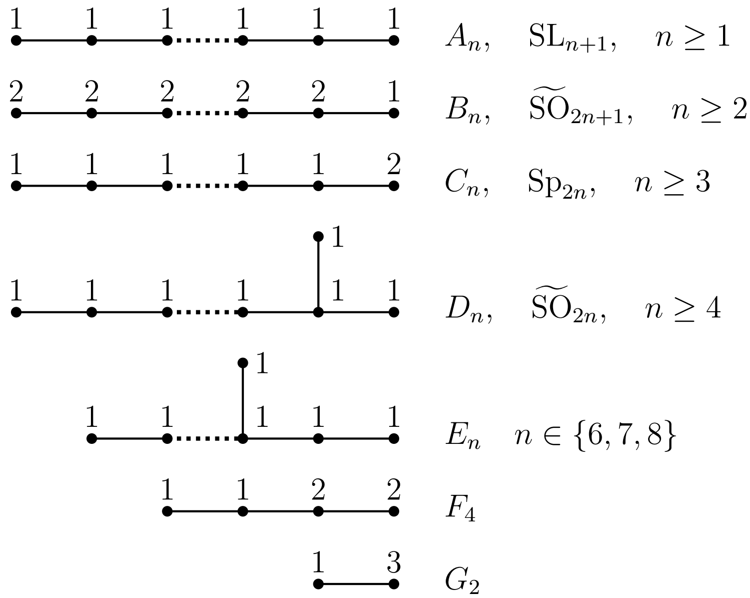
## 4.16 Subgroup schemes containing $T$, Borel groups, simple roots

Let $G$ be reductive over $k$, $T$ a split maximal torus, $R$ the set of roots. Then the map that sends smooth subgroup scheme $H$ of $G$ containing $T$ to the set of $\alpha \in R$ such that $\mathrm{Lie}(H)_\alpha \neq 0$ is an injection to the power set of $R$.

Let $B$ be a Borel subgroup containing $T$ (a smooth subgroup scheme, minimal parabolic), in $\mathrm{GL}_n$ typically the upper triangular elements). Then the set of $\alpha \in R$ such that $\mathrm{Lie}(B)_\alpha \neq 0$ is called the *set of positive roots* $R^+$ attached to $(G, T, B)$. It has a subset $\Delta$, called the set of simple roots, with the property that each $\alpha$ in $R^+$ can be uniquely written as a sum of elements of $\Delta$. For $\mathrm{GL}_n$, these are the $e_i - e_{i+1}$. The set of $B$ containing $T$ is a torsor under the Weyl group. Each $B$ has an opposite $B^-$ such that $B \cap B^- = T$. Then $R = R^+ \cup R^-$.

One attaches to $\Delta$ a *Dynkin diagram*: the graph with set of vertices $\Delta$, with distinct vertices $\alpha$ and $\beta$ connected if and only if $\langle \alpha^\vee, \beta \rangle \neq 0$, and with the function $\mathrm{length}$ from $\Delta$ to $\{1, 2, 3\}$ that sends each $\alpha$ to its "length", determined by $\mathrm{length}(\alpha)\langle \alpha^\vee, \beta \rangle = \mathrm{length}(\beta)\langle \beta^\vee, \alpha \rangle$, for all $\alpha$ and $\beta$ in $\Delta$. Root data that are simply connected or adjoint are uniquely determined by their Dynkin diagram.

Here is the list of all connected Dynkin diagrams coming from simply connected (almost) simple groups, copied from SGA3, Exp. 21.




## 4.17 Decomposition of adjoint or simply connected groups

Let $G$ be reductive over a field $k = k^{\mathrm{sep}}$, and suppose that $G$ is adjoint or simply connected. Then $G$ is a product of (almost) simple factors, each corresponding to a connected Dynkin diagram. Over a general field $k$, $G$, is a product $G = \prod_i G_i$, with each $G_i$ (almost) simple (over $k$), and such that $G_{i,k^{\mathrm{sep}}}$ is a product of (almost) simple groups over $k^{\mathrm{sep}}$ that are all isomorphic ($G_i$ is isotypical). In other words, $G_i$ is a *twist* of a product of split almost simple groups over $k$ that are all of the same type.

## 4.18 Galois action on Dynkin diagram

Let $G$ be reductive over a field $k$. Then there exists a maximal torus $T$ of $G$, not necessarily split ($T_{\overline{k}}$ is maximal in $G_{\overline{k}}$, this is a theorem of Grothendieck, SGA3, Exp.XIV); $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ acts on $X^*(T)$, on $R$ (because $\mathrm{Lie}(G)$ is in $\mathrm{Rep}(T)$), on the root datum of $(G, T)$ over $k^{\mathrm{sep}}$. Let $B$ be a Borel subgroup of $G_{k^{\mathrm{sep}}}$ containing $T_{k^{\mathrm{sep}}}$, and $\Delta \subset R$ the set of simple roots. For $\sigma$ in $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ we have $\sigma(B)$, $\sigma(\Delta)$, and $\sigma\colon \Delta \to \sigma(\Delta)$, and there is a $g$ in $G(k^{\mathrm{sep}})$ that conjugates $(T_{k^{\mathrm{sep}}}, \sigma(B), \sigma(\Delta))$ to $(T_{k^{\mathrm{sep}}}, B, \Delta)$ (we could also take an element of the Weyl group, which is moreover unique). This is the action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on $\Delta$.

Exercise: show that the complex conjugation acts on the Dynkin diagram $A_{n-1}$ of $\mathrm{SU}_{n,\mathbb{R}}$ as the reversion.

## 4.19 Reductive group as stabiliser of element

We have seen above that for $k \supset \mathbb{Q}$ each closed subgroup scheme $G$ of $\mathbf{GL}(V)$ is the stabiliser of some line $L$ in a suitable $k$-vector space $W$ obtained from $V$ by direct sums, duals and tensor products. Assume now that $G$ is reductive. Then, as representation of $G$, we have $W = L \oplus W'$, and $W^\vee = L^\vee \oplus (W')^\vee$, hence $G$ is the stabiliser of any non-zero element of the line $L \otimes L^\vee$ in the representation $W \otimes_k W^\vee$ of $\mathbf{GL}(V)$. Exercise: given lines $L \subset V$ and $M \subset W$, then $(\mathbf{GL}(V) \times \mathbf{GL}(W))_{L\otimes M} = \mathbf{GL}(V)_L \times \mathbf{GL}(W)_M$.

### 4.20 Simply connectedness over $\mathbb{R}$ and connectedness

Let $G$ be a connected linear algebraic group over $\mathbb{R}$, semi-simple, and simply connected. Then $G(\mathbb{R})$ is connected. Reference: Borel-Tits, Cor. 4.7, Publ. Math. IHES, 1972, "Compléments à l'article Groupes réductifs". Deligne says "Puisque $\widetilde{G}(k_v)$ est connexe…" right after Cor. 2.0.7.

### 4.21 Compact over $\mathbb{R}$ and connectedness

Let $G$ be a connected linear $\mathbb{R}$-algebraic group with $G(\mathbb{R})$ compact. Then $G(\mathbb{R}) = G(\mathbb{R})^+$. Let me give the argument from Platonov-Rapinchuk, page 121, for this. Let $g$ be in $G(\mathbb{R})$. We have the Jordan decomposition $g = g_s g_u$ with $g_s$ and $g_u$ in $G(\mathbb{R})$. As $G(\mathbb{R})$ is compact, $g_u = 1$. Hence $g$ is contained in some torus $T$ in $G$ (Springer, 13.3.8: every semi-simple $g$ in $G(\mathbb{R})$ is contained in a maximal torus $T$ of $G$). As $T(\mathbb{R})$ is compact, it is connected connected (it is a product of circles).

## 5 Mumford-Tate groups

For $h\colon \mathbb{S} \to \mathbf{GL}(V)$ a $\mathbb{Q}$-HS, $h(\mathbb{S})$ is a closed subgroupscheme of $\mathbf{GL}(V_\mathbb{R}) = \mathbf{GL}(V)_\mathbb{R}$.

**5.1 Definition.** We define:

$$\mathrm{MT}(V) = \bigcap \{H : H \subset \mathbf{GL}(V) \text{ closed subgroupscheme (over } \mathbb{Q}!) \text{ such that } h(\mathbb{S}) \subset H_\mathbb{R}\}.$$

Hence $\mathrm{MT}(V)$, also denoted $\mathrm{MT}(h)$, $\mathrm{MT}(V, h)$, is the smallest algebraic subgroup of $\mathbf{GL}(V)$ such that $h\colon \mathbb{S} \to \mathbf{GL}(V_\mathbb{R})$ factors through $\mathrm{MT}(V)_\mathbb{R}$. It is connected, because $\mathbb{S}$ is (if $h(\mathbb{S}) \subset H_\mathbb{R}$, then $h(\mathbb{S}) \subset (H^0)_\mathbb{R}$). It is called the *Mumford-Tate group* of $(V, h)$.

**5.2 Proposition.** *Another description is the following:*

$$\mathrm{MT}(V)(\mathbb{C}) = \left\langle \bigcup \{\sigma(h(\mathbb{S}(\mathbb{C}))) : \sigma \in \mathrm{Aut}(\mathbb{C})\} \right\rangle^{\mathrm{Zar}} \subset \mathrm{GL}(V_\mathbb{C}).$$

**Proof.** To prove this, one observes that the right hand side is the Zariski closure of an $\mathrm{Aut}(\mathbb{C})$-invariant subgroup of $\mathrm{GL}(V_\mathbb{C})$, hence a closed subgroup of $\mathrm{GL}(V_\mathbb{C})$, "defined over $\mathbb{Q}$", meaning that it comes from $\mathbb{Q}$ by base change. More precisely: let $I$ be the ideal of the right hand side in $\mathcal{O}(\mathbf{GL}(V_\mathbb{C})) = \mathcal{O}(\mathbf{GL}(V))_\mathbb{C}$; then $I$ is an $\mathrm{Aut}(\mathbb{C})$-invariant sub-$\mathbb{C}$-vector space, and therefore generated by $I_\mathbb{Q} := I \cap \mathcal{O}(\mathbf{GL}(V))$, and $I_\mathbb{Q}$ is the ideal of $\mathrm{MT}(V)$. In fact: one does not need to take Zariski closure here (use Prop.2.2.6 of Springer's LAG). $\square$

Now I copy from Moonen's notes. The key property of $\mathrm{MT}(V)$ is that in any tensor construction obtained from $V$ it cuts out exactly the sub-HS's. Notation: for $\nu = \{(a_i, b_i)\}_i$ a finite collection of pairs of integers $a_i \geq 0$, $b_i \geq 0$, we let

$$T^\nu := \bigoplus_i V^{\otimes a_i} \otimes (V^\vee)^{\otimes b_i}, \quad \text{(called "tensor space obtained from } V\text{")}$$

which inherits a HS from $V$, and we have

$$\mathbb{S} \longrightarrow \mathrm{MT}(V)_\mathbb{R} \hookrightarrow \mathbf{GL}(V)_\mathbb{R} \longrightarrow \mathbf{GL}(T^\nu)_\mathbb{R}.$$

**5.3 Proposition.** *Let $V$ be a $\mathbb{Q}$-HS, $T^\nu$ as above. Let $W \subset T^\nu$ be a subspace. Then $W$ is a sub $\mathbb{Q}$-HS if and only if it is $\mathrm{MT}(V)$-invariant. Let $t \in T^\nu$. Then $t$ is a Hodge class if and only if it is $\mathrm{MT}(V)$-invariant.*

**Proof.** Let $H = \mathbf{GL}(V)_W$, the stabiliser. Then $H$ is a closed subgroupscheme of $\mathbf{GL}(V)$. Suppose that $W$ is a sub-HS. Then $W_\mathbb{R}$ is $\mathbb{S}$-invariant, hence $h(\mathbb{S}) \subset H_\mathbb{R}$, and hence $\mathrm{MT}(V) \subset H$ and $W$ is $\mathrm{MT}(V)$-invariant. Suppose that $W$ is $\mathrm{MT}(V)$-invariant. Then $W_\mathbb{R}$ is $\mathbb{S}$-invariant because $\mathbb{S}$ acts on $T^\nu$ via $\mathrm{MT}(V)_\mathbb{R}$. For the second assertion, apply the first result to $T^{\nu'} := \mathbb{Q}(0) \oplus T^\nu$, and note that $t$ is a Hodge class if and only if $\mathbb{Q}\cdot(1, t) \subset T^{\nu'}$ is a sub-HS. □

**5.4 Example.** Let $A$ be an abelian variety over $\mathbb{C}$, and let $V = \mathrm{H}_1(A, \mathbb{Z})$. Then

$$\mathrm{End}(A) = \mathrm{End}_\mathbb{Z}(V)^{\mathrm{MT}(V)} = (V^\vee \otimes V)^{\mathrm{MT}(V)}.$$

Let $E/\mathbb{C}$ be an elliptic curve, $V = \mathrm{H}_1(E, \mathbb{Q})$, and $\mathrm{MT}(E) := \mathrm{MT}(V)$. If $\mathrm{End}(E) = \mathbb{Z}$, then $\mathrm{MT}(E) = \mathbf{GL}(V)$. If $\mathrm{End}(E)_\mathbb{Q} = F$, an imaginary quadratic extension of $\mathbb{Q}$, then $\mathrm{MT}(E) = T_F$, with $T_F = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m}F}$.

**5.5 Corollary.** *Let $V$ be a $\mathbb{Q}$-HS. Then*

$$\mathrm{MT}(V) = \bigcap \{\mathbf{GL}(V)_L : L = \mathbb{Q}\cdot t \text{ with } t \in T^\nu \cap (T^\nu)^{(p,p)}_\mathbb{C}, \nu \text{ varying}\}$$

**Proof.** For each such $\nu$, and $L$, $L$ is a sub $\mathbb{Q}$-HS, hence $\mathrm{MT}(V)$-invariant. Hence $\mathrm{MT}(V)$ is contained in the right hand side. Now we prove equality, by showing that $\mathrm{MT}(V)$ is one of these stabilisers. As $\mathrm{MT}(V)$ is a subgroup scheme of $\mathbf{GL}(V)$, and our base field is of characteristic zero, it is the stabiliser of a line $L$ in some $T^\nu$. Then $L_\mathbb{R}$ is $\mathbb{S}$-invariant, hence $\mathbb{S}$ acts on it via a character $\mathbb{S} \to \mathbb{G}_{\mathrm{m}\mathbb{R}}$, $z \mapsto (z\bar{z})^p$ for some $p$ in $\mathbb{Z}$. □

**5.6 Corollary.** *Let $V$ be a $\mathbb{Q}$-HS. The functor*

$$\mathrm{Rep}(\mathrm{MT}(V)) \to \mathbb{Q}\text{-HS}, \quad W \mapsto (W, h\colon \mathbb{S} \to \mathrm{MT}(V)_\mathbb{R} \to \mathbf{GL}(W)_\mathbb{R})$$

*is fully faithful, and its essential image is the full subcategory $\langle V \rangle^\otimes$ of $\mathbb{Q}$-HS of objects isomorphic to subquotients of $T^\nu$'s.*

**Proof.** We prove the fully faithfulness. Let $W_1$ and $W_2$ be in $\mathrm{Rep}(\mathrm{MT}(V))$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Q}\text{-HS}}(W_1, W_2) &= \mathrm{Hom}_{\mathbb{Q}}(W_1, W_2) \cap (\mathrm{Hom}_{\mathbb{Q}}(W_1, W_2)_{\mathbb{R}})^{\mathbb{S}} = \mathrm{Hom}_{\mathbb{Q}}(W_1, W_2)^{\mathrm{MT}(V)} \\ &= \mathrm{Hom}_{\mathrm{Rep}(\mathrm{MT}(V))}(W_1, W_2).\end{aligned}$$

For the second equality, note that if $\phi$ is in the LHS, then $\mathbf{GL}(V)_\phi$ is a subgroup over $\mathbb{Q}$ containing $h(\mathbb{S})$ over $\mathbb{R}$, and that if $\phi$ is in the RHS, then $\phi$ is $\mathbb{S}$-invariant because $\mathbb{S}$ acts via $\mathrm{MT}(V)_{\mathbb{R}}$.

Now the essential image. As $V$ is a faithful representation of $\mathrm{MT}(V)$, every $W$ in $\mathrm{Rep}(\mathrm{MT}(V))$ is isomorphic to a subquotient of some $T^\nu$, and by the Proposition, the subquotients are the same in $\mathbb{Q}$-HS and in $\mathrm{Rep}(\mathrm{MT}(V))$. $\square$

**5.7 Corollary.** *Let $V$ be a polarisable $\mathbb{Q}$-HS. Then $\mathrm{MT}(V)$ is reductive.*

**Proof.** The group $\mathrm{MT}(V)$ is connected, and $\mathrm{Rep}(\mathrm{MT}(V)) = \langle V\rangle^{\otimes}$ is semisimple, because all $W$ in $\langle V\rangle^{\otimes}$ get a polarisation from one on $V$.

Another proof is to show that a polarisation leads to a compact form over $\mathbb{R}$. Let $\psi$ be a polarisation on $V$ and let $G$ be connected component of the algebraic subgroup of $\mathbf{GL}(V)$ of elements that preserve $\psi$ up to a multiple (the group of similitudes of $\psi$). Then we claim that $\tau := \mathrm{inn}(h(i))$ is a Cartan involution of $G^{\mathrm{der}}$, that is, the twist $G^{\mathrm{der},\tau}$ of $G$ over $\mathbb{C}$ by $\tau$ has the property that $G^{\mathrm{der},\tau}(\mathbb{R})$ is compact. By definition of twisting, we have:

$$G^{\mathrm{der},\tau}(\mathbb{R}) = \{g \in G^{\mathrm{der}}(\mathbb{C}) : \tau(\overline{g}) = g\} = \{g \in G^{\mathrm{der}}(\mathbb{C}) : h(i)\overline{g} = gh(i)\}.$$

Let us show that such $g$ fix the positive definite hermitian form $H\colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ given by $(x, y) \mapsto \psi(x, h(i)\overline{y})$. Note that $G^{\mathrm{der}}$ fixes $\psi$ because it acts on $\mathbb{Q}{\cdot}\psi$ and has no nontrivial character. So, for $g \in G^{\mathrm{der},\tau}(\mathbb{R})$ and $x$ and $y$ in $V_{\mathbb{C}}$ we have:

$$H(gx, gy) = \psi(gx, h(i)\overline{gy}) = \psi(gx, h(i)\overline{g}\overline{y}) = \psi(gx, gh(i)\overline{y}) = \psi(x, h(i)\overline{y}) = H(x, y).$$

Hence $G^{\mathrm{der},\tau}$ is a subgroup of the algebraic subgroup of $\mathbf{GL}(V)$ that preserves $H$, and therefore $G^{\mathrm{der},\tau}(\mathbb{R})$ is closed in the unitary group of $H$ which is compact.

It follows that $\mathrm{MT}(V) \cap G^{\mathrm{der}}$ is reductive, and hence $\mathrm{MT}(V)$ is (it is an extension of a torus by this intersection). $\square$

**5.8 Lemma.** *Let $V_1$ and $V_2$ be $\mathbb{Q}$-HS's. Then $\mathrm{MT}(V_1 \oplus V_2)$ is contained in $\mathrm{MT}(V_1) \times \mathrm{MT}(V_2)$ and both its projections are surjective.*

**Proof.** Easy exercise. $\square$

**5.9 Definition.** Let $A$ be an abelian variety over $\mathbb{C}$. Then $A$ is a CM-abelian variety if $\mathrm{MT}(A)$ is commutative, hence a torus, because it is reductive.

**5.10 Proposition.** *Let $A/\mathbb{C}$ be an abelian variety. Then $A$ is isogeneous to $\prod_i A_i^{n_i}$ with the $A_i$ simple and mutually non-isogeneous. Then $A$ is a CM-abelian variety if and only if for all $i$, $\mathrm{End}(A_i)_{\mathbb{Q}}$ is a field of degree $2\dim A_i$ over $\mathbb{Q}$.*

**Proof.** Assume that $\mathrm{MT}(A)$ is a torus. Let $B$ be one of the $A_i$'s. Then $\mathrm{MT}(B)$ is a torus, because it is a quotient of $\mathrm{MT}(A)$. Let $V := \mathrm{H}_1(B, \mathbb{Q})$. Then $\mathrm{End}(B)_{\mathbb{Q}} = \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{MT}(B)}$ is a division algebra; let $F$ be its center. Then $\mathrm{MT}(B) \subset T_F$, and therefore $\mathrm{End}(B)_{\mathbb{Q}} \supset \mathrm{End}_F(V)$, hence $\dim_F(V) = 1$, and $\dim_{\mathbb{Q}}(F) = \dim_{\mathbb{Q}}(V) = 2\dim(B)$.

On the other hand, assume that for all $i$, $F_i := \mathrm{End}(A_i)_{\mathbb{Q}}$ is a field of degree $2\dim A_i$ over $\mathbb{Q}$. Then for all $i$, $\mathrm{MT}(A_i) \subset T_{F_i}$, and $\mathrm{MT}(A) \subset \prod_i \mathrm{MT}(A_i)$ is a torus. $\square$

**5.11 Definition. (extended Mumford-Tate group)** Sometimes it is convenient to consider a slightly larger Mumford-Tate group, to not only consider $T^\nu$'s and their subquotients, but also Tate twists. For $V$ a $\mathbb{Q}$-HS, we define

$$\mathrm{MT}^{\#}(V) := \mathrm{MT}(V \oplus \mathbb{Q}(1)) \subset \mathrm{MT}(V) \times \mathbb{G}_{\mathrm{m}\mathbb{Q}}.$$

Then $\mathrm{Rep}(\mathrm{MT}^{\#}(V))$ is the full subcategory of $\mathbb{Q}$-HS consisting of subquotients of the $T^\nu(r)$.

# 6 Motivation for Deligne's definition of Shimura datum

At this point we can really start working on the title of my series of lectures: Introduction to Shimura varieties. Our goal is now to understand Deligne's motivation for his definition of "Shimura datum".

The starting point is that we have some very nice and important examples of quotients $\Gamma\backslash X$ of hermitian symmetric domains by discrete arithmetic groups, that have an interpretation as a moduli space and play an important role in number theory (Langlands program, for example). The first example is $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ with $\mathbb{H}$ the complex upper half plane, on which $\mathrm{SL}_2(\mathbb{R})$ acts transitively. It is the moduli space of elliptic curves. Of course, congruence subgroups are also essential. The next example is $\mathrm{Sp}_{2g}(\mathbb{Z})$ acting on $\mathbb{H}_g$ the Siegel half space consisting of symmetric $g$ by $g$ complex matrices $\tau$ such that $\Im(\tau)$ is positive definite. This gives the moduli space $A_{g,1}(\mathbb{C})$ of principally polarised complex abelian varieties of dimension $g$. Inside (or better, "almost" inside) $A_{g,1}(\mathbb{C})$, one finds many more examples (Hilbert modular varieties, or more generally subvarieties parametrising abelian varieties with certain endomorphisms). Shimura showed in many cases that such quotients can be defined over number fields, in the 1960's. Deligne, in his Bourbaki lecture "Travaux de Shimura", 1971, reshaped and generalised it, into what is now called the theory of Shimura varieties.

As complex abelian varieties correspond, via $A \mapsto \mathrm{H}_1(A, \mathbb{Z})$, to polarisable $\mathbb{Z}$-Hodge structures of type $(-1, 0), (0, -1)$, and subvarieties of $A_{g,1}$ that we are interested in are given in Hodge theoretical terms (classes being of type $(0, 0)$), the idea is to study the problem in terms of Hodge structures. For example, $\mathbb{H}$ can be seen as the set of complex structures on the fixed $\mathbb{R}$-vector space $\mathbb{R}^2$, or, equivalently, as half of the set of Hodge structures on $\mathbb{R}^2$ of type $(-1, 0), (0, -1)$ (in stead of viewing $\mathbb{H}$ as the set of lattices in $\mathbb{C}$ with basis $(1, \tau)$): $V_\tau^{0,-1} \subset \mathbb{C}^2$ is the line $\{(x, \tau x) : x \in \mathbb{C}\}$. Similarly, $\mathbb{H}_g$ is the set of Hodge structures on $\mathbb{R}^{2g}$ of type $(-1, 0), (0, -1)$ for which the standard symplectic form $\psi$ on $\mathbb{Z}^{2g}$ is a polarisation.

Let $(V_i)_{i\in I}$ be a finite set of finite dimensional $\mathbb{R}$-vector spaces, and let $(s_j)_{j\in J}$ be a set of elements in $\mathbb{R}$-vector spaces obtained from the $V_i$ by direct sum, duals and tensor products. We are then interested in the set $S$ of Hodge structures $(h_i\colon \mathbb{S} \to \mathbf{GL}(V_i))_{i\in I}$ on the $V_i$ such that for all $j$, $s_j$ is of type $(0,0)$, that is, is $\mathbb{S}$-invariant. Note that the set of Hodge structures on an $\mathbb{R}$-vector space $V$, such that the $V^{p,q}$ have fixed dimensions, form one orbit under $\mathrm{GL}(V)$ (compose by inner automorphisms, two such $(V,h)$ are isomorphic as representations of $\mathbb{S}$). This gives $S$ the structure of the set of real points of an $\mathbb{R}$-scheme that is locally of finite type.

For example, for abelian varieties of dimension $g$, we would take $V_1 = \mathbb{R}^{2g}$, and $V_2 = \mathbb{R}$, and $s = \psi\colon V_1 \otimes V_1 \to V_2$ the standard symplectic pairing, that is, $s \in V_1^\vee \otimes V_1^\vee \otimes V_2$, and consider the set of $h_1\colon \mathbb{S} \to \mathbf{GL}(V_1)$ and $h_2\colon \mathbb{S} \to \mathbf{GL}(V_2)$ for which $V_1$ be of type $(-1,0),(0,-1)$, $V_2$ of type $(-1,-1)$, and $s$ is of type $(0,0)$, that is, $\psi$ is a morphism of Hodge structures.

Back to the general case. Let now $G$ be the algebraic group over $\mathbb{R}$ obtained as intersection, in $\prod_{i\in I}\mathbf{GL}(V_i)$, of the stabilisers of the $s_j$:

$$G := \bigcap_{j\in J}\left(\prod_{i\in I}\mathbf{GL}(V_i)\right)_{s_j}.$$

Then $S$ is the set of $h = (h_i)_{i\in I}$ such that $h\colon \mathbb{S} \to \prod_{i\in I}\mathbf{GL}(V_i)$ factors through $G$. Hence:

$$S = \mathrm{Hom}(\mathbb{S}, G)$$

The idea is now to consider $G$ as the primordial object, and not the $V_i$ and $s_j$. An element $h$ in $\mathrm{Hom}(\mathbb{S},G)$ gives each representation $V$ of $G$ a Hodge structure, compatible with morphisms of representations and with tensor products. Conversely, each such compatible system of Hodge structures on $\mathrm{Rep}(G)$ comes from a unique morphism $h\colon \mathbb{S} \to G$, because for each $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$, the action of $z$ on the objects in $\mathrm{Rep}(G)$ is an automorphism of the forget functor from $\mathrm{Rep}(G)$ to the category of $\mathbb{R}$-vector spaces (Tannakian result). For $h$ in $\mathrm{Hom}(\mathbb{S},G)$ we have $w_h\colon \mathbb{G}_{\mathrm{m}\mathbb{R}} \to G$ and $\mu_h\colon \mathbb{G}_{\mathrm{m}\mathbb{C}} \to G_{\mathbb{C}}$.

So, let us forget about the $G$ above, and let now $G$ be a connected (I added this!) linear algebraic group over $\mathbb{R}$. As $\mathbb{S}$ is a torus, the functor $\mathbf{Hom}(\mathbb{S},G)$, sending $\mathbb{R} \to A$ to $\mathrm{Hom}(\mathbb{S}_A, G_A)$, is representable, by a smooth $\mathbb{R}$-scheme that we denote $\mathbf{Hom}(\mathbb{S},G)$, with the property that each connected component is a $G$-orbit, for $G$ acting by composition with inner automorphisms. Reference for conjugacy: SGA3, Exp.IX, Cor. 3.3. For representability and smoothness: SGA3, Exp.XI, Cor. 4.2 Let $X^+$ be a connected component of $\mathrm{Hom}(\mathbb{S},G) = \mathbf{Hom}(\mathbb{S},G)(\mathbb{R})$. Then $X^+$ is a $G(\mathbb{R})^+$-orbit. Let $G_1$ be the smallest algebraic subgroup of $G$ through which all $h$ in $X^+$ factor. Then $G_1$ is a normal subgroup of $G$, and, applying the same theorem again, $X^+$ is a $G_1(\mathbb{R})^+$-orbit.

Recall that variations of Hodge structure that come from via cohomology from geometry are polarisable, have holomorphically varying Hodge filtration and satisfy Griffiths transversality. Any $V$ in $\mathrm{Rep}(G)$ gives a family of Hodge structures on $X^+$: the fiber at $h$ is $(V,h)$. Therefore, Deligne imposes the following conditions on $X$. For any $V$ in $\mathrm{Rep}(G)$:

($\alpha$) The weight decomposition $V = \oplus_n V^n$ is independent of $h \in X^+$. (Equivalently: $w_h$ factors through $Z_G$.)

($\beta$) There is a complex structure on $X^+$, independent of $V$, such that the Hodge filtration $\mathrm{Fil}^\bullet V_{\mathbb{C}}$ varies holomorphically, and satisfies Griffiths transversality.

($\gamma$) For each $n$, there exists a $\psi^n \colon V^n \otimes V^n \to \mathbb{R}(-n)$ that is a polarisation for all $h \in X^+$.

And then he proves the following proposition, translating these conditions in terms of properties of $G$, $G_1$ and $X^+$.

**6.1 Proposition.** *Assume that ($\alpha$) is satisfied.*

1. *There is a unique complex structure on $X^+$ such that for any $V$ the Hodge filtration varies holomorphically on $X^+$.*

2. *Condition ($\beta$) is satisfied if and only if for one (equivalently, all, because these representations of $\mathbb{S}$ are all isomorphic via the $G(\mathbb{R})^+$-action) $h \in X^+$, $\mathrm{Lie}(G)$ is of type $(-1,1),(0,0),(1,-1)$. If ($\beta$) is satisfied then the complex structure on $T_{X^+}(h)$ is given by the $S^1$-action induced by $h \colon \mathbb{C}^\times \to G(\mathbb{R})_h \to \mathrm{Aut}_\mathbb{R}(T_{X^+}(h))$ factoring through $\mathbb{C}^\times \to S^1$, $z \mapsto z/\overline{z}$.*

3. *Condition ($\gamma$) is satisfied if and only if $G_1$ is reductive and for one (equivalently, all) $h \in X^+$, the involution $\mathrm{Inn}(h(i))$ is a Cartan involution of $G_1^{\mathrm{ad}}$.*

**Proof.** We start with (1). Let $V$ be a faithful representation of $G$. All representations of $G$ can be obtained from $V$ by direct sums, duals, tensor products, and subquotients, hence it suffices to prove the result for $V$. We consider the map $\phi$ that sends $h \in X^+$ to the Hodge filtration plus weight decomposition of $V_\mathbb{C}$ that it gives. By condition ($\alpha$), this is a map from $X^+$ to the homogenous $\mathrm{GL}(V_\mathbb{C}, w_h)$-space $H = \mathrm{GL}(V_\mathbb{C}, w_h)/\mathrm{GL}(V_\mathbb{C}, w_h)_{\mathrm{Fil}^\bullet_{h_0}}$, where $h_0$ is any element in $X^+$, and where $\mathrm{GL}(V_\mathbb{C}, w_h)$ denotes the subgroup of $\mathrm{GL}(V_\mathbb{C})$ that preserves the weight decomposition (which is independent of $h$). It is injective because the Hodge decomposition of $V_\mathbb{C}$ at $h$ is determined by the weight decomposition together with the Hodge filtration. We will show that $\phi$ identifies $X^+$ with a complex submanifold of $H$, which will prove (1).

Let $p \colon \mathrm{Lie}(G) \to \mathrm{End}(V, w_h)$ be the derivative of $\rho \colon G \to \mathbf{GL}(V, w_h)$. Let $G$ act on itself by inner automorphisms, and on $\mathbf{GL}(V, w_h)$ via $\rho$, followed by conjugation. Then $\rho$ is a morphism of $G$-varieties, and hence $p$ is a morphism of $G$-modules, and therefore, at each $h \in X^+$, a morphism of Hodge structures. Let $h$ be in $X^+$. Then

$$T_{X^+}(h) = \mathrm{Lie}(G)/\mathrm{Lie}(G_h) = \mathrm{Lie}(G)/\mathrm{Lie}(G)^{0,0}$$

(these are $\mathbb{R}$-vector spaces; indeed, the $(0,0)$-part makes sense and is the right thing). The tangent space of $H$ at $\phi(h)$ is

$$T_H(\phi(h)) = \mathrm{Lie}(\mathrm{GL}(V_\mathbb{C}, w_h))/\mathrm{Lie}(\mathrm{GL}(V_\mathbb{C}, w_h)_{\mathrm{Fil}^\bullet_h}) = \mathrm{End}(V_\mathbb{C}, w_h)/\mathrm{Fil}^0\mathrm{End}(V_\mathbb{C}, w_h),$$

because to stabilise $\phi(h)$ means stabilising the weight decomposition of $(V, h)$ and the Hodge filtration $\mathrm{Fil}^\bullet_h$ of $(V_\mathbb{C}, h)$, and

$$\mathrm{End}(V^n_\mathbb{C})^{i,j} = \bigoplus_{\substack{-p+p'=i \\ -q+q'=j \\ p+q=n=p'+q'}} \mathrm{Hom}(V^{p,q}, V^{p',q'})$$

hence

$$\mathrm{Fil}^0\mathrm{End}(V^n_{\mathbb{C}}) = \bigoplus_{\substack{p'\geq p\\ p+q=n\\ p'+q'=n}} \mathrm{Hom}(V^{p,q}, V^{p',q'}).$$

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Lie}(G)/\mathrm{Lie}(G)^{0,0} \arrow[r, hook] \arrow[d, "\sim"] \arrow[dr, "d\phi"] & \mathrm{End}(V, w_h)/\mathrm{End}(V, w_h)^{0,0} \arrow[d, "\sim"] \\
\mathrm{Lie}(G)_{\mathbb{C}}/\mathrm{Fil}^0\mathrm{Lie}(G)_{\mathbb{C}} \arrow[r, hook] & \mathrm{End}(V_{\mathbb{C}}, w_h)/\mathrm{Fil}^0\mathrm{End}(V_{\mathbb{C}}, w_h)
\end{tikzcd}$$

where the vertical arrows are isomorphisms of $\mathbb{R}$-vector spaces because $\mathrm{Lie}(G)$ and $\mathrm{End}(V)^0$ are Hodge structures of weight $0$: for $W$ of weight $0$ and $p > 0$ we have $(W^{-p,p} \oplus W^{p,-p})_{\mathbb{R}} \to W^{p,-p}$ an isomorphism. As a consequence, as $\phi$ is injective, the horizontal maps are injective.

This shows that, in a neighborhood of $h$, the map $\phi\colon X^+ \to H$ factors through what is called the open embedding in the compact dual $X^+ = G(\mathbb{R})^+/G(\mathbb{R})^+_h \to G(\mathbb{C})/G(\mathbb{C})_{\mathrm{Fil}^\bullet_h}$, where $G(\mathbb{C})_{\mathrm{Fil}^\bullet_h}$ is the stabiliser in $G(\mathbb{C})$ of $\mathrm{Fil}^\bullet_h$ for its action on $\mathrm{Lie}(G)_{\mathbb{C}}$. Note that $G(\mathbb{C})/G(\mathbb{C})_{\mathrm{Fil}^\bullet_h}$ is a complex projective variety; this embedding generalises $\mathbb{H} \to \mathbb{P}^1(\mathbb{C})$ in the case of $X^+ = \mathbb{H}$ and $G = \mathrm{GL}_{2,\mathbb{R}}$. In fact, from the actions by $G(\mathbb{R})$, $G(\mathbb{C})$ and $\mathrm{GL}(V, w_h)$ and $\mathrm{GL}(V_{\mathbb{C}}, w_h)$ on $\mathrm{Lie}(G) \to \mathrm{End}(V, w_h)$ and its complexification, we get a commutative diagram

$$\begin{tikzcd}
G(\mathbb{R})/G(\mathbb{R})_h \arrow[r, hook] \arrow[d, hook] \arrow[dr, "\phi"] & \mathrm{GL}(V, w_h)/\mathrm{GL}(V, w_h)_h \arrow[d, hook] \\
G(\mathbb{C})/G(\mathbb{C})_{\mathrm{Fil}^\bullet_h} \arrow[r, hook] & \mathrm{GL}(V_{\mathbb{C}}, w_h)/\mathrm{GL}(V_{\mathbb{C}}, w_h)_{\mathrm{Fil}^\bullet_h}
\end{tikzcd}$$

where the vertical maps are open immersions. This clearly shows (1).

The proof of (2) is short. Griffiths transversality means that the image of $d\phi$ is in $\mathrm{Fil}^{-1}\mathrm{End}(V_{\mathbb{C}}, w_h)/\mathrm{Fil}^0\mathrm{End}(V_{\mathbb{C}}, w_h)$, and that means that $\mathrm{Lie}(G)_{\mathbb{C}} = \mathrm{Fil}^{-1}\mathrm{Lie}(G)_{\mathbb{C}}$. Now the second claim. In the proof of (1) we have seen that $T_{X^+}(h) = \mathrm{Lie}(G)/\mathrm{Lie}(G)^{0,0}$, and that the $\mathbb{C}$-structure on $T_{X^+}(h)$ is induced by the isomorphism of $\mathbb{R}$-vector spaces

$$\mathrm{Lie}(G)/\mathrm{Lie}(G)^{0,0} \xrightarrow{\sim} \mathrm{Lie}(G)_{\mathbb{C}}/\mathrm{Fil}^0\mathrm{Lie}(G)_{\mathbb{C}}.$$

Now note that $\mathbb{S}$ acts, via $h$, on the right hand side by the character $z \mapsto z/\overline{z}$.

Then (3). Let $V$ be a faithful representation of $G$. As in the proof of (1), if for $V$ and all $n$ a $\psi^n$ as in $(\gamma)$ exists, then this gives polarisations for all $W$ in $\mathrm{Rep}(G)$. So let us assume that the $\psi^n$ are as in $(\gamma)$, and deduce the conclusion of (3) (after that we will prove the converse implication). It was already noticed that $X^+$ is a $G_1(\mathbb{R})^+$-orbit. Hence, for all $h_0 \in X^+$, $\mathrm{Inn}(h_0(i))$ is a Cartan involution of $G_1^{\mathrm{ad}}$ if and only this is so for all $h \in X^+$. So we have $\psi^n$ that are polarisations for all $h \in X^+$. Then $\psi^n\colon V^n \otimes V^n \to \mathbb{R}(-n)$, being a morphism of Hodge structures for all $h \in X^+$, is $\mathbb{S}$-invariant for all $h \in X^+$. Hence $\psi^n$ is $S^1$-invariant for all $h \in X^+$ ($S^1$ is the unit circle in $\mathbb{C}^\times$). Note that for all $h \in X^+$, $S^1$ acts trivially on $\mathbb{R}(-n)$. Let $G_2$ be the smallest $\mathbb{R}$-subgroup scheme of $G_1$ through wich all $h|_{S^1}$,

$h \in X^+$, factor. Then all $\psi^n$ are $G_2$-invariant. Let now $h \in X^+$. We show that $\tau := \mathrm{Inn}(h(i))$ is a Cartan involution of $G_2$, that is, the group

$$G_2^\tau(\mathbb{R}) = \{g \in G_2(\mathbb{C}) : h(i)\overline{g}h(i)^{-1} = g\}$$

is compact. For each $n$, the sesquilinear form

$$\psi_C^n \colon V_\mathbb{C}^n \times V_\mathbb{C}^n \longrightarrow \mathbb{C}, \quad (x, y) \mapsto \psi^n(x, h(i)\overline{y})$$

is hermitian and positive definite. For $g$ in $G_2^\tau(\mathbb{R})$, and $x$ and $y$ in $V_\mathbb{C}^n$ we have

$$\begin{aligned}\psi_C^n(gx, gy) &= \psi^n(gx, h(i)\overline{gy}) = \psi^n(gx, h(i)\overline{g}\,\overline{y}) = \psi^n(g^{-1}gx, g^{-1}h(i)\overline{g}\,\overline{y})\\ &= \psi^n(x, g^{-1}h(i)\overline{g}\,\overline{y}) = \psi^n(x, g^{-1}h(i)\overline{g}h(i)^{-1}h(i)\overline{y}) = \psi^n(x, g^{-1}gh(i)\overline{y})\\ &= \psi_C^n(x, y)\end{aligned}$$

Hence all $g$ in $G_2^\tau(\mathbb{R})$ fix all $\psi_C^n$ and therefore $G_2^\tau(\mathbb{R})$ is contained in the product of the unitary groups given by the $\psi_C^n$ and therefore $G_2^\tau(\mathbb{R})$, being closed in that product, is compact. Hence, for each $h \in X^+$, $\mathrm{Inn}(h(i))$ is a Cartan involution of $G_2$. Hence $G_2$ is reductive.

Let us now consider the difference between $G_2$ and $G_1$. For each $h \in X^+$, $w_h \colon \mathbb{G}_{\mathrm{m}\mathbb{R}} \to G$ factors through $Z_G$, and all $w_h$ are equal. Hence $G_2$ and the image of $w_h$, a central torus, generate $G_1$, and therefore $G_2^{\mathrm{ad}} \to G_1^{\mathrm{ad}}$ is an isomorphism, and $G_1$ is reductive because $G_2$ is. As $G^{\mathrm{ad}}$ is a quotient of $G_2$, all $\mathrm{Inn}(h(i))$ are Cartan involutions of $G_2^{\mathrm{ad}}$, and therefore of $G_1^{\mathrm{ad}}$.

It remains to prove the converse implication. So we assume now that $\mathrm{Inn}(h_0(i))$ is a Cartan involution of $G_1^{\mathrm{ad}}$, for some $h_0 \in X^+$, and we must prove that for each $n$, there exists a $\psi^n \colon V^n \otimes V^n \to \mathbb{R}(-n)$ that is a polarisation for all $h \in X^+$. For this, it suffices that $\psi^n$ is a polarisation for $h_0$ and that $\psi^n$ is $G_2(\mathbb{R})^+$-invariant, because then, for all $g \in G_2(\mathbb{R})^+$ and $x$ and $y$ in $V$ we have

$$\psi^n(x, gh_0(i)g^{-1}y) = \psi^n(g^{-1}x, g^{-1}gh_0(i)g^{-1}y) = \psi^n(g^{-1}x, h_0(i)g^{-1}y)$$

is symmetric and positive definite. Also, note that $G_2(\mathbb{R})^+$ acts transitively on $X^+$, as $G_2(\mathbb{R})^+ \to G_1^{\mathrm{ad}}(\mathbb{R})^+$ is surjective.

As $G_2 \to G_1^{\mathrm{ad}}$ is an isogeny, $\tau := \mathrm{Inn}(h_0(i))$ is a Cartan involution of $G_2$. Let $n$ be in $\mathbb{Z}$. Consider the $\mathbb{R}$-vector space $\Phi$ of $h_0(i)$-invariant bilinear forms $\phi \colon V^n \times V^n \to \mathbb{R}$ that are $(-1)^n$-symmetric, and the $\mathbb{R}$-vector space $\Phi_C$ of $h_0(i)$-invariant hermitian sesquilinear $H \colon V_\mathbb{C}^n \times V_\mathbb{C}^n \to \mathbb{C}$. Then $\phi \mapsto \phi_C$ with $\phi_C(x, y) = \phi(x, h_0(i)\overline{y})$ and $H \mapsto \phi_H$, with $\phi_H(x, y) = H(x, h_0(i)^{-1}\overline{y})$, are inverse maps between $\Phi$ and $\Phi_C$.

We need to see that there are $G_2$-invariant $\phi$ in $\Phi$ such that $\phi_C \colon V_\mathbb{C}^n \times V_\mathbb{C}^n \to \mathbb{C}$, $(x, y) \mapsto \phi(x, h_0(i)\overline{y})$ is positive definite. We claim that for $\phi \in \Phi$, $\phi$ is $G_2$-invariant if and only if $\phi_C$ is $G_2^\tau(\mathbb{R})$-invariant. First we prove that $\phi_C$ is $G_2^\tau(\mathbb{R})$-invariant if and only if $\phi_\mathbb{C} \colon V_\mathbb{C}^n \times V_\mathbb{C}^n \to \mathbb{C}$ is $G_2^\tau(\mathbb{R})$-invariant:

$$\text{for all } x, y \text{ in } V_\mathbb{C}^n, \quad \phi_C(gx, gy) = \phi_\mathbb{C}(gx, h(i)\overline{gy}) = \phi_\mathbb{C}(gx, h(i)\overline{g}\,\overline{y}) = \phi_\mathbb{C}(gx, g\,h(i)\overline{y}),$$

because for $g \in G_2^\tau(\mathbb{R})$ we have $g = h(i)\overline{g}h(i)^{-1}$, hence $g\,h(i) = h(i)\overline{g}$. Then, as $G_2^\tau(\mathbb{R})$ is Zariski dense in $G_2(\mathbb{C})$, $\phi_\mathbb{C}$ is $G_2^\tau$-invariant if and only if $\phi_\mathbb{C}$ is $G_2(\mathbb{C})$-invariant. Finally, $\phi$ is $G_2$-invariant if and only if it is $G_2(\mathbb{C})$-invariant, by definition of the notion of being invariant under an algebraic group. So, our bijection between $\Phi$ and $\Phi_C$ induces a bijection between $\Phi^{G_2}$ and $\Phi_C^{G_2^\tau(\mathbb{R})}$. The compactness of $G_2^\tau(\mathbb{R})$ gives us the existence of positive definite $H$ in $\Phi_C^{G_2^\tau(\mathbb{R})}$, hence of $G_2$-invariant polarisations $\phi_H$ in $\Phi$. $\square$

We have now the most important ingredients of Deligne's definition of Shimura datum at our disposal: a pair $(G, X^+)$, with $G$ a reductive linear $\mathbb{R}$-group with $X^+$ a $G(\mathbb{R})$-orbit in $\mathrm{Hom}(\mathbb{S}, G)$ that satisfies the following conditions:

1. for all $h \in X^+$, $w_h$ factors through $Z_G$,
2. for all $h \in X^+$, $\mathrm{Lie}(G)$ is of type $(-1,1), (0,0), (1,-1)$.
3. for all $h \in X^+$, $\mathrm{Inn}(h(i))$ is a Cartan involution of $G^{\mathrm{ad}}$.

Under these conditions, every $V$ in $\mathrm{Rep}(G)$ gives a variation of $\mathbb{R}$-Hodge structures over $X^+$: holomorphically varying Hodge filtration, Griffiths transversality, and polarisable. Note: we really want variations of $\mathbb{Q}$-Hodge structures, with these three properties; only the polarisability is an issue and that will be addresses after the definition of shimura datum.

## 6.2 The Siegel case

After the $\mathrm{GL}_{2,\mathbb{R}}$-case, the best known example is that of the group of symplectic similitudes. In the end, it is given by the same formula. So let $n \in \mathbb{Z}_{\geq 1}$, let $V = \mathbb{R}^{2n}$, and $\psi$ be the symplectic bilinear form on $V$ given by

$$\psi\colon V \times V \to \mathbb{R}, \quad (x,y) \mapsto x^t J y, \quad \text{with} \quad J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

Let $G = \mathrm{GSp}(\psi)$ be the linear algebraic $\mathbb{R}$-group given by

$$G(\mathbb{R}) = \{(g,c) \in (\mathrm{GL}(V), \mathbb{R}^\times) : \text{for all } x, y \in V, \psi(gx, gy) = c{\cdot}\psi(x,y)\}.$$

Then we have

$$h_0\colon \mathbb{C}^\times \to G(\mathbb{R}), \quad z = a + bi \mapsto a + bJ = \begin{pmatrix} a_n & b_n \\ -b_n & a_n \end{pmatrix}.$$

Then $h_0(\mathbb{R}^\times)$ is in the center. The characters of $\mathbb{S}$ appearing the left and right multiplication on $\mathrm{End}(V)_\mathbb{C}$ are $z$ and $\overline{z}$, hence the characters in the action on $\mathrm{End}(V)$ by conjugation are $1$, $z\overline{z}^{-1}$ and $z^{-1}\overline{z}$. Therefore, all three conditions are satisfied (of course, we just wanted to know what the Siegel case looks like, from this perspective).

Let us also show that $X^+ := G(\mathbb{R})^+{\cdot}h_0$ is identified with the Siegel upper half space $\mathbb{H}_n$, by the map that sends $h$ in $X$ to $\mathrm{Fil}^0(V_{\mathbb{C},h})$ (the embedding in the compact dual). Let $g$ be in $\mathrm{Sp}_{2n}(\mathbb{R})$, and let $h := \mathrm{Inn}(g) \circ h_0$. Then $\mathrm{Fil}^0(V_{\mathbb{C},h}) = g{\cdot}\mathrm{Fil}^0(V_{\mathbb{C},h_0})$. We compute $\mathrm{Fil}^0(V_{\mathbb{C},h_0})$. We have $V_{\mathbb{C},h_0} = V_{\mathbb{C},h_0}^{-1,0} \oplus V_{\mathbb{C},h_0}^{0,-1}$, the decomposition in the two subspaces on which $J$ acts as $i$, and as $-i$, respectively. Then

$$\mathrm{Fil}^0(V_{\mathbb{C},h_0}) = V_{\mathbb{C},h_0}^{0,-1} = \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{C}^{2n} : J\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = -i\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\} = \{\left(\begin{smallmatrix} iy \\ y \end{smallmatrix}\right) : y \in \mathbb{C}^n\}.$$

In terms of the usual description of the open part of the Grassmannian $G_n(V_{\mathbb{C}})$ of $n$-dimensional subspaces $F$ that intersect $\mathbb{C}^n \times \{0\}$ and $\{0\} \times \mathbb{C}^n$ trivially as graphs of isomorphisms, $\mathrm{Fil}^0(V_{\mathbb{C},h_0})$ is the isomorphism $(0, y) \mapsto (iy, 0)$. Indeed our $F$ have this property, because they satisfy $F \cap \overline{F} = \{0\}$. Now we apply $g$ to $\mathrm{Fil}^0(V_{\mathbb{C},h_0})$, writing $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$:

$$\mathrm{Fil}^0(V_{\mathbb{C},h}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} iy \\ y \end{smallmatrix}\right) : y \in \mathbb{C}^n\} = \left(\begin{smallmatrix} (ai+b)y \\ (ci+d)y \end{smallmatrix}\right) : y \in \mathbb{C}^n\} = \left(\begin{smallmatrix} (ai+b)(ci+d)^{-1}y \\ y \end{smallmatrix}\right) : y \in \mathbb{C}^n\}.$$

So we see that $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is sent to $\tau_g := (ai + b)(ci + d)^{-1}$ in $\mathrm{M}_n(\mathbb{C})$. A simple computation shows that $g^t J g = J$ implies $\tau_g^t = \tau_g$. Let us also show that $\Im(\tau_g)$ is positive definite. We observe that for each $g \in \mathrm{Sp}_{2n}(\mathbb{R})$, $\Im(\tau_g)$ is invertible because $\mathrm{Fil}^0(V_{\mathbb{C},h}) \cap \{0\} \times \mathbb{C}^n = \{0\}\}$, so that the signature of $\Im(\tau_g)$ is constant ($\mathrm{Sp}_{2n}(\mathbb{R})$ is connected). And at $h_0$ we have $\tau_1 = i$, hence $\Im(\tau_1) = 1$, which is positive definite.

## 6.3 Some unitary cases

Let $n_1$ and $n_2$ be in $\mathbb{Z}_{\geq 0}$ and let $n = n_1 + n_2$. Let $V := \mathbb{C}^n$ and let $\psi \colon V \times V \to \mathbb{C}$ be the skew-hermitian form given by

$$\psi(x, y) = x^t Q \overline{y}, \quad Q = \begin{pmatrix} i_{n_1} & 0 \\ 0 & -i_{n_2} \end{pmatrix}$$

Let $G$ be the linear algebraic $\mathbb{R}$-group of similitudes of $\psi$:

$$G(\mathbb{R}) = \{(g, c) \in (\mathrm{GL}_n(\mathbb{C}), \mathbb{R}^\times) : \text{for all } x, y \in V,\ \psi(gx, gy) = c{\cdot}\psi(x, y)\}$$

In other words, $G(\mathbb{R})$ is the subgroup of $g$ in $\mathrm{GL}_n(\mathbb{C})$ such that there is a $c$ in $\mathbb{R}^\times$ such that $g^t Q \overline{g} = cQ$. Then we have

$$h \colon \mathbb{C}^\times \to G(\mathbb{R}), \quad z \mapsto \begin{pmatrix} \overline{z}_{n_1} & 0 \\ 0 & z_{n_2} \end{pmatrix}.$$

Then $h(\mathbb{R}^\times)$ is in the center, and conjugation by $h(z)$ on $\mathrm{End}(V)$ is given by the characters 1, $z\overline{z}^{-1}$ and $z^{-1}\overline{z}$. Finally, twisting $(V, \psi)$ by its automorphism $h(i)$ gives the positive definite hermitian form $\psi'(x, y) = x^t \overline{y}$, hence the twist of $G$ by its inner automorphism $\mathrm{Inn}(h(i))$ gives the group of similitudes of $\psi'$, and hence $\mathrm{Inn}(h(i))$ is a Cartan involution of $G^{\mathrm{ad}}$. So the three conditions are satisfied.

Another way to think about this twist is that the complex conjugation on $G^\tau(\mathbb{C})$ is given by $g \mapsto h(i)\overline{g}h(i)^{-1}$, where $\overline{g}$ is in $G(\mathbb{C})$. Then one sees that $G^\tau(\mathbb{R})$ consists of the $g$ in $\mathrm{GL}_n(\mathbb{C})$ such that $g^t Q h(i) \overline{g} = Q h(i)$.

To finish: note that if $n_1 = 0$ or $n_2 = 0$, then $X^+$ is a point. We do not consider such cases, unless $G^{\mathrm{ad}}$ is trivial.

## 6.4 Hermitian symmetric domains

Deligne shows in his Corvallis article, Cor. 1.1.17, that for $(G, X^+)$ with $G$ a reductive $\mathbb{R}$-group and $X^+$ a $G(\mathbb{R})^+$-orbit in $\mathrm{Hom}(\mathbb{S}, G)$ satisfying conditions 1–3, $X^+$ is a hermitian symmetric domain (of non-compact type, i.e., negative curvature), and that all of them are obtained like this. For at least some details references about this notion we refer to Milne's notes. Let it suffice to say that $X^+$ is a complex manifold, that $G(\mathbb{R})^+$ acts transitively on it, via $G^{\mathrm{ad}}(\mathbb{R})$, by the way, and that for each $h$ in $X^+$ the stabiliser $G^{\mathrm{ad}}(\mathbb{R})_h$ in $G^{\mathrm{ad}}(\mathbb{R})$ is compact, because $\tau := \mathrm{Inn}(h(i))$ is a Cartan involution of $G^{\mathrm{ad}}(\mathbb{R})$. Namely, for $g$ in $G(\mathbb{R})$ we have

$$g{\cdot}h = h \Leftrightarrow \mathrm{Inn}(g) \circ h = h \Rightarrow g{\cdot}h(i){\cdot}g^{-1} = h(i) \Leftrightarrow h(i)g^{-1}h(i)^{-1} = g^{-1} \Rightarrow g^{-1} \in G^{\tau}(\mathbb{R}).$$

Hence $X^+$ has a $G(\mathbb{R})^+$-invariant hermitian metric.

The word "symmetric" has to do with the existence, for each $h$ in $X^+$, of an isometry that induces $-1$ on its tangent space (and this is what $\mathrm{Inn}(h(i))$ does, because $T_{X^+}(h) = \mathrm{Lie}(G)/\mathrm{Lie}(G)^{0,0}$). Or, it has to do with the automorphism group acting transitively (symmetric space). The word "domain" means "connected open subset of some $\mathbb{C}^n$". They are simply connected.

## 6.5 Classification

The action of $G(\mathbb{R})^+$ factors through $G^{\mathrm{ad}}$, which is a product $G_1 \times \cdots \times G_r$ of simple $\mathbb{R}$-groups. Therefore $X^+$ decomposes as a product $X_1^+ \times \cdots \times X_r^+$, with $X_i^+$ the hermitian symmetric space attached to $G_i$. The possible $X_i^+$, or, equivalently, the $G_i$, have been classified (see Deligne, Corvallis for the list): there are the unitary cases that we have seen above (Dynkin types $A_n$), orthogonal groups in even and odd dimensions ($B_n$ and $D_n$), the symplectic groups that we have seen ($C_n$), and then $E_6$ and $E_7$. In particular: $E_8$, $F_4$ and $G_2$ do not occur.

# 7 Definition of Shimura varieties

**7.1 Definition.** A *Shimura datum* is a pair $(G, X)$, with $G$ a reductive algebraic group over $\mathbb{Q}$, and $X$ a $G(\mathbb{R})$-orbit in the set of morphisms of algebraic groups $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$, such that for some $h$ in $X^+$ the following conditions hold:

**SD1** $\mathrm{Lie}(G)_{\mathbb{R}}$ is of type $\{(-1,1),(0,0),(1,-1)\}$.

**SD2** $\mathrm{Inn}(h(i))$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$: $G^{\mathrm{ad},\tau}(\mathbb{R}) = \{g \in G^{\mathrm{ad}}(\mathbb{C}) : h(i)\overline{g}h(i)^{-1} = g\}$ is compact.

**SD3** for every simple factor $H$ of $G^{\mathrm{ad}}$, the composition of $h\colon \mathbb{S} \to G_{\mathbb{R}}$ with $G_{\mathbb{R}} \to H_{\mathbb{R}}$ is not trivial.

As $G(\mathbb{R})$ acts transitively on $X$, each of these conditions are "constant" on $X$, that is, it is satisfied by all $h$ if and only if it is satisfied by some $h$, no-matter which.

The first condition implies that the weight morphism $w_h\colon \mathbb{G}_{\mathrm{m}\mathbb{R}} \to G_{\mathbb{R}}$ has image in the center $Z_{G,\mathbb{R}}$, and therefore does not depend on $h$. (Note: it is not necessarily defined over $\mathbb{Q}$.) We have seen that SD1 implies that $X$ has a unique complex structure such that every $V$ in $\mathrm{Rep}(G)$ gives a variation of $\mathbb{Q}$-Hodge structures over $X$ with holomorphically varying Hodge filtration, and satisfying Grifiths transversality.

About condition SD3. Let $G^{\mathrm{ad}}_{\mathbb{R}} \to H$ be a simple factor such that $H(\mathbb{R})$ is compact. Then $\mathrm{Inn}(h(i))$ induces the identity on $H$, hence, $\mathrm{Lie}(H)$ being of type $(-1,1),(0,0),(1,-1)$, is of type $(0,0)$, hence trivial ($H$ being adjoint), and the composition of $h$ with $G_{\mathbb{R}}^{\mathrm{ad}} \to H$ is trivial. Condition SD3 means that for all simple factors $H$ of $G^{\mathrm{ad}}$ (over $\mathbb{Q}$), $H(\mathbb{R})$ is not compact. Under this condition, the universal cover $\widetilde{H} \to H$ satisfies strong approximation; a very useful fact.

## 7.2 Existence of polarisations

Let $(G, X)$ be a Shimura datum. Deligne shows (Corvallis, 1.1.18(a) and (b)) that under the following two assumptions these are polarisable. (And I think that without SD5 one gets a counterexample with $G$ a simple torus of dimension two, such that $G(\mathbb{R})$ is isomorphic to $\mathbb{C}^\times$ (start with a cubic field)).

**SD4** The weight morphism $w_{\mathbb{R}} \colon \mathbb{G}_{\mathrm{m}\mathbb{R}} \to G_{\mathbb{R}}$, which is constant on $X^+$, is defined over $\mathbb{Q}$.

**SD5** For all (or, equivalently, for some) $h \in X$, $\mathrm{Inn}(h(i))$ induces a Cartan involution on $(G/w(\mathbb{G}_{\mathrm{m}\mathbb{Q}}))_{\mathbb{R}}$.

Under these conditions, $X(Z_G^0)_{\mathbb{Q}}$ contains the trivial representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with multiplicity at most one, depending on $w(\mathbb{G}_{\mathrm{m}\mathbb{Q}})$ being trivial or not. Let $T$ denote the subtorus of $Z_G^0$ such that $X(Z_G^0) \to X(T)$ is the quotient by the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariants in $X(Z_G^0)$. Then $w(\mathbb{G}_{\mathrm{m}\mathbb{Q}}) \times T \times G^{\mathrm{der}} \to G$ is an isogeny. Let $G_2$ be the image of $T \times G^{\mathrm{der}}$ in $G$. Then $w(\mathbb{G}_{\mathrm{m}\mathbb{Q}}) \times G_2 \to G$ is an isogeny, and for every $h \in X$, $\mathrm{Inn}(h(i))$ induces a Cartan involution of $G_{2,\mathbb{R}}$. We can use the group $G_2$ as in the last part of the proof of Proposition 6.1(3). All $h|_{S^1}$ factor through $G_{2,\mathbb{R}}$, because $(G/G_2)_{\mathbb{R}}$ is either trivial or $\mathbb{G}_{\mathrm{m}\mathbb{R}}$. Let $V$ be in $\mathrm{Rep}(G)$. Then $w \colon \mathbb{G}_{\mathrm{m}\mathbb{Q}} \to G$ decomposes $V$ as $V = \oplus_n V^n$. We may and do assume that $V = V^n$. Let $\Phi$ be the $\mathbb{Q}$-vector space of $(-1)^n$ symmetric bilinear forms $\psi \colon V \times V \to \mathbb{Q}$, and $\Phi^{G_2}$ the sub $\mathbb{Q}$-vector space of those that are $G_2$-invariant. Then, as in the proof referred to,

$$(\Phi^{G_2})_{\mathbb{R}} = (\Phi_{\mathbb{R}})^{G_{2,\mathbb{R}}} = \Phi_C^{G_2^\tau(\mathbb{R})}$$

and so we see that the $\psi$ in $(\Phi^{G_2})_{\mathbb{R}}$ such that $\psi_C$ is positive definite form a non-empty open subset, and therefore there are such $\psi$ in $\Phi^{G_2}$.

## 7.3 A few examples

We have already seen $(\mathrm{GL}_{2,\mathbb{Q}}, \mathbb{H}^\pm)$ ($\mathbb{H}^\pm$ is called the "double half space") and $(\mathrm{GSp}(\psi)_{\mathbb{Q}}, \mathbb{H}_n^\pm)$. These also satisfy SD3 and SD4. Let us give one example that does not satisfy SD4. Let $\mathbb{Q} \to F$ be a cubic totally real field, and let $B$ be a quaternion algebra over $F$ that is everywhere split except at two real places: $\mathbb{R} \otimes_{\mathbb{Q}} B$ is isomorphic to $\mathrm{M}_2(\mathbb{R}) \times \mathbb{H} \times \mathbb{H}$ (where now $\mathbb{H}$ denotes the quaternions over $\mathbb{R}$, sorry). Then one can take $G$ the group over $\mathbb{Q}$ such that for all $\mathbb{Q}$-algebras $A$ one has $G(A) = (A \otimes_{\mathbb{Q}} B)^\times$, and for $X$ the orbit of

$$h \colon \mathbb{S} \to G_{\mathbb{R}}, \quad a + bi \mapsto \left( \left( \begin{smallmatrix} a & b \\ -b & a \end{smallmatrix} \right), 1, 1 \right) \quad \text{in } \mathrm{GL}_2(\mathbb{R}) \times \mathbb{H}^\times \times \mathbb{H}^\times$$

(one chooses any isomorphism from $\mathbb{R} \otimes_{\mathbb{Q}} B$ to $\mathrm{M}_2(\mathbb{R}) \times \mathbb{H} \times \mathbb{H}$, the conjugacy class is independent of it by the Skolem-Noether theorem). The weight morphism $w \colon \mathbb{G}_{\mathrm{m}\mathbb{R}} \to G_{\mathbb{R}}$ is then given by $a \mapsto (a^{-1}, 1, 1)$, $a \in \mathbb{R}^\times$, and $(a, 1, 1)$ in $(\mathbb{R} \otimes F)^\times$. This is not defined over $\mathbb{Q}$ because $F$ is a field, and not $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$. The field of definition of $w$ is $F \to \mathbb{R}$ via the place where $B$ splits.

**7.4 Exercise.** Show that there is no Shimura datum $(G, X)$ with $G = \mathrm{SL}_{2,\mathbb{Q}}$. (But there is a so-called connected Shimura datum; see Milne's notes.)

## 7.5 Morphisms of Shimura data

Let $(G_1, X_1)$ and $(G_2, X_2)$ be Shimura data. A morphism from $(G_1, X_1)$ to $(G_2, X_2)$ is a morphism $f \colon G_1 \to G_2$ such that for each $h$ in $X_1$, the composition $f \circ h \colon \mathbb{S} \to G_{2,\mathbb{R}}$ is in $X_2$.

## 7.6 The adjoint Shimura datum

Let $(G, X)$ be a Shimura datum. Let $X^{\mathrm{ad}}$ be the $G^{\mathrm{ad}}(\mathbb{R})$-orbit in $\mathrm{Hom}(\mathbb{S}, G^{\mathrm{ad}}_{\mathbb{R}})$ that contains the image of $X$ in $\mathrm{Hom}(\mathbb{S}, G^{\mathrm{ad}}_{\mathbb{R}})$. Then $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is a Shimura datum, and the quotient morphism $G \to G^{\mathrm{ad}}$ is a morphism of Shimura data $(G, X) \to (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. The map $X \to X^{\mathrm{ad}}$ is a closed and open immersion ($G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})$ need not be surjective, one gets examples from Hilbert modular varieties, that is, $\mathrm{Res}_{F/\mathbb{Q}}G$, with $G(\mathbb{Q}) = \{g \in \mathrm{GL}_2(F) : \det(g) \in \mathbb{Q}^\times\}$). This is a useful construction, for example because $G^{\mathrm{ad}}$ decomposes as a product, over $\mathbb{Q}$, and even more over $\mathbb{R}$. On the other hand, if $(G, X)$ has a moduli interpretation, it may become harder to understand it for $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$. Note that $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ satisfies SD4 and SD5. In the context of the André-Oort conjecture, or Zilber-Pink, the difference between $(G, X)$ and $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is irrelevant (but passing to subvarieties does not preserve this propery).

## 7.7 Intermezzo on adèles

For $p$ a prime number, let $\mathbb{Z}_p = \lim_n(\mathbb{Z}/p^n\mathbb{Z})$, and $\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$. Together with $\mathbb{R}$, this gives all completions of $\mathbb{Q}$ (locally compact fields, non-discrete). We define $\mathbb{A}_f$, the topological $\mathbb{Q}$-algebra of the *finite adèles* of $\mathbb{Q}$, be the *restricted product*

$$\mathbb{A}_f = {\prod_p}' \mathbb{Q}_p = \{(x_p)_p : \text{for almost all } p,\, x_p \in \mathbb{Z}_p\}$$

with the following topology: $\hat{\mathbb{Z}} = \lim_n(\mathbb{Z}/n\mathbb{Z}) = \prod_p \mathbb{Z}_p \subset \mathbb{A}_f$, and all its translations, are open, and carry the product topology, hence are compact. We have $\mathbb{A}_f = \mathbb{Q} \otimes \hat{\mathbb{Z}}$, it is sometimes written $\hat{\mathbb{Q}}$. Then the complete $\mathbb{Q}$-algebra of adèles of $\mathbb{Q}$ is defined as:

$$\mathbb{A} = \mathbb{A}_f \times \mathbb{R}, \quad \text{with the product topology.}$$

The image of $\mathbb{Q}$ is discrete: in $\hat{\mathbb{Q}}$ we have $\mathbb{Q} \cap \hat{\mathbb{Z}} = \mathbb{Z}$, hence in $\mathbb{A}$ we have $\mathbb{Q} \cap \hat{\mathbb{Z}} \times (-1, 1) = \{0\}$. And it is co-compact: $\hat{\mathbb{Z}} \times [0, 1)$ is a fundamental domain, $\hat{\mathbb{Z}} \times [0, 1]$ surjects to $\mathbb{A}/\mathbb{Q}$.

**7.7.1 Exercise.** $\mathbb{A}/\mathbb{Q}$ is the profinite universal cover of $\mathbb{R}/\mathbb{Z}$, called solenoid:

$$\mathbb{A}/\mathbb{Q} = (\hat{\mathbb{Z}} \times \mathbb{R})/\mathbb{Z} = \lim_n (\mathbb{Z}/n\mathbb{Z} \times \mathbb{R})/\mathbb{Z} = \lim_n \mathbb{R}/n\mathbb{Z}.$$

**7.7.2 Exercise.** Prove that $\mathrm{End}(\mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}}$, and that $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Q}}$.

For $F$ a finite dimensional $\mathbb{Q}$-algebra we define $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F$, etc.

For $X$ a closed subscheme of $\mathbb{A}^n_{\mathbb{Q}}$, we give $X(\mathbb{A}) \subset \mathbb{A}^n$ the induced topology. And similarly for any topological $\mathbb{Q}$-algebra. For example, $\mathbb{A}^\times = \mathbb{G}_{\mathrm{m}}(\mathbb{A})$ gets the topology from its embedding into $\mathbb{A}^2$ as the subset of $(x, y)$ with $xy = 1$.

In this way, for $G$ a linear algebraic group over $\mathbb{Q}$, $G(\mathbb{A})$ becomes a locally compact topological group (equivalently: locally profinite). For example, $G(\mathbb{A}_f)$ is locally compact and totally disconnected, it has a basis of $1$ consisting of compact open subgroups: embed $G$ into $\mathrm{GL}_{n,\mathbb{Q}}$, use the kernels of $\mathrm{GL}_n(\hat{\mathbb{Z}}) \to \mathrm{GL}_n(\mathbb{Z}/n\mathbb{Z})$.

If $K_1$ and $K_2$ are compact open subgroups of $G(\mathbb{A}_f)$, then $K_1 \cap K_2$ is a compact open subgroup, of finite index in both $K_1$ and $K_2$.

For $K \subset \mathbb{A}_f$ a compact open subgroup, let, for $p$ prime, $K_p$ be its image under projection to $\mathbb{Q}_p$, in which it is compact open ($\mathbb{A}_f = \mathbb{Q}^p \times \mathbb{A}_f^p$, product topology). Then (exercise), for almost all $p$ we have $K_p = \mathbb{Z}_p$, and $K = \prod_p K_p$.

Let now $G$ be a linear algebraic group over $\mathbb{Q}$, and $K$ a compact open subgroup of $G(\mathbb{A}_f)$, with projections $K_p \subset G(\mathbb{Q}_p)$. Then (exercise) $\prod_p K_p$ is contained in $G(\mathbb{A}_f)$ and is compact open ($K$ contains a $K'$ obtained by intersecting $G(\mathbb{A}_f)$ with a suitable compact open $\ker(\mathrm{GL}_n(\hat{\mathbb{Z}}) \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z}))$, and $K' = \prod_p K'_p$).

Be careful with notation as $G(\hat{\mathbb{Z}})$ or $G(\mathbb{Z}_p)$, these do not make sense for $G$ over $\mathbb{Q}$, one has to specify a model over $\mathbb{Z}$ or so first.

## 7.8 Congruence subgroups, arithmetic subgroups

Let $G$ be a linear algebraic group over $\mathbb{Q}$. The subgroups obtained as $G(\mathbb{Q}) \cap K$, with $K$ compact open in $G(\mathbb{A}_f)$ are called *congruence subgroups*), and a subgroup of $G(\mathbb{Q})$ is called *arithmetic* if it is commensurable with a congruence subgroup: $\Gamma_1$ and $\Gamma_2$ are commensurable if $\Gamma_1 \cap \Gamma_2$ has finite index in $\Gamma_1$ and in $\Gamma_2$.

Let $K \subset G(\mathbb{A}_f)$ be compact open, and $\Gamma := G(\mathbb{Q}) \cap K$. As $G(\mathbb{Q})$ is discrete in $G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$, $G(\mathbb{Q}) \cap (G(\mathbb{R}) \times K)$ is discrete, and as $K$ is compact, $\Gamma$ is discrete in $G(\mathbb{R})$: the projection

$$G(\mathbb{R}) \times K \to G(\mathbb{R})$$

is closed, hence every subset of the image of $G(\mathbb{Q}) \cap (G(\mathbb{R}) \times K)$ is closed. Another way to see this is to embed $G$ in some $\mathrm{GL}_n$ and to compare with $\mathrm{GL}_n(\mathbb{Z})$ in $\mathrm{GL}_n(\mathbb{R})$.

One has to be careful about the distinction between arithmetic subgroups and congruence subgroups. The subgroup $\Gamma(2)$ of $\mathrm{SL}_2(\mathbb{Z})$ has the property that its quotient by $\{\pm 1\}$ is $\pi_1(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\})$, and so, by Belyi's theorem, all curves over $\overline{\mathbb{Q}}$ are quotients of $\mathbb{H}$ by arithmetic subgroups of $\mathrm{SL}_2(\mathbb{Q})$. Whereas modular curves, quotients by congruence subgroups, are defined over cyclotomic fields. On the other hand, in many algebraic groups all arithmetic subgroups are congruence subgroups by Bass, Milnor and Serre (search for "Congruence subgroup problem"). A second way in which $\mathrm{SL}_2$ is not typical is that

$\mathrm{SL}_2(\mathbb{R})$ has many discrete subgoups of finite covolume, for example the fundamental groups of compact Riemann surfaces with $\mathbb{H}$ as universal cover. It is a theorem of Margulis that for $H$ a simple real Lie group not isogenous to $\mathrm{SO}(1,n)$ of $\mathrm{SU}(1,n)$, every discrete subgroup of finite covolume is conjugate to an arithmetic subgroup (note that $\mathrm{SL}_2(\mathbb{R})$ is isogeneous to $\mathrm{SO}(1,2)$).

## 7.9 Finiteness results

It is a theorem of Borel that for any linear algebraic group $G$ over $\mathbb{Q}$, and for every compact open subgroup $K$ of $G(\mathbb{A}_f)$, the set $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ is finite (Theorem 5.1 in 'Some finiteness properties in adele groups over number fields', IHES PM 16). Of course, the set $G(\mathbb{A}_f)/K$ is discrete because $K$ is open. This result can be seen as a generalisation of the finiteness of class numbers of number fields:

$$\mathrm{Pic}(O_F) = \frac{\text{divisors on } \mathrm{Spec}(O_F)}{\text{principal divisors}} = F^\times\backslash\mathbb{A}_{F,f}^\times/\hat{O}_F^\times.$$

To get some more feeling for such quotients, let us consider the case of $\mathrm{GL}_{n,F}$ with $F$ a number field. Let $O = O_F$. Instead of working over $F$, we work over $\mathbb{Q}$, or rather over $\mathbb{Z}$, with the group $G = \mathrm{Res}_{O/\mathbb{Z}}\mathrm{GL}_n$. Let us show that

$$\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_{F,f})/\mathrm{GL}_n(\hat{O}) = \frac{\{\text{locally free } O\text{-modules of rank } n\}}{\cong}.$$

Let $L$ be a locally free $O$-module of rank $n$. Then $L_F := F \otimes_O L$ is free of rank $n$. We choose an isomorphism

$$\phi\colon F^n \to L_F, \quad \text{unique up to } \phi' = \phi g, \text{ with } g \in \mathrm{GL}_n(F).$$

For each maximal ideal $m$ of $O$, let $O_m = \lim O/m^r$ be the completion of $O$ at $m$, and $F_m$ the completion of $F$ at $m$. Then for each $m$, $L_m$ is free of rank $n$ as $O_m$-module, and we choose an isomorphism

$$\phi_m\colon O_m^n \to L_m, \quad \text{unique up to } \phi_m' = \phi_m g_m, \text{ with } g_m \in \mathrm{GL}_n(O_m).$$

Then, for each $m$, we have $\phi^{-1}\phi_m$ in $\mathrm{GL}_n(F_m)$, and we have a well-defined element

$$[(\phi^{-1}\phi_m)_m] \in \mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_{F,f})/\mathrm{GL}_n(\hat{O}).$$

And vice versa, if we start with an element $g$ in $\mathrm{GL}_n(\mathbb{A}_{F,f})$, we have the $O$-module

$$(g\hat{O}^n) \cap F^n \subset F^n,$$

whose isomorphism class depends only on $\overline{g}$ in $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_{F,f})/\mathrm{GL}_n(\hat{O})$.

To prove that these constructions are inverses of each other (we leave that to the audience), it seems most convenient to use the bijection between "lattices" in $\mathbb{Q}^n$ and in $\mathbb{A}_f^n$.

We define a $\mathbb{Z}$-lattice in $\mathbb{Q}^n$ to be a sub $\mathbb{Z}$-module $L$ of $\mathbb{Q}^n$ that is finitely generated and satisfies $\mathbb{Q}{\cdot}L = \mathbb{Q}^n$. Such $L$ are automatically free of rank $n$. The group $\mathrm{GL}_n(\mathbb{Q})$ acts on the set of such $L$ by $(g, L) \mapsto gL$. This action is transitive and the stabiliser of $\mathbb{Z}^n$ is $\mathrm{GL}_n(\mathbb{Z})$. Hence

$$\mathrm{GL}_n(\mathbb{Q})/\mathrm{GL}_n(\mathbb{Z}) = \mathbb{Z}\text{-lattices in } \mathbb{Q}^n.$$

We define a $\hat{\mathbb{Z}}$-lattice in $\mathbb{A}_f^n$ to be a compact open subgroup. Equivalently: finitely generated sub $\hat{\mathbb{Z}}$-modules $M$ with $\mathbb{A}_f{\cdot}M = \mathbb{A}_f^n$, or also: free sub $\hat{\mathbb{Z}}$-modules $M \subset \mathbb{A}_f^n$ of rank $n$. The group $\mathrm{GL}_n(\mathbb{A}_f)$ acts on the set of $M$, transitively, and with stabiliser $\mathrm{GL}_n(\hat{\mathbb{Z}})$ at $\hat{\mathbb{Z}}^n$. Therefore,

$$\mathrm{GL}_n(\mathbb{A}_f)/\mathrm{GL}_n(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}\text{-lattices in } \mathbb{A}_f^n.$$

We have two maps:

$$\{\mathbb{Z}\text{-lattices in } \mathbb{Q}^n\} \longleftrightarrow \{\hat{\mathbb{Z}}\text{-lattices in } \mathbb{A}_f^n\} \qquad L \longmapsto \hat{\mathbb{Z}}{\cdot}L \qquad M\cap\mathbb{Q}^n \longleftarrow\!\!\shortmid M$$

These maps are inverses. To see it, start with an $L$. Take a $\mathbb{Z}$-basis for $L$, take that basis as $\mathbb{Q}$-basis for $\mathbb{Q}^n$, and use that $\hat{\mathbb{Z}}\cap\mathbb{Q} = \mathbb{Z}$ to conclude that $(\hat{\mathbb{Z}}{\cdot}L)\cap\mathbb{Q}^n = L$. For every $m \in \mathbb{Z}_{\geq 1}$ we have on each side the subsets of lattices between "at most $m$ away from the standard lattice", that is:

$$\{L : m\mathbb{Z}^n \subset L \subset m^{-1}\mathbb{Z}^n\} \quad \text{and} \quad \{M : m\hat{\mathbb{Z}}^n \subset M \subset m^{-1}\hat{\mathbb{Z}}^n\}.$$

Both sets are preserved by our two maps, and are the same as the finite set of all subgroups of $(\mathbb{Z}/m^2\mathbb{Z})^n$. That finishes the argument that the two maps are inverses of each other (details left to audience).

There are the following stronger general results, by Borel. The quotient $G(\mathbb{Q})\backslash G(\mathbb{A})$ is of finite volume if and only if $\mathrm{Hom}(G^0, \mathbb{G}_{\mathrm{m}\mathbb{Q}})$ is trivial. And $G(\mathbb{Q})\backslash G(\mathbb{A})$ is compact if and only if $\mathrm{Hom}(G^0, \mathbb{G}_{\mathrm{m}\mathbb{Q}})$ is trivial and every unipotent element in $G(\mathbb{Q})$ is in the unipotent radical of $G$. Even more general for connected $G$: $G(\mathbb{A})_1 := \cap_\chi \ker(\|\cdot\| \circ \chi\colon G(\mathbb{A}) \to \mathbb{R})$ (with $\chi$ ranging over $\mathrm{Hom}(G, \mathbb{G}_{\mathrm{m}\mathbb{Q}})$) is unimodular, contains $G(\mathbb{Q})$, and $G(\mathbb{Q})\backslash G(\mathbb{A})_1$ is of finite volume, and is compact if and only if every unipotent element in $G(\mathbb{Q})$ is in the unipotent radical of $G$.

## 7.10 Neat subgroups

References: Milne's notes, §3, and Pink's thesis (on his webpage), §0.6 for the adelic case. And for more details, Borel's "Introduction aux groupes arithmétiques." (I have not seen this book! Pink refers to it.)

Let $G$ be a linear algebraic group over $\mathbb{Q}$. An element $g$ of $G(\mathbb{Q})$ is called *neat* if the subgroup of $\overline{\mathbb{Q}}^\times$ generated by the eigenvalues of $g$ in some faithful $V$ in $\mathrm{Rep}(G)$ is free (that is, no nontrivial elements of finite order). This is independent of $V$, as all $W$ in $\mathrm{Rep}(G)$ are obtained from $V$ via sums, tensor products, duals and subquotients, hence the group in question *is* the set of eigenvalues that occur in the $W$.

The importance of this notion is that if $g$ is neat and of finite order, then it is the identity element: its eigenvalues are all equal to 1, therefore $g$ is unipotent, say $1 + a$ in some faithful representation, with minimal polynomial $(x-1)^n$, then $n = 1$ (the field is $\mathbb{Q}$).

It is easy to guarantee elements to be neat by imposing congruences. Let $p$ be prime, and $K \subset \mathrm{GL}_n(\mathbb{Q}_p)$ the kernel of $\mathrm{GL}_n(\mathbb{Z}_p) \to \mathrm{GL}_n(\mathbb{Z}/p^r\mathbb{Z})$, with $r \geq 1$, and $r \geq 2$ if $p = 2$. Then for $g$ in $K$ we have $g = 1 + p^r a$, with $a \in \mathrm{M}_n(\mathbb{Z}_p)$. The eigenvalues $a$ (in $\overline{\mathbb{Q}}_p$, to which we extend the valuation $v\colon \mathbb{Q}_p^\times \to \mathbb{Z}$, keeping $v(p) = 1$) are satisfy $v_p(\lambda) \geq 0$. Hence the eigenvalues $\mu$ of $g$, being of the form $1+p^r\lambda$, satisfy $v(1-\mu) \geq r$. The only element of finite order in $\overline{\mathbb{Q}}_p^\times$ with this property is 1. For example, for $\zeta \neq 1$ of finite order prime to $p$, $1-\zeta$ is a unit, and for $\zeta$ of order $p^e$, $v(1-\zeta) = 1/((p-1)p^{e-1})$ (the ramification index, and the ramification is total).

We conclude: for $G$ a linear algebraic group over $\mathbb{Q}$, and for $K$ a compact open subgroup of $G(\mathbb{A}_f)$, there is a compact open subgroup $K'$ of $K$ such that $G(\mathbb{Q}) \cap K'$ is neat. Namely: take $K'$ of the form $\prod_p K'_p$, such that at least one factor $K'_p$ is obtained as intersection with $\ker(\mathrm{GL}_n(\mathbb{Z}_p) \to \mathrm{GL}_n(\mathbb{Z}/p^r\mathbb{Z}))$ under some embedding of $G_{\mathbb{Q}_p}$ into $\mathrm{GL}_{n,\mathbb{Q}_p}$.

**7.11 Definition. (Shimura variety, finally!)** Let $(G, X)$ be a Shimura datum (that is, satisfying SD1–3), and let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. Then the *Shimura variety attached to* $(G, X)$ *and* $K$ is

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) := G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K),$$

with $G(\mathbb{Q})$ and $K$ acting as follows:

$$q{\cdot}(x, g) = (qx, qg), \quad (x, g){\cdot}k = (x, gk).$$

We recall that $X = G(\mathbb{R})/G(\mathbb{R})_{h_0}$ is a finite union of hermitian symmetric domains, on which $G(\mathbb{R})$ acts via $G^{\mathrm{ad}}(\mathbb{R})$. Considering the $G(\mathbb{Q})$-action on $G(\mathbb{A}_f)/K$ we see:

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K) = \coprod_{g \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K} \Gamma_g\backslash X, \quad \Gamma_g = G(\mathbb{Q}) \cap gKg^{-1}.$$

In words: the Shimura variety is a finite union of quotients of finite unions of hermitian symmetric domains by congruence subgroups.

We want to see that $\mathrm{Sh}_K(G, X)(\mathbb{C})$ is a complex analytic variety. This means that the action of the $\Gamma_g$ on $X$ should be sufficiently nice. Let $K' \subset K$ be a normal neat open subgroup of $K$ (these are easy to get). Then $K/K'$ is a finite group, and

$$\mathrm{Sh}_{K'}(G, X)(\mathbb{C}) \longrightarrow \mathrm{Sh}_K(G, X)(\mathbb{C}) \quad \text{is the quotient for the } K/K'\text{-action}$$

So we may assume that $K$ is neat. Let $\overline{\Gamma_g}$ be the image of $\Gamma_g$ in $G^{\mathrm{ad}}(\mathbb{R})$. Then $\overline{\Gamma_g}$ acts faithfully on $X$ by SD3 (if $G^{\mathrm{ad}} = \prod_i G_i$ and $G_i = \prod_{j \in J_i} G_{i,j}$, then the kernel of the action of $G^{\mathrm{ad}}(\mathbb{R})$ on $X^{\mathrm{ad}}$ is the product of the $G_{i,j}(\mathbb{R})$ that are compact). Let $\gamma \in \Gamma_g$, and $x \in X$ with $\gamma{\cdot}x = x$. Then $\overline{\gamma}$ is in $(\overline{\Gamma}_g)_x$ which is discrete in the compact group $G^{\mathrm{ad}}(\mathbb{R})_x$ (it is compact because of SD2, or, if you like, because the product of the non-compact $G_{i,j}(\mathbb{R})_x$ acts faithfully on $T_X(x)$, preserving the hermitian inner product), hence of finite order. As $\gamma \in G(\mathbb{Q})$ is neat, $\overline{\gamma} \in G^{\mathrm{ad}}(\mathbb{Q})$ is neat as well, and therefore trivial. So we have:

$$\overline{\Gamma_g} \text{ acts freely on } X \text{ (assuming } K \text{ is neat).}$$

Now we show that $\mathrm{Sh}_{K'}(X, G)(\mathbb{C})$ is a complex manifold. Let $x$ be in $X$. We get a fundamental domain $F$ containing $x$ for the $\overline{\Gamma}_g$-action

$$F := \{y \in X : \text{for all } \gamma \in \overline{\Gamma_g} - \{1\},\, d(y, x) < d(y, \gamma{\cdot}x)\}$$

Then $F \to \overline{\Gamma_g}\backslash X$ is a chart at the image of $x$. Hence:

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) \text{ is a complex analytic variety, non-singular if } K \text{ is neat.}$$

## 7.12 The set of connected components

It will take us some time to understand the set of connected components of a Shimura variety, but it is important because we need it for the Galois action on special points, when we take $G$ a torus; then the Shimura variety is zero-dimensional, and it *is* its $\pi_0$.

Let $(G, X)$ be a Shimura datum, and $K \subset G(\mathbb{A}_f)$ an compact open subgroup. Let $X^+$ be a connected component of $X$. The stabiliser of $X^+$, as an element of $\pi_0(X)$, say, in $G^{\mathrm{ad}}(\mathbb{R})$ is exactly $G^{\mathrm{ad}}(\mathbb{R})^+$. This is not a triviality. To see it, write $G^{\mathrm{ad}}_{\mathbb{R}} = \prod_i G_i$ with $G_i$ simple, use ((1.2.7) of Deligne, Corvallis) for the $i$ with $G_i(\mathbb{R})$ not compact, and use that for $H$ a linear algebraic group over $\mathbb{R}$ with $H(\mathbb{R})$ compact, $H(\mathbb{R})$ is connected (see notes above on algebraic groups).

We can also see it like this. Let $h \in X^+$. Then $G^{\mathrm{ad}}_{\mathbb{R},h}$ is the centraliser of a torus, hence it is connected (Springer, 6.4.7); $G^{\mathrm{ad}}_{\mathbb{R},h}(\mathbb{R})$ is compact by SD2, hence connected; and then $G^{\mathrm{ad}}(\mathbb{R})_{X^+}$ is connected because $G^{\mathrm{ad}}(\mathbb{R})^+$ acts transitively on $X^+$ and $G^{\mathrm{ad}}(\mathbb{R})_h$ is connected.

We define

$$G(\mathbb{R})_+ := q^{-1}(G^{\mathrm{ad}}(\mathbb{R})^+), \quad \text{where } q\colon G \to G^{\mathrm{ad}}.$$

Then $G(\mathbb{R})_+$ is the stabiliser in $G(\mathbb{R})$ of $X^+ \in \pi_0(X)$. In other words: $\pi_0(X)$ is a $G(\mathbb{R})/G(\mathbb{R})_+$-torsor. We define

$$G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+.$$

Then, because $G(\mathbb{Q}) \subset G(\mathbb{R})$ is dense (this is called *real approximation*, valid for all connected linear algebraic $\mathbb{Q}$-groups; Platonov-Rapinchuk, Theorem 7.7, p.415, see also Milne's notes) $G(\mathbb{Q})$ acts transitively on $\pi_0(X)$. Hence:

$$\pi_0(\mathrm{Sh}_K(G, X)(\mathbb{C})) = G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K.$$

It is not obvious that this finite set is in fact a commutative group! As $K$ is open, all $G(\mathbb{Q})_+ \times K$-orbits in $G(\mathbb{A}_f)$ are open, hence, being the complement of the union of the other orbits, closed. Hence, with $\overline{G(\mathbb{Q})_+}$ the closure of $G(\mathbb{Q})_+$ in $G(\mathbb{A}_f)$:

$$G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K = \overline{G(\mathbb{Q})_+} \backslash G(\mathbb{A}_f)/K.$$

Let $\rho\colon \widetilde{G} \to G^{\mathrm{der}}$ be the universal cover. Then $\widetilde{G}(\mathbb{R})$ is connected (see notes above), hence

$$\rho(\widetilde{G}(\mathbb{Q})) \subset G(\mathbb{Q})^+ \subset G(\mathbb{Q})_+.$$

Now we use strong approximation in $\widetilde{G}$; note that SD3 say that $G^{\mathrm{ad}}$ has no simple factor $G_i$ with $G_i(\mathbb{R})$ compact, so neither has $\widetilde{G}$, and therefore the strong approximation theorem applies (see Platonov-Rapinchuk, Thm 7.12, page 427):

**7.13 Theorem. (Strong approximation)** *Let $G$ be a reductive algebraic group over a number field $F$, and $S$ a finite set of places of $F$. Then $G(F)$ is dense in $G(\mathbb{A}_F^S) = \prod'_{w \notin S} G(F_w)$ if and only if 1: $G$ is semi-simple and simply connected and 2: $G$ has no simple factor $G_i$ with $G_i(\prod_{v \in S} F_v)$ compact.*

Hence:

$$\overline{\widetilde{G}(\mathbb{Q})} = \widetilde{G}(\mathbb{A}_f), \quad \text{and} \quad \rho(\widetilde{G}(\mathbb{A}_f)) \subset \overline{G(\mathbb{Q})_+} \subset G(\mathbb{A}_f).$$

As $\rho\colon \widetilde{G} \to G$ is proper, $\rho\colon \widetilde{G}(\mathbb{A}_f) \to G(\mathbb{A}_f)$ is proper, hence $\rho(\widetilde{G}(\mathbb{A}_f))$ is closed in $G(\mathbb{A}_f)$, and therefore $\rho(\widetilde{G}(\mathbb{A}_f))\backslash G(\mathbb{A}_f)$ has a decent topology (locally profinite). So, $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$ is a quotient of $\rho(\widetilde{G}(\mathbb{A}_f))\backslash G(\mathbb{A}_f)$. Deligne shows that this is a commutative group. At least $\rho(\widetilde{G}(\mathbb{A}_f))$ is a subgroup of $G(\mathbb{A}_f)$. It is normal because the action of $\widetilde{G}$ on itself by conjugation factors through $G^{\mathrm{ad}}$:

$$\widetilde{G} \to G^{\mathrm{ad}} \to \mathbf{Aut}(\widetilde{G}), \quad \text{hence} \quad G \to G^{\mathrm{ad}} \to \mathbf{Aut}(\widetilde{G}).$$

Next we consider commutator maps (morphisms of varieties over $\mathbb{Q}$):

$$[\cdot,\cdot]\colon G \times G \to G, \quad \widetilde{G} \times \widetilde{G} \to \widetilde{G}, \quad G^{\mathrm{ad}} \times G^{\mathrm{ad}} \to G^{\mathrm{ad}}.$$

All these factor through $G^{\mathrm{ad}} \times G^{\mathrm{ad}}$, hence we get:

$$[\cdot,\cdot]\colon G \times G \to G^{\mathrm{ad}} \times G^{\mathrm{ad}} \to \widetilde{G} \to G$$

which shows that indeed that $\rho(\widetilde{G}(\mathbb{A}_f))$ contains all commutators in $G(\mathbb{A}_f)$. We have:

$$\begin{aligned}
\pi_0(\mathrm{Sh}_K(G,X)(\mathbb{C})) &= \overline{G(\mathbb{Q})_+}\backslash G(\mathbb{A}_f)/K = \overline{G(\mathbb{Q})_+}\rho(\widetilde{G}(\mathbb{A}_f))\backslash G(\mathbb{A}_f)/K \\
&= G(\mathbb{A}_f)/\rho(\widetilde{G}(\mathbb{A}_f))G(\mathbb{Q}_+)K \\
&= G(\mathbb{A}_f)/G(\mathbb{Q})_+K \quad (G(\mathbb{Q})_+K \text{ is open normal, with abelian quotient})
\end{aligned}$$

Similarly (well, see pages 262–264 in Deligne, Corvallis, especially Cor. 2.0.8), $G(\mathbb{Q})\rho(\widetilde{G}(\mathbb{A}))$ is a closed normal subgroup of $G(\mathbb{A})$ with abelian quotient. With

$$\pi(G) := G(\mathbb{A})/G(\mathbb{Q})\rho(\widetilde{G}(\mathbb{A})), \quad \overline{\pi}_0\pi(G) := \pi_0\pi(G)/\pi_0(G(\mathbb{R})_+),$$

we have

$$\begin{aligned}
\pi_0(\mathrm{Sh}_K(G,X)(\mathbb{C})) &= G(\mathbb{A})/\rho\widetilde{G}(\mathbb{A})G(\mathbb{Q})G(\mathbb{R})_+ \times K \\
&= \pi(G)/G(\mathbb{R})_+ \times K = \overline{\pi}_0\pi(G)/K.
\end{aligned}$$

Let us finish with an example. We take $(G,X) = (\mathrm{GL}_2, \mathbb{H}^{\pm})$. Then for $K$ compact open in $\mathrm{GL}_2(\mathbb{A}_f)$ we have $\det(K)$ open compact in $\hat{\mathbb{Z}}^\times \subset \mathbb{A}_f^\times$,

$$\mathrm{SL}_2(\mathbb{Q}){\cdot}K = \mathrm{SL}_2(\mathbb{A}_f), \quad \text{and } \det\colon \mathrm{GL}_2 \to \mathbb{G}_{\mathrm{m}} \text{ is split, with kernel } \mathrm{SL}_2$$

hence

$$\begin{aligned}
\mathrm{Sh}_K(G,X)(\mathbb{C}) &= \mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)/K) = \mathrm{GL}_2(\mathbb{Q})^+\backslash(\mathbb{H} \times \mathrm{GL}_2(\mathbb{A}_f)/K) \\
\pi_0(\mathrm{Sh}_K(G,X)(\mathbb{C})) &= \mathbb{Q}_{>0}^\times\backslash\mathbb{A}_f^\times/\det(K) = \hat{\mathbb{Z}}^\times/\det(K) \\
&= \mathbb{Q}^\times\backslash\mathbb{A}^\times/(\mathbb{R}^\times \times \det(K)) = \pi_0(\mathbb{Q}^\times\backslash\mathbb{A}^\times/\det(K)).
\end{aligned}$$

Let us compare with the general result. We have $\pi_0(\mathrm{GL}_2(\mathbb{R})_+) = \{1\}$, and

$$\pi(\mathrm{GL}_2) = \mathrm{GL}_2(\mathbb{A})/\mathrm{SL}_2(\mathbb{A})\mathrm{GL}_2(\mathbb{Q}) = \mathbb{A}^\times/\mathbb{Q}^\times, \quad \pi_0\pi(\mathrm{GL}_2) = \mathbb{A}_f^\times/\mathbb{Q}_{>0}^\times = \overline{\pi}_0\pi(\mathrm{GL}_2).$$

Let us also give the result for the adjoint Shimura datum $(\mathrm{PGL}_2, \mathbb{H}^\pm)$. Then $G^{\mathrm{der}} = \mathrm{PGL}_2$, and $\widetilde{G} = \mathrm{SL}_2$:

$$1 \to \mu_2 \to \mathrm{SL}_2 \to \mathrm{PGL}_2 \to 1.$$

Then, for each prime $p$, we have

$$1 \to \{\pm 1\} \to \mathrm{SL}_2(\mathbb{Q}_p) \to \mathrm{PGL}_2(\mathbb{Q}_p) \to \mathrm{H}^1(\mathbb{Q}_p, \mu_2) \to \mathrm{H}^1(\mathbb{Q}_p, \mathrm{SL}_2) = 1$$

(Indeed, $\mathrm{SL}_2$ is the automorphism groupscheme of $\mathbb{Z}^2$ plus a trivialisation of $\wedge^2(\mathbb{Z}^2)$, and this has no non-trivial twists over a field.) The Kummer sequence gives $\mathrm{H}^1(\mathbb{Q}_p, \mu_2) = \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$. Similarly, over $\mathbb{R}$:

$$1 \to \{\pm 1\} \to \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PGL}_2(\mathbb{R}) \to \mathbb{R}^\times/\mathbb{R}^\times_{>0} \to 1$$

We have:

$$\begin{aligned}\pi(\mathrm{PGL}_2) &= \mathrm{PGL}_2(\mathbb{A})/\rho(\mathrm{SL}_2(\mathbb{A}))\mathrm{PGL}_2(\mathbb{Q}) = (\mathbb{F}_2 \otimes \mathbb{A}^\times)/\mathrm{PGL}_2(\mathbb{Q}) = (\mathbb{F}_2 \otimes \mathbb{A}^\times)/\mathbb{Q}^\times \\ &= \mathbb{F}_2 \otimes (\mathbb{R}^\times_{>0} \times \hat{\mathbb{Z}}^\times) = \mathbb{F}_2 \otimes \hat{\mathbb{Z}}^\times.\end{aligned}$$

Observe that this is profinite, and that $\pi_0(\mathrm{PGL}_2(\mathbb{R})^+) = 1$, hence

$$\pi_0\pi(\mathrm{PGL}_2) = \pi(\mathrm{PGL}_2), \quad \overline{\pi}_0\pi(\mathrm{PGL}_2) = \pi(\mathrm{PGL}_2).$$

So, for $K \subset \mathrm{PGL}_2(\mathbb{A}_f)$ compact open, $\pi_0(\mathrm{Sh}_K(\mathrm{PGL}_2, \mathbb{H}^\pm)) = (\mathbb{F}_2 \otimes \hat{\mathbb{Z}}^\times)/K$. Note also that in this case, for each prime $p$, there are two $\mathrm{PGL}_2(\mathbb{Q}_p)$-conjugacy classes of maximal compact subgroups: one can stabilise a vertex, or an edge, in the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$ (the set is vertices is $\mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$, the set of lattices in $\mathbb{Q}_p^2$ up to $\mathbb{Q}_p^\times$).

## 7.14 The theorem of Baily and Borel

Let $(G, X)$ be a Shimura datum, and $K \subset G(\mathbb{A}_f)$ a compact open subgroup. Baily and Borel have shown (1966):

> $\mathrm{Sh}_K(G, X)(\mathbb{C})$ is canonically the analytification of a quasi-projective complex algebraic variety $\mathrm{Sh}_K(G, X)_\mathbb{C}$.

The proof is by compactification and projective embedding via sufficiently many sections (modular forms!) of a high enough power of $\Omega^d$ (where $d = \dim(X)$). Boundary components correspond to parabolic subgroups of $G$ (think of $\mathbb{P}^1(\mathbb{Q})$ and the stabilisers its points in $\mathrm{GL}_{2,\mathbb{Q}}$). For $G = \mathrm{GSp}_{2n}$, the compactification is named after Satake.

If you need an ample line bundle on a Shimura variety, then first try the one from the Baily-Borel compactification. The Baily-Borel compactification is normal, but usually singular (already for $\mathrm{GSp}_4$, moduli of abelian surfaces, and even for Hilbert modular surfaces, where the boundary is zero-dimensional). There are better compactifications, toroidal, and they become more and more canonical (Alexeev's and Olsson's work on $A_g$, for example).

The modular forms used by Baily and Borel can be used to understand algebraic functions. Example: $j = c_4^3/\Delta$.

## 7.15 Moduli interpretation in the Siegel case

For details, and certainly for more generality, see §1 and §4 of Deligne's "Travaux de Shimura".

Let $n \in \mathbb{Z}_{\geq 1}$, let $V = \mathbb{Z}^{2n}$, and $\psi$ be the symplectic bilinear form on $V$ given by

$$\psi\colon V \times V \to \mathbb{Z}, \quad (x, y) \mapsto x^t J y, \quad \text{with} \quad J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

Let $G = \mathrm{GSp}(\psi)$ be the linear algebraic $\mathbb{Z}$-group such that for all $\mathbb{Z}$-algebra's $A$

$$G(A) = \{(g, c) \in (\mathrm{GL}(V_A), A^\times) : \text{for all } A \to B,\, x, y \in V_B,\, \psi(gx, gy) = c{\cdot}\psi(x, y)\}.$$

The morphism

$$\nu\colon G \to \mathbb{G}_{\mathrm{m}}, \quad (g, c) \mapsto c$$

is called the multiplier character.

Let $X := \mathbb{H}_n^\pm$ the set of $h\colon \mathbb{S} \to G_\mathbb{R}$ that are Hodge structures of weight $-1$ on $\mathbb{R}^{2n}$ such that $\psi$ is a polarization up to a sign. Then $X$ is the $G(\mathbb{R})$-orbit of

$$h_0\colon \mathbb{C}^\times \to G(\mathbb{R}), \quad z = a + bi \mapsto a + bJ = \begin{pmatrix} a_n & b_n \\ -b_n & a_n \end{pmatrix}.$$

and it is called the Siegel double space. Let us consider:

$$A_n(\mathbb{C}) := G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_\mathrm{f})/G(\hat{\mathbb{Z}})).$$

What we want to show is that $A_n(\mathbb{C})$ is the set of isomorphisms classes of pairs $(A, \lambda)$ of principally polarized abelian varieties of dimension $n$. We already know what the interpretation of $X$ is: it is the set of Hodge structures of weight $-1$ on $\mathbb{R}^{2n}$ such that $\psi$ is a polarization up to a sign. Let us now interpret $G(\mathbb{A}_\mathrm{f})/G(\hat{\mathbb{Z}})$. Consider the action of $G(\mathbb{A}_\mathrm{f})$ on the set of lattices in $\mathbb{A}_\mathrm{f}^{2n}$. The stabilizer of the standard lattice $\hat{\mathbb{Z}}^{2n}$ is $G(\hat{\mathbb{Z}})$. Hence $G(\mathbb{A}_\mathrm{f})/G(\hat{\mathbb{Z}})$ is the set of lattices of the form $x\hat{\mathbb{Z}}^2$, with $x$ in $G(\mathbb{A}_\mathrm{f})$. We claim that this is the set of lattices $L$ on which a suitable multiple of $\psi$ induces a perfect pairing. For $x$ in $G(\mathbb{A}_\mathrm{f})$ we have: $\psi(xu, xv) = \nu(x)\psi(u, v)$, which proves that $\nu(x)^{-1}\psi$ is a perfect pairing on $x\hat{\mathbb{Z}}^2$. On the other hand, let $L$ be a lattice and $a$ in $\mathbb{A}_\mathrm{f}^\times$ be such that $a\psi$ is a perfect pairing on $L$. Then take a $\hat{\mathbb{Z}}$-basis $l_1, \ldots, l_{2n}$ of $L$ such that $a\psi$ is in standard form, i.e., given by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then the element $x$ of $\mathrm{GL}_{2n}(\mathbb{A}_\mathrm{f})$ with $xe_i = l_i$ is in $G(\mathbb{A}_\mathrm{f})$. This finishes the proof of the fact that $G(\mathbb{A}_\mathrm{f})/G(\hat{\mathbb{Z}})$ is the set of lattices on which a multiple of $\psi$ is perfect.

Let us now describe the constructions that give $A_n(\mathbb{C})$ the interpretation as the set of isomorphism classes of abelian varieties of dimension $n$, with a principal polarization.

Suppose $(A, \lambda)$ is given. Then choose an isomorphism $f\colon \mathbb{Q}^{2n} \to \mathrm{H}_1(A, \mathbb{Q})$ such that $\psi$ corresponds to a multiple of $\lambda$ (such an $f$ is unique up to an element of $G(\mathbb{Q})$). Let $x$ be the element of $X$ that is given by the Hodge structure on $\mathbb{Q}^{2n}$ induced from $A$ via $f$. Let $L$ in $G(\mathbb{A}_\mathrm{f})/G(\hat{\mathbb{Z}})$ be the lattice corresponding to $\mathbb{Z}^{2n}$ via $f$. The class of $(x, L)$ modulo $G(\mathbb{Q})$ depends only on the isomorphism class of $(A, \lambda)$.

Suppose now that we have $(x, L)$ in $X \times G(\mathbb{A}_{\mathrm{f}})/G(\hat{\mathbb{Z}})$. Then let $A$ be $(\mathbb{R} \otimes L)/L$ with the complex structure given by the Hodge structure corresponding to $x$. Let $a$ be the element of $\mathbb{Q}^\times$ such that $a\psi$ is perfect on $L$ (this fixes $a$ up to sign) and is a polarization $\lambda$ on $A$ (this fixes the sign). For $g$ in $G(\mathbb{Q})$, multiplication by $g$ gives an isomorphism from $(A, \lambda)$ to the $(A', \lambda')$ obtained from $(gx, gL)$.

## 7.16 Limit over $K$, $G(\mathbb{A}_f)$-action

Let $(G, K)$ be a Shimura datum. For every inclusion $K_1 \subset K_2$ of compact open subgroups we have a morphism of complex algebraic varieties (also morphisms are algebraic by a result of Borel):

$$\mathrm{Sh}_{K_1}(G, X)_{\mathbb{C}} \to \mathrm{Sh}_{K_2}(G, X)_{\mathbb{C}}.$$

If $K_1$ is normal in $K_2$ then it is the quotient for the action by the finite group $K_2/K_1$, and therefore these morphisms are finite (as morphisms of schemes). That means that we can take the limit (projective limit) of the system of these, in the category of schemes:

$$\mathrm{Sh}(G, X)_{\mathbb{C}} := \lim_{K} \mathrm{Sh}_K(G, X)_{\mathbb{C}}.$$

The system has a right-action by $G(\mathbb{A}_f)$:

$$\begin{tikzcd}
G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)) \arrow[loop above, "K"] \arrow[r, "\cdot g", "\sim"'] \arrow[d, "\mathrm{quot}"] & G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)) \arrow[loop above, "g^{-1}Kg"] \arrow[d, "\mathrm{quot}"] \\
\mathrm{Sh}_K(G, X)(\mathbb{C}) \arrow[r, "\cdot g", "\sim"'] & \mathrm{Sh}_{g^{-1}Kg}(G, X)(\mathbb{C})
\end{tikzcd}$$

inducing

$$\cdot g \colon \mathrm{Sh}_K(G, X)_{\mathbb{C}} \xrightarrow{\sim} \mathrm{Sh}_{g^{-1}Kg}(G, X)_{\mathbb{C}}.$$

These are compatible with the transition maps in the system, and so give us a right-action of $G(\mathbb{A}_f)$ on $\mathrm{Sh}(G, X)_{\mathbb{C}}$. This action is continuous in the sense that $\mathrm{Sh}(G, X)_{\mathbb{C}}$ has a cover by open affine subschemes $U_i$ such that each $U_i$ is stabilised by some open subgroup $K_i$, and each $f \in \mathcal{O}(U_i)$ has open stabiliser in $K_i$ (indeed, let $K$ be compact open, then $\mathrm{Sh}(G, X)_{\mathbb{C}} \to \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ is affine etc.). From this action we recover the system of the $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ because $\mathrm{Sh}_K(G, X)_{\mathbb{C}} = \mathrm{Sh}(G, X)_{\mathbb{C}}/K$.

In the Langlands program, the $G(\mathbb{A}_f)$-action on the cohomology of Shimura varieties is very important because it brings the representation theory of $G(\mathbb{A}_f)$ into the picture.

Deligne gives two results that describe $\mathrm{Sh}(G, X)(\mathbb{C})$. We state them here.

**7.17 Proposition. (Deligne, Corvallis, 2.1.10.)** *Let $(G, X)$ be a Shimura datum. Then:*

$$\mathrm{Sh}(G, X)(\mathbb{C}) = \frac{G(\mathbb{Q})}{Z_G(\mathbb{Q})} \backslash \left( X \times G(\mathbb{A}_f)/\overline{Z_G(\mathbb{Q})} \right)$$

It is indeed clear that the map from $G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f))$ factors through the indicated quotient by $Z(\mathbb{Q})$ and $\overline{Z(\mathbb{Q})}$ (consider the action of $Z(\mathbb{Q})$ on the left).

**7.18 Corollary. (Deligne, Corvallis, 2.1.11)** *Let $(G, X)$ be a Shimura datum that satisfies D4 and D5. Then*

$$\mathrm{Sh}(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f))$$

We remark that this corollary applies to $(\mathrm{GSp}, \mathbb{H}^{\pm})$. See also Milne's notes, Proposition 4.19 for the case of "connected Shimura varieties. Let us give a proof of the last result in case $G = G^{\mathrm{ad}}$ (then SD4–5 hold, trivially). We write $X = G(\mathbb{R})/K_{\mathbb{R}}$, and $K_{\mathbb{R}}$ is a compact subgroup of $G(\mathbb{R})$. We observe that $G(\mathbb{Q})$ acts faithfully on $X$, because, writing $G = \prod_i G_i$ with the $G_i$ simple, and $(G_i)_{\mathbb{R}} = \prod_j G_{i,j}$, there is no $i$ such that for all $j$ the $G_{i,j}(\mathbb{R})$ are compact; for each $i$ there is at least one $j$ such that $G_{i,j}(\mathbb{R})$ acts faithfully on its hermitian symmetric domain. More of relevance is that the map

$$G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f) \to (G(\mathbb{R})/K_{\mathbb{R}}) \times G(\mathbb{A}_f) = X \times G(\mathbb{A}_f)$$

is proper, hence closed, so that all $G(\mathbb{Q})$-orbits in $X \times G(\mathbb{A}_f)$ are discrete. So we get, for each $(x, g)$ in $X \times G(\mathbb{A}_f)$, a compact neighborhood $\overline{B}(x, r) \times gK$, with $\overline{B}(x, r)$ the closed ball of radius $r$, and $K \subset G(\mathbb{A}_f)$ a compact open subgroup. Shrinking these, we get an open immersion of $B(x, r) \times gK$ into $G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f))$. The image is $K$-stable, and for every $K' \subset K$ it has quotient $B(x, r) \times K/K'$ in $\mathrm{Sh}_{K'}(G, X)(\mathbb{C})$. The we have

$$\lim_{K'}(B(x, r) \times gK/K') = B(x, r) \times g \lim_{K'} K/K' = B(x, r) \times gK$$

because $K \to \lim_{K'} K/K'$ is from compact to Hausdorff etc.

## 7.19 Hecke correspondences

At finite level, the $G(\mathbb{A}_f)$-action induces correspondences, as follows. For $K$, $K'$ compact open, and $g$ in $G(\mathbb{A}_f)$, we have :

$$\begin{tikzcd}
\mathrm{Sh}(G, X)_{\mathbb{C}} \arrow[loop above, "K"] \arrow[r, "\cdot g", "\sim"'] \arrow[d, "\mathrm{quot}"] \arrow[dr, "q"] & \mathrm{Sh}(G, X)_{\mathbb{C}} \arrow[loop above, "K'"] \arrow[d, "\mathrm{quot}"] \\
\mathrm{Sh}_K(G, X)_{\mathbb{C}} & \mathrm{Sh}_{K'}(G, X)_{\mathbb{C}}
\end{tikzcd}$$

where $q$ is a quotient for the action of $gK'g^{-1}$. This diagram induces the Hecke correspondence:

$$\begin{tikzcd}
& \mathrm{Sh}_{K \cap gK'g^{-1}}(G, X)_{\mathbb{C}} \arrow[dl] \arrow[dr] & \\
\mathrm{Sh}_K(G, X)_{\mathbb{C}} & & \mathrm{Sh}_{K'}(G, X)_{\mathbb{C}}
\end{tikzcd}$$

## 7.20 Moduli of elliptic curves, "up to isogeny"

We give an example. Let $G = \mathrm{GL}_{2,\mathbb{Q}}$ and $X = \mathbb{H}^{\pm}$ as usual. As SD4–5 hold we have

$$\mathrm{Sh}(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f))$$

We want to interpret this limit as a moduli space: the points should correspond to elliptic curves with some data. But then the right-action by $G(\mathbb{A}_f)$ must also have such an interpretation, and therefore its quotient. The quotient $\mathrm{Sh}_{\mathrm{GL}_2(\hat{\mathbb{Z}})}(G,X)$ by the action of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ should be the moduli space of elliptic curves without any extra structure. But then the quotient by $\mathrm{GL}_2(\mathbb{A}_f)$ should be elliptic curves up to isogeny (Hecke correspondences do change elliptic curves by isogenies). And therefore, finally, we should interpret $\mathrm{Sh}(G,X)(\mathbb{C})$ as the moduli space of "elliptic curves up to isogeny with some extra data". Let us define this properly.

Let $\mathrm{Ell}(\mathbb{C})$ be the category of complex elliptic curves, its Hom-sets are finitely generated free $\mathbb{Z}$-modules of rank $0$, $1$ or $2$, and composition is bilinear (this is called pre-additive). The category $\mathbb{Q}\otimes\mathrm{Ell}(\mathbb{C})$ of complex elliptic curves up to isogeny has the same objects as $\mathrm{Ell}(\mathbb{C})$ but the Hom-sets are tensored with $\mathbb{Q}$. The functor $\mathrm{Ell}(\mathbb{C}) \to \mathbb{Q}\otimes\mathrm{Ell}(\mathbb{C})$ is the localisation that inverts the set of isogenies. Let us write $\mathbb{Q}\otimes E$ for the image of $E$ in $\mathbb{Q}\otimes\mathrm{Ell}(\mathbb{C})$. The functor

$$T\colon \mathrm{Ell}(\mathbb{C}) \to \hat{\mathbb{Z}}\text{-Mod}, \quad E \mapsto T(E) = \lim_n E[n] = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, E_{\mathrm{tors}})$$

does not factor through this localisation: $T(E)$ is a free $\hat{\mathbb{Z}}$-module of rank 2, but not a $\mathbb{Q}$-module. But the functor

$$V\colon \mathrm{Ell}(\mathbb{C}) \to \mathbb{A}_f\text{-Mod}, \quad E \mapsto V(E) = \mathbb{Q}\otimes T(E) = \mathrm{Hom}(\mathbb{Q}, E_{\mathrm{tors}})$$

does. And so we have

$$V\colon \mathbb{Q}\otimes\mathrm{Ell}(\mathbb{C}) \to \mathbb{A}_f\text{-Mod}, \quad \mathbb{Q}\otimes E \mapsto V(E).$$

We note that applying $\mathrm{Hom}(-, E_{\mathrm{tors}})$ to $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ gives

$$E_{\mathrm{tors}} = V(E)/T(E).$$

For $\phi\colon E \to F$ an isogeny we have

$$\ker\phi = \phi^{-1}T(F)/T(E) \subset V(E)/T(E).$$

Similarly, the functor $E \mapsto \mathrm{H}_1(E,\mathbb{Z})$ does not factor through the localisation, but we have the functor

$$H\colon \mathbb{Q}\otimes\mathrm{Ell}(\mathbb{C}) \to \mathbb{Q}\text{-Mod}, \quad \mathbb{Q}\otimes E \mapsto \mathrm{H}_1(E,\mathbb{Q}).$$

The functors $V$ and $H$ are related, for every $E$ we have an isomorphism

$$V(E) = \mathbb{A}_f \otimes_{\mathbb{Q}} H(E).$$

It is still useful to have $V$, as it is defined algebraically, for elliptic curves over arbitrary fields, whereas $H$ is not. Note that $H(E)$ has a $\mathbb{Q}$-Hodge structure, we have $h\colon \mathbb{S} \to \mathbf{GL}(H(E)_{\mathbb{R}})$.

**7.20.1 Proposition.** *We have:*

$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)) = \{(\mathbb{Q}\otimes E, \alpha)\}/\cong, \quad \mathbb{Q}\otimes E \textit{ in } \mathbb{Q}\otimes\mathrm{Ell}(\mathbb{C}),\ \alpha\colon \mathbb{A}_f^2 \xrightarrow{\sim} V(E).$$

**Proof.** We start with $(\mathbb{Q} \otimes E, \alpha)$. We choose an isomorphism $\phi\colon \mathbb{Q}^2 \to H(\mathbb{Q} \otimes E)$; it is unique up to $\phi' = \phi \circ q$, $q \in \mathrm{GL}_2(\mathbb{Q})$. Then we have

$$\phi\colon \mathbb{A}_f^2 \xrightarrow{\sim} \mathbb{A}_f \otimes_{\mathbb{Q}} H(E) = V(E), \quad \alpha\colon \mathbb{A}_f^2 \xrightarrow{\sim} V(E), \quad \phi^{-1} \circ \alpha \in \mathrm{GL}_2(\mathbb{A}_f).$$

And we have

$$\mathbb{S} \xrightarrow{h} \mathbf{GL}(H(E)_{\mathbb{R}}) \xrightarrow{\phi^{-1} * \phi} \mathrm{GL}_{2,\mathbb{R}}, \quad s \mapsto h(s) \mapsto \phi^{-1} \circ h(s) \circ \phi.$$

And we get a well-defined element

$$(\phi^{-1} h(*) \phi, \phi^{-1} \circ \alpha) \quad \text{in } \mathrm{GL}_2(\mathbb{Q}) \backslash (\mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)).$$

And vice versa, for $(x, g)$ in $\mathrm{GL}_2(\mathbb{Q}) \backslash (\mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f))$ we equip $\mathbb{R}^2$ with the complex structure corresponding to $x$. This gives $E := \mathbb{R}^2/\mathbb{Z}^2$, with $\mathrm{H}_1(E, \mathbb{Z}) = \mathbb{Z}^2$, hence $V(E) = \mathbb{A}_f^2$. We get the pair

$$(\mathbb{Q} \otimes (\mathbb{R}^2/\mathbb{Z}^2, x), g) \quad \text{in } \{(\mathbb{Q} \otimes E, \alpha)\}/\cong.$$

For $q \in \mathrm{GL}_2(\mathbb{Q})$, $q(x, g) = (q \circ x \circ q^{-1}, qg)$, which is sent to $(\mathbb{Q} \otimes (\mathbb{R}^2/\mathbb{Z}^2, qxq^{-1}), qg)$, and indeed

$$q\colon (\mathbb{Q} \otimes (\mathbb{R}^2/\mathbb{Z}^2, x), g) \xrightarrow{\sim} (\mathbb{Q} \otimes (\mathbb{R}^2/\mathbb{Z}^2, qxq^{-1}), qg).$$

$\square$

Our next step is to interpret, for $K \subset \mathrm{GL}_2(\mathbb{A}_f)$, the quotient $\mathrm{Sh}_K(G, X)$:

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) = \{(\mathbb{Q} \otimes E, \overline{\alpha})\}/\cong, \quad \overline{\alpha} \in \mathrm{Isom}(\mathbb{A}_f^2, V(E))/K$$

For $K = \mathrm{GL}_2(\hat{\mathbb{Z}})$, we have, for each $E$

$$\mathrm{Isom}(\mathbb{A}_f^2, V(E))/\mathrm{GL}_2(\hat{\mathbb{Z}}) = \{\hat{\mathbb{Z}}\text{-lattices in } V(E)\}, \quad \overline{\alpha} \mapsto \alpha(\hat{\mathbb{Z}}^2).$$

**7.20.2 Proposition.** *The functor*

$$\mathrm{Ell}(\mathbb{C}) \to \{(\mathbb{Q} \otimes E, L)\}, \quad \textit{with } L \textit{ a } \hat{\mathbb{Z}}\textit{-lattice in } V(E), \quad E \mapsto (\mathbb{Q} \otimes E, T(E))$$

*is an equivalence of categories, where morphisms in the target category are those that send the lattice in the lattice.*

**Proof.** Let $E$ and $F$ be in $\mathrm{Ell}(\mathbb{C})$. Let $\phi \in \mathbb{Q} \otimes \mathrm{Hom}(E, F)$ be a morphism from $\mathbb{Q} \otimes E$ to $\mathbb{Q} \otimes F$ with $\phi(T(E)) \subset T(F)$. Let $n \in \mathbb{Z}_{\geq 1}$ be such that $\alpha := n\phi$ is in $\mathrm{Hom}(E, F)$. Then $\alpha(T(E)) \subset nT(F)$, hence $\ker(\alpha) = \alpha^{-1}T(F)/T(E) \supset n^{-1}T(E)/T(E) = E[n]$. This proves that $\alpha = n\phi$ with $\phi \in \mathrm{Hom}(E, F)$. So, the functor is fully faithful. Now we show that it is essentially surjective. Let $E$ be in $\mathrm{Ell}(\mathbb{C})$ and $L \subset V(E)$ a $\hat{\mathbb{Z}}$-lattice. Let $n \in \mathbb{Z}_{\geq 1}$ such that $T(E) \subset n^{-1}L$. Then $n\cdot\colon E \to E$ induces an isomorphism from $(\mathbb{Q} \otimes E, n^{-1}L)$ to $(\mathbb{Q} \otimes E, L)$. The quotient $E \to F$ by $n^{-1}L/T(E)$ induces an isomorphism from $(\mathbb{Q} \otimes E, n^{-1}L)$ to $(\mathbb{Q} \otimes F, T(F))$. $\square$

So, we conclude that

$$\mathrm{Sh}_{\mathrm{GL}_2(\hat{\mathbb{Z}})}(G,X)(\mathbb{C}) = \mathrm{Ob}(\mathrm{Ell}(\mathbb{C}))/\cong.$$

We can now also understand Hecke correspondences better. Let $K = K' = \mathrm{GL}_2(\hat{\mathbb{Z}})$, and let $p$ be prime and $g \in \mathrm{GL}_2(\mathbb{A}_f)$ be such that $g_p = \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and for all $l \neq p$, $g_l = 1$. Then the correspondence on the $j$-line $\mathrm{Sh}_K(G,X)_{\mathbb{C}} = \mathbb{C}$ given by $g$ sends $j(E)$ in $\mathbb{C}$ to the formal sum of the $j(E/G)$, where $G$ varies over the $p+1$ subgroups of order $p$ of $E$. The reason is that $\mathbb{Z}_p^2 \subset g_p\mathbb{Z}_p^2$ with quotient of order $p$. We leave the details to the audience.

## 7.21 Morphisms of Shimura varieties

Let $f\colon (G_1, X_1) \to (G_2, X_2)$ be a morphism of Shimura data. Recall that this means that $f\colon G_1 \to G_2$ is a morphism of reductive linear algebraic groups over $\mathbb{Q}$, such that

$$f\circ\colon \mathrm{Hom}(\mathbb{S}, G_{1,\mathbb{R}}) \to \mathrm{Hom}(\mathbb{S}, G_{2,\mathbb{R}}) \quad \text{sends } X_1 \text{ to } X_2.$$

For this, it suffices that one $h$ in $X_1$ has $f \circ h \in X_2$, the $G_1(\mathbb{R})$-action then does it for all $\mathrm{inn}_g \circ h$, $g \in G_1(\mathbb{R})$. For $K_1 \subset G_1(\mathbb{A}_f)$ and $K_2 \subset G_2(\mathbb{A}_f)$ compact open subgroups such that $f(K_1) \subset K_2$, we get a map

$$f_{K_1,K_2}\colon \mathrm{Sh}_{K_1}(G_1,X_1)(\mathbb{C}) \to \mathrm{Sh}_{K_2}(G_2,X_2)(\mathbb{C})$$

By a theorem of Borel (see Milne's notes, Theorem 3.14), this map is a morphism of algebraic varieties

$$f_{K_1,K_2}\colon \mathrm{Sh}_{K_1}(G_1,X_1)_{\mathbb{C}} \to \mathrm{Sh}_{K_2}(G_2,X_2)_{\mathbb{C}}.$$

Taking limits, we have

$$f\colon \mathrm{Sh}(G_1,X_1)_{\mathbb{C}} \to \mathrm{Sh}(G_2,X_2)_{\mathbb{C}}$$

One can factor $f = f_2 \circ f_1$ with $f_1$ surjective and $f_2$ a closed immersion.

**7.22 Proposition. (Deligne (Travaux de Shimura, Prop.1.15))** *If $G_1$ is a subgroup of $G_2$, then for all $K_1$ there is a $K_2$ such that $f_{K_1,K_2}$ is a closed immersion. The limit $f$ is injective.*

The proof takes more than one page, I do not go into it.

## 7.23 Special subvarieties

There are two ways of looking at this: as images of morphisms, or as Hodge class loci and Mumford-Tate groups. We give the definition in terms of morphisms.

**7.24 Definition.** Let $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ be a Shimura variety. A closed irreducible subvariety $Z$ is called *special* if there exists a morphism of Shimura data $f\colon (G',X') \to (G,X)$ and a $g$ in $G(\mathbb{A}_f)$, such that $Z$ is an irreducible component of the image of

$$\mathrm{Sh}(G',X')_{\mathbb{C}} \xrightarrow{f} \mathrm{Sh}(G,X)_{\mathbb{C}} \xrightarrow{\cdot g} \mathrm{Sh}(G,X)_{\mathbb{C}} \xrightarrow{\mathrm{quot}} \mathrm{Sh}_K(G,X)_{\mathbb{C}}.$$

This is equivalent to: $Z$ is an irreducible component of the image of $T_g \circ f_{K',K}$, for a suitable $K'$. The *special points* in $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ are the zero-dimensional special subvarieties. Note that the set of special subvarieties of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ is countable.

**7.25 Definition.** For $h \in X$, $\mathrm{MT}(h) \subset G$ is the smallest subgroupscheme $H \subset G$ such that $h\colon \mathbb{S} \to G_{\mathbb{R}}$ factors through $H_{\mathbb{R}}$. For $z \in \mathrm{Sh}_K(G,X)(\mathbb{C})$, each pre-image $(h,g)$ in $X \times G(\mathbb{A}_f)$ gives $\mathrm{MT}(h) \subset G$, and together this gives a well-defined group $\mathrm{MT}(z)$ with a $G(\mathbb{Q})$-conjugacy class of embeddings in $G$.

**7.26 Proposition.** *Let $S := \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ be a Shimura variety, and $z$ in $S(\mathbb{C})$. Let $(\tilde{s}, g)$ be a pre-image of $s$ in $X \times G(\mathbb{A}_f)$, and let $M := \mathrm{MT}(\tilde{s})$, and $X_M := M(\mathbb{R})\cdot\tilde{s} \subset X$ and $X_M^+ := M(\mathbb{R})^+\cdot\tilde{s}$. Then the smallest special subvariety of $S$ containing $s$ is the image of*

$$X_M^+ \times \{1\} \longrightarrow X \times \{g\} \to S.$$

**Proof.** Because $\tilde{s}$ is Hodge generic on $M(\mathbb{R})^+\cdot\tilde{s}$. ◻

## 7.27 Canonical models

Let $(G,X)$ be a Shimura datum. The $\mathbb{C}$-scheme $\mathrm{Sh}(G,X)_{\mathbb{C}}$ together with its action by $G(\mathbb{A}_f)$ can be naturally defined over a number field $E = E(G,X) \subset \mathbb{C}$ called the *reflex field* of $(G,X)$. By this we mean that there is an $E$-scheme $\mathrm{Sh}(G,X)_E$ with an action by $G(\mathbb{A}_f)$ that by base change to $\mathbb{C}$ gives $\mathrm{Sh}(G,X)_{\mathbb{C}}$ together with its action by $G(\mathbb{A}_f)$.

The subfield $E$ of $\mathbb{C}$ is defined as follows. Let $h \in X$. Recall that we have

$$\begin{array}{rcl} \mu\colon \mathbb{G}_{\mathrm{m}\mathbb{C}} & \longrightarrow & \mathbb{G}_{\mathrm{m}\mathbb{C}} \times \mathbb{G}_{\mathrm{m}\mathbb{C}} \xrightarrow{(z,\bar{z})^{-1}} \mathbb{S}_{\mathbb{C}} \\ a & \longmapsto & (a,1) \end{array}$$

Then we have

$$\mu_h\colon \mathbb{G}_{\mathrm{m}\mathbb{C}} \xrightarrow{\mu} \mathbb{S}_{\mathbb{C}} \xrightarrow{h_{\mathbb{C}}} G_{\mathbb{C}}$$

When $h$ varies over $X$, $\mu_h$ varies in the $G(\mathbb{C})$-orbit of $\mu_h$, where $G(\mathbb{C})$ acts by composition with inner automorphisms. Let $c(X)$ be this orbit:

$$c(X) \in G(\mathbb{C})\backslash\mathrm{Hom}_{\mathbb{C}}(\mathbb{G}_{\mathrm{m}\mathbb{C}}, G_{\mathbb{C}}).$$

Note that $\mathrm{Aut}(\mathbb{C})$ acts on $\mathrm{Hom}_{\mathbb{C}}(\mathbb{G}_{\mathrm{m}\mathbb{C}}, G_{\mathbb{C}})$, and on $G(\mathbb{C})\backslash\mathrm{Hom}_{\mathbb{C}}(\mathbb{G}_{\mathrm{m}\mathbb{C}}, G_{\mathbb{C}})$. Then

$$E(G,X) := \mathbb{C}^{\mathrm{Aut}(\mathbb{C})_{c(X)}}$$

the subfield of $\mathbb{C}$ fixed, pointwise, by the stabiliser in $\mathrm{Aut}(\mathbb{C})$ of $c(X)$, that is, "the field of definition of $c(X)$".

Let us state it in scheme theoretic terms. The functor $\mathbf{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}}, G)$ is representable by a smooth $\mathbb{Q}$-scheme, and the connected components are precisely the $G$-orbits (I gave the references to SGA3 before).

So $G\backslash\mathbf{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}}, G)$ is an etale $\mathbb{Q}$-scheme (a disjoint union of spectra of finite field extensions of $\mathbb{Q}$), and $c(X)$ is a $\mathbb{C}$-valued point of it

$$c(X)\colon\ \mathrm{Spec}(\mathbb{C}) \to G\backslash\mathbf{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}}, G).$$

Then $E$ is the residue field at $c(X)$, together with its embedding into $\mathbb{C}$.

We can also state it more concretely (see Milne, chapter 12, for more details). Let $k \subset \mathbb{C}$ be a finite Galois extension of $\mathbb{Q}$ such that $G_k$ contains a split torus $T$. Then

$$G(\mathbb{C})\backslash\mathrm{Hom}_{\mathbb{C}}(\mathbb{G}_{\mathrm{m}\mathbb{C}}, G_{\mathbb{C}}) = W\backslash\mathrm{Hom}_{\mathbb{C}}(\mathbb{G}_{\mathrm{m}\mathbb{C}}, T_{\mathbb{C}}) = W\backslash\mathrm{Hom}_k(\mathbb{G}_{\mathrm{m}k}, T) = W\backslash X^*(T)$$

where $W = N_{G_k}(T)/T$, the Weyl group of $T$. Note that $W$ is a finite group (in fact a constant group-scheme over $k$). Then $\mathrm{Gal}(k/\mathbb{Q})$ acts on $W\backslash X^*(T)$ (this is a bit tricky, $T$ is not necessarily fixed by $\mathrm{Gal}(k/\mathbb{Q})$), and

$$E = k^{\mathrm{Gal}(k/\mathbb{Q})_{c(X)}}.$$

A few examples. The reflex field of $(\mathrm{GSp}(\psi), \mathbb{H}^{\pm})$ is $\mathbb{Q}$, as the group has a split maximal torus over $\mathbb{Q}$. (Exercise: try to see it directly, start with $\mathrm{GL}_2$.) The moduli interpretation gives the canonical model over $\mathbb{Q}$.

Second example. Hilbert modular varieties: $E = \mathbb{Q}$, but the group has no split maximal torus over $\mathbb{Q}$. Again: moduli interpretation.

Third example (we've seen it already before, as example where the weight morphism is not defined over $\mathbb{Q}$). Let $\mathbb{Q} \to F$ be a totally real field of degree $d$, say, and let $B$ be a quaternion algebra over $F$, and $G = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{GL}_1(B)$. Let $\Sigma := \mathrm{Hom}(F, \mathbb{R})$. Then

$$G(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{R})^{\Sigma_0} \times (\mathbb{H}^{\times})^{\Sigma_1}.$$

With $X$ as described before. Then $E$ is the field of definition of the subsets $\Sigma_0$ and $\Sigma_1$ of $\Sigma$ (let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}$, then $\Sigma = \mathrm{Hom}(F, \overline{\mathbb{Q}})$, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts, $E$ is fixed field of the stabiliser of $\Sigma_0$).

## 7.28 Canonical model when $G$ is a torus

Let $T$ be a torus over $\mathbb{Q}$, and $h\colon \mathbb{S} \to T_{\mathbb{R}}$. Then $(T, \{h\})$ is a Shimura datum (satisfying SD1–3, not necessarily SD4–5). Then

$$\mathrm{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{C}}, T_{\mathbb{C}}) = \mathrm{Hom}(\mathbb{G}_{\mathrm{m}\overline{\mathbb{Q}}}, T_{\overline{\mathbb{Q}}}) = X_*(T)$$

the co-character group with its $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. Then $\mathbf{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}}, T)$ is the etale $\mathbb{Q}$-scheme corresponding to $X_*(T)$, and $E \subset \mathbb{C}$ is the residue field at $\mu_h \in \mathbf{Hom}(\mathbb{G}_{\mathrm{m}\mathbb{Q}}, T)(\mathbb{C})$. This gives us

$$\mu_h\colon \mathbb{G}_{\mathrm{m}E} \to T_E.$$

We have

$$\mathrm{Sh}(T, \{h\})(\mathbb{C}) = \lim_{K} T(\mathbb{A}_f)/T(\mathbb{Q})K = T(\mathbb{A}_f)/\overline{T(\mathbb{Q})}.$$

So $T(\mathbb{A}_f)$ acts transitively. If $\mathrm{Sh}(T,\{h\})_E$ is a model over $E$, then $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ acts on $\mathrm{Sh}(T,\{h\})_E(\overline{\mathbb{Q}}) = \mathrm{Sh}(T,\{h\})_E(\mathbb{C})$, commuting with the $T(\mathbb{A}_f)$-action. But that means that $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ acts via a continuous morphism of groups $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \to T(\mathbb{A}_f)/\overline{T(\mathbb{Q})}$. And vice versa, such a morphism defines a model over $E$; the coordinate ring is:

$$\{f\colon T(\mathbb{A}_f)/\overline{T(\mathbb{Q})} \to \overline{\mathbb{Q}} : f \text{ locally constant}\}^{\mathrm{Gal}(\overline{\mathbb{Q}}/E)}.$$

**Class field theory** gives us what we need. Let $E^{\mathrm{ab}}$ be the maximal abelian Galois extension of $E$ in $\overline{\mathbb{Q}}$, and $\mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}} = \mathrm{Gal}(E^{\mathrm{ab}}/E)$. Then there is a unique isomorphism of topological groups

$$\mathrm{art}_E\colon \pi_0(\mathbb{A}_E^\times/E^\times) = (\mathbb{A}_E^\times/E_{\mathbb{R}}^{\times,+})/\overline{E^\times} \xrightarrow{\sim} \mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}}$$

such that for all finite extensions $E \subset L \subset E^{\mathrm{ab}}$ and for all finite places $v$ of $E$ where $L$ is unramified, any uniformiser $\pi_v$ in $E_v^\times \subset \mathbb{A}_E^\times$ is sent to the *inverse* of the arithmetic Frobenius element $\mathrm{Frob}_v$ ($\mathrm{Frob}_v$ induces the $q_v$th power map on the residue field of $L$ at all places over $v$) (we follow Deligne, Corvallis, 0.8 here; see Milne's notes, Chapter 11 for many more details).

We can now define the morphism $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \to T(\mathbb{A}_f)/\overline{T(\mathbb{Q})}$ that defines the model $\mathrm{Sh}(T,\{h\})_E$. We define

$$r(T,h)\colon \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{\mathrm{m}E} \xrightarrow{\mathrm{Res}\mu_h} \mathrm{Res}_{E/\mathbb{Q}}T_E \xrightarrow{\mathrm{Norm}} T$$

where, for every $\mathbb{Q}$-algebra $A$, $\mathrm{Norm}$ is given by

$$\begin{tikzcd}
(\mathrm{Res}_{E/\mathbb{Q}}T_E)A \arrow[r, equal] & T(E\otimes_{\mathbb{Q}} A) \arrow[r] & T(\overline{\mathbb{Q}}\otimes_{\mathbb{Q}} A)^{\mathrm{Hom}(E,\overline{\mathbb{Q}})} \arrow[r, "\text{product}"] & T(A) \\
& t \arrow[r, mapsto] & (\sigma(t))_{\sigma\colon E\to\overline{\mathbb{Q}}} \arrow[r, mapsto] & \prod_\sigma \sigma(t)
\end{tikzcd}$$

Taking $\mathbb{A}_f$-points gives

$$\begin{tikzcd}
\mathbb{A}_E^\times \arrow[r, "r(T,h)"] \arrow[d, two heads, "\mathrm{art}_E"] & T(\mathbb{A}) \arrow[d, two heads] & \\
\mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}} \arrow[r] & T(\mathbb{A}_f)/\overline{T(\mathbb{Q})} \arrow[r, equal, "\text{def}"] & \mathrm{Sh}(T,\{h\})_E(\overline{\mathbb{Q}})
\end{tikzcd}$$

Explicitly, $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ acts on $\mathrm{Sh}(T,\{h\})_E(\overline{\mathbb{Q}})$ as follows

$$\text{for } a \in \mathbb{A}_E^\times, \text{ and } t \in T(\mathbb{A}_f)\text{: } \mathrm{art}_E(a)\cdot[(h,t)] = [(h, r(T,h)(a)t)].$$

## 7.29 CM elliptic curves

Let $E \subset \mathbb{C}$ be an imaginary quadratic field, $T := \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{\mathrm{m}E}$, and $h\colon \mathbb{S} \to T_{\mathbb{R}}$ such that $h\colon \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times = T(\mathbb{R})$ is the identity. Then $E(T,h) = E$, and

$$\mathrm{Sh}(T,\{h\})_E(\overline{\mathbb{Q}}) = E^\times\backslash\mathbb{A}_{E,f}^\times.$$

Let $\mathrm{Ell}_E(\overline{\mathbb{Q}})$ be the category of $(F, \alpha)$, with $F$ an elliptic curve over $\overline{\mathbb{Q}}$, and $\alpha\colon E \xrightarrow{\sim} \mathbb{Q}\otimes\mathrm{End}(F)$, such that the action of $E$ on the tangent space $T_F(0)$ gives the given embedding $E \subset \overline{\mathbb{Q}} \subset \mathbb{C}$. Let $\mathbb{Q}\otimes\mathrm{Ell}_E(\overline{\mathbb{Q}})$ be the category obtained by tensoring the Hom's with $\mathbb{Q}$. Note that all objects in $\mathbb{Q}\otimes\mathrm{Ell}_E(\overline{\mathbb{Q}})$ are isomorphic: all $F$ with CM by $E$ are isogeneous, and $\mathrm{Aut}(\mathbb{Q}\otimes(F,\alpha])) = E^\times$. Then we have a bijection

$$\mathrm{Sh}(T,\{h\})_E(\overline{\mathbb{Q}}) \xrightarrow{\sim} \{(\mathbb{Q}\otimes(F,\alpha),\phi) : \text{with } \phi\colon \mathbb{A}_{E,f} \xrightarrow{\sim} V(F)\}/\cong,$$

sending $[(h,t)]$ to the elliptic curve $(\mathbb{R}\otimes_{\mathbb{Q}} E)/O_E$ with $\phi = t$. Both sides have an action by $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$. The theory of complex multiplication says that these actions are equal, and that justifies the choices that were made in defining the canonical model $\mathrm{Sh}(T,\{h\}_E)$. The same is true for general CM-abelian varieties by the theorem of Shimura-Taniyama. See Milne's notes, Ch. 10–11 for details.

The algebraic definition of the $(\mathbb{Q}\otimes(F,\alpha),\phi)$ in fact gives directly a morphism $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \to \mathbb{A}_{E,f}^\times/E^\times$, because the $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$-action commutes with the $\mathbb{A}_{E,f}^\times/E^\times$-action and that action is free and transitive.

## 7.30 Back to general theory.

For $f\colon (G_1,X_1) \to (G_2,X_2)$ a morphism of Shimura data, we have $E_2 \subset E_1$: this follows directly from the definition.

**7.31 Theorem.** *Let $(G,X)$ be a Shimura datum. Let $E := E(G,X) \subset \mathbb{C}$. There is a unique $E$-scheme $S$ with a right action by $G(\mathbb{A}_f)$, together with a $G(\mathbb{A}_f)$-equivariant isomorphism $S\otimes_E\mathbb{C} \xrightarrow{\sim} \mathrm{Sh}(G,X)_{\mathbb{C}}$, such that for all closed immersions of Shimura data $i\colon (T,\{h\})\hookrightarrow(G,X)$ with $T$ a torus, the induced $T(\mathbb{A}_f)$-equivariant morphism $i\colon \mathrm{Sh}(T,\{h\})_{\mathbb{C}} \to \mathrm{Sh}(G,X)_{\mathbb{C}}$ comes via base change $E(T,h)\to\mathbb{C}$ from a morphism $\mathrm{Sh}(T,\{h\})_{E(T,h)} \to S_{E(T,h)}$ of $E(T,h)$-schemes.*

*For a morphism of Shimura data $f\colon (G',X') \to (G,X)$, and $E' = E(G',X')$, the induced morphism $\mathrm{Sh}(G',X')_{\mathbb{C}} \to \mathrm{Sh}(G,X)_{\mathbb{C}}$ comes by base change $E' \to \mathbb{C}$ from a unique morphism $\mathrm{Sh}(G',X')_{E'} \to \mathrm{Sh}(G,X)_{E'}$.*

For a discussion of the proof see Section 2 of Moonen's article "Models of Shimura varieties…". This theorem tells us how a large part of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on special points.

**7.32 Corollary.** *Let $(T,\{h\})$ and $(G,X)$ be as in the theorem. Let $g\in G(\mathbb{A}_f)$ and $t\in T(\mathbb{A}_f)$, and $a\in\mathbb{A}_E^\times$. Then $[(i(h),i(t)g)] \in \mathrm{Sh}(G,X)_E(E(T,h)^{\mathrm{ab}})$, and*

$$\mathrm{art}(a)\cdot[(i(h),i(t)g)] = [(i(h),i(r(T,h)a)i(t)g)].$$

Theorem 2.6.3 in Deligne's Corvallis paper gives, for $(G,X)$ a Shimura datum, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on $\pi_0(\mathrm{Sh}(G,X)_{\overline{\mathbb{Q}}})$, a torsor under the commutative group $G(\mathbb{A}_f)/\overline{G(\mathbb{Q})_+}$ (Prop. 2.1.14). The definition of $r(G,X)$ is very complicated. Klingler and Yafaev do not use it, but use $(G,X)\to(G^{\mathrm{ab}},\{\cdot\})$, and $G^{\mathrm{ab}}$ is a torus.

# 8 Where to put????

## 8.1 PEL type, simple of type A or C

(For details, see Milne, 8.) Let $B$ be a simple finite dimensional $\mathbb{Q}$-algebra (that is, $\mathrm{M}_n(D)$ for some division algebra of finite dimension over $\mathbb{Q}$) and $*$ a positive involution (for every nonzero $b \in B$, $\mathrm{trace}(b^* \cdot b) > 0$), and such that .

## 8.2 A Hodge type example

In his paper "A note of Shimura's paper…" in 1969, Mumford gives an example of a Shimura datum $(G, X)$ that is of Hodge type, but not of PEL type, meaning that $(G, X)$ can be embedded in $(\mathrm{GSp}_{2n}, \mathbb{H}_n^{\pm})$ for some $n$, but that morphisms between abelian varieties are not sufficient to define $G$ as stabilising subgroup in $\mathrm{GSp}_{2n}$. We describe his example.

Let $\mathbb{Q} \to F$ be a totally real extension of degree three, and $B$ a quaternion algebra over $F$ that is ramified at two of the three real places of $F$ and unramified elsewhere. Then the corestriction $\mathrm{Cor}_{F/\mathbb{Q}}(B)$ of $B$ from $F$ to $\mathbb{Q}$ is isomorphic to $\mathrm{M}_8(\mathbb{Q})$ (it is the tensor product along the fibres $\mathrm{Spec}(F) \to \mathrm{Spec}(\mathbb{Q})$; Weil restriction is product along the fibres). Here is a description. We have a semi-linear action:

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \bigotimes_{\phi\colon F \to \overline{\mathbb{Q}}} \phi B \to \bigotimes_{\phi\colon F \to \overline{\mathbb{Q}}} \phi B, \quad \sigma \otimes \mathrm{id}\colon \phi B \to \sigma\phi B,$$

and

$$\mathrm{Cor}_{F/\mathbb{Q}} B = \left( \bigotimes_{\phi\colon F \to \overline{\mathbb{Q}}} \phi B \right)^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}.$$

We have a norm map

$$\mathrm{Norm}\colon B \to \mathrm{Cor}_{F/\mathbb{Q}} B, \quad b \mapsto \otimes_\phi (1 \otimes b).$$

Doing the same over $\mathbb{Q}$-algebras gives a morphism of linear algebraic groups over $\mathbb{Q}$ which on $\mathbb{Q}$-points is:

$$\mathrm{Norm}\colon B^\times \to \mathrm{Cor}_{F/\mathbb{Q}}(B)^\times \cong \mathrm{GL}_{8,\mathbb{Q}}.$$

We let $G$ be the image, hence a subgroup of $\mathrm{GL}_{8,\mathbb{Q}}$.

Geometrically: over $\overline{\mathbb{Q}}$, $G$ is isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$, and $G$ is its image under the representation on $W \otimes W \otimes W$, with $W$ the standard representation of $\mathrm{GL}_2$. So, $G$ is the quotient of $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ by the subgroup given as the kernel of $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$, $(x, y, z) \mapsto xyz$. Over $\mathbb{Q}$, this kernel is the norm 1 subgroup of $F^\times$ (well, the algebraic group).

Let us write, over $\mathbb{C}$, an $h\colon \mathbb{S}(\mathbb{R}) \to G(\mathbb{C})$, of which the $G(\mathbb{R})$-orbit gives the Shimura datum. We first give $h$ on $\mathbb{R}^\times$ and on $S^1$ seperately, as morphisms to $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$:

$$h\colon \mathbb{R}^\times \ni \lambda \mapsto (\lambda, \lambda, \lambda), \quad S^1 \ni z \mapsto (1, 1, \left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)).$$

Note that the images of $-1$ do not agree, they are $(-1-1-1)$ and $(1,1,-1)$, but their difference $(-1-1,1)$ is zero in $G$, and therefore $h$ factors as $\mathbb{R}^\times \times S^1 \to \mathbb{C}^\times \to G(\mathbb{C})$. The weight morphism is defined over $\mathbb{Q}$. The representation $W \otimes W \otimes W$ has, up to scalar multiple, a unique symplectic form that is invariant up to multiple (it is the tensor product of a symplectic form on each factor $W$). Over $\mathbb{R}$, we have $\mathbb{G}_\mathrm{m} \times \mathrm{SO}_4 \times \mathrm{SL}_2 \twoheadrightarrow G_\mathbb{R}$.

These Shimura varieties parametrise 4-dimensional abelian varieties with extra Hodge classes. See Moonen's notes on MT-groups, (5.9) that there are none in $\mathrm{H}^\bullet(X,\mathbb{Q})$, but also that they occur in $\mathrm{H}^4(X\times X,\mathbb{Q})$.

Mumford also shows that there are special points, as follows. Let $h$ be in $X$, and $T$ a maximal torus of $G_\mathbb{R}$ containing $h(\mathbb{S})$. Let $a \in T(\mathbb{R})$ be a regular element (not in kernel of any root; the set of regular elements in $G$ is open and dense (multiplicity of eigenvalue $1$ in char pol of $g$ acting on $\mathrm{Lie}(G)$ is minimal)). The centraliser of $a$ is $T$, and there is an open $U \subset G(\mathbb{R})$ containing $a$ such that for every $b$ in $U$ the centraliser of $b$ is a conjugate of $T$ (take $U$ the connected component of $a$ in $G^{\mathrm{reg}}(\mathbb{R})$). Now take $b$ in $U(\mathbb{Q})$ (we know it is dense in $U(\mathbb{R})$), and let $g \in G(\mathbb{R})$ such that $gTg^{-1}$ is the centraliser of $b$. Then $g{\cdot}h$ is special, as $(g{\cdot}h)\mathbb{S} \subset (Z_G(b))\mathbb{R}$. Of course, this is a very general argument.