May 7, 2013. III-seminar. Bas Edixhoven.

Algebraic de Rham cohomology. 2x 45 minutes.

2 reasons for this talk: 1. lack of HS, VHS in the "Shafarevich seminar"
(that seminar is about families of alg. var's being constant or not, and so it is good to think of what constant means for the cohomology + HS, i.e., VHS).
2. Upcoming lectures of Barghav Bhatt on comparison crystalline–de Rham.

References: Grothendieck's letter to Atiyah, 1966
Voisin's book "Hodge theory & complex alg. geom I" (analytic theory).
The first hits when googling "algebr. de Rham cohom".
Deligne's Théorèmes de Lefschetz et critères de dégén. ...
Théorie de Hodge I, II, III.
EGA III for hypercohomology.
Katz & Oda, On the differentiation of de Rham coh.... (1968)
André & Baldassarri: de Rham cohom. of diff. modules.
Deligne: Equations différentielles à points singuliers réguliers.
Deligne–Illusie: Relèvement mod. p^2 et décomp. du complexe...
(or Oesterlé's Bourbaki lecture on this).
General books on homological algebra: Beilinson–Manin, Methods of ....

Let's start.

Let X be a C^∞-manifold real. $\forall\, U \subset X$: $\mathcal{O}(U) = \{f: U \to \mathbb{R} : f \text{ is } C^\infty\}$.
We have $H^\cdot_{sing}(X)^{\mathbb{Z}}$, $H_\cdot^{sing}(X)^{\mathbb{Z}}$, but also $H^\cdot(X, \mathbb{Z}_X)$, $H^\cdot_{sing}(X;\mathbb{Z}) \xrightarrow{\sim} H^\cdot(X, \mathbb{Z}_X)$
functorial comparison isomorphism.
$H^\cdot(X, \mathbb{R}_X) \leftarrow \mathbb{R} \otimes H^\cdot(X, \mathbb{Z}_X)$ is an isomorphism if X has a finite good cover.
(consider $X = \mathbb{N}$ + discr. top.)
Also $H^\cdot_{sing}(X;\mathbb{R}) \xrightarrow{\sim} H^\cdot(X, \mathbb{R}_X)$. But now we also have the de Rham complex: $(\Omega^\cdot_{X/\mathbb{R}}, d)$, it is a resolution of $\mathbb{R}_X = \ker(\Omega^0_X \to \Omega^1_X)$ (Poincaré lemma). or soft
If X is Hausdorff and paracompact then the sheaves Ω^j_X are fine, hence $\forall\, i>0$, $H^i(X, \Omega^j) = 0$ and $(\Omega^\cdot_X, d)$ is a $\Gamma(X,-)$ acyclic resolution of $\mathbb{R}_X$,
hence $H^\cdot(X, \mathbb{R}_X) = H^\cdot(\Omega^\cdot_X(X), d) =: H^\cdot_{dR}(X)$ (de Rham comparison thm).
This is compatible with:

$$H_\cdot^{sing}(X, \mathbb{R}) \times H^\cdot_{dR}(X) \longrightarrow \mathbb{R}, \quad (\underset{\text{smooth}}{\gamma: \Delta_k \to X}, \underset{\Omega^k(X)}{\omega}) \mapsto \int_{\Delta_q} \gamma^*\omega.$$

Nice: $(\Omega^\cdot(X), d)$ gives us $H^\cdot_{sing}(X, \mathbb{R})$.

But $(\Omega^\bullet(X), d)$ is still horribly big!

Let now X be a ~~smooth~~ nonsingular complex algebraic variety.
(i.e., a $\mathbb{C}$-scheme that is smooth, of finite type and separated).
Then we have Kähler diff. $d: \mathcal{O}_X \to \Omega^1_X$.
This can be extended: $\forall i \in \mathbb{Z}$: $\Omega^i_X := \Lambda^i \Omega^1_X$,

$d: \Omega^i_X \to \Omega^{i+1}_X$: $d(\omega \cdot \eta) = (d\omega)\cdot\eta + (-1)^i \omega \cdot (d\eta)$ $\quad (\omega \in \Omega^i_X(U) \ldots)$

$\Omega^\bullet_X := \bigoplus_{i \in \mathbb{Z}} \Omega^i_X$, graded comm. algebra over $\Omega^0_X = \mathcal{O}_X$ with differential d. We have $d^2 = 0$.

We would like to do the same as before: $(\Omega^\bullet(X), d)$, cohom.,
but there are 3 problems:
0. $(\Omega^\bullet, d)$ is not a resolution at any $x \in X$ s.t. $\dim_x(X) > 0$
1. we are working here with the Zariski topology, so $H^\bullet(X, \mathbb{C}_X)$ is not what we want
2. the sheaves Ω^j_X are not nec. acyclic for $\Gamma(X, -)$ (but they are if X is affine).

Nevertheless there is this fantastic theorem by Grothendieck.

<u>Thm.</u> Let X be an <u>affine</u> non-singular complex algebraic variety.
Then $H^\bullet(X^{an}, \mathbb{C}_{X^{an}}) = H^\bullet(\Omega^\bullet(X), d) =: H_{dR}(X)$ (<u>algebraic de Rham cohom</u>)
(this means: functorial isom., here are 2 of them, and they are compatible:
1. on X^{an}, $(\Omega^\bullet_{X^{an}}, d)$ is a resolution of $\mathbb{C}_{X^{an}}$, and the $\Omega^j_{X^{an}}$ are $\Gamma(X^{an}, -)$-acyclic, hence $H^\bullet(X^{an}, \mathbb{C}_{X^{an}}) = H^\bullet(\Omega^\bullet(X^{an}), d) \leftarrow H^\bullet(\Omega^\bullet(X), d)$.
2. $H^{sing}_\bullet(X^{an}, \mathbb{C}) \times H_{dR}(X)$, $(\gamma: \Delta_k \to X^{an}, \omega) \mapsto \int_{\Delta_k} \gamma^* \omega$.

The proof uses: definition of algebr. de Rham cohom. for general X, compactification, resolution of singularities, GAGA.

So, of our 3 problems, only 2 is serious. And there is a standard way of dealing with this in homological algebra: extend derived functors from objects to complexes, that is "derived categories", of which the objects are complexes. The philosophy is: if $(A^\bullet, d)$ is a complex in an ab. cat. $\mathcal{A}$, ~~F~~ $F: \mathcal{A} \to \mathcal{B}$ is a left exact functor, then take $A^\bullet \to B^\bullet$ a quasi-isomorphism, s.t. the B^j are F-acyclic, and put $(RF)A^\cdot = F(B^\cdot)$.

In our case: $X/\mathbb{C}$ smooth, separated, finite type, this gives the following recipe.
Let $(U_i)_{0\le i\le d}$ be an affine open cover ($d = \dim X$),

consider the double complex (Čech–de Rham) of $\mathbb{C}$-vector spaces:

$$(\mathcal{C}^p\Omega^q)(X) = \bigoplus_{i_0<\dots<i_p} \Omega^q(U_{i_0}\cap\dots\cap U_{i_p})$$

$$d : (\mathcal{C}^p\Omega^q)(X) \to (\mathcal{C}^p\Omega^{q+1})(X) : \quad d : \Omega^q \longrightarrow \Omega^{q+1}$$

$$\delta : (\mathcal{C}^p\Omega^q)(X) \to (\mathcal{C}^{p+1}\Omega^q)(X) : (\delta\omega)_{i_0,\dots,i_{p+1}} = (-1)^q \sum_j (-1)^{j+1}\, \omega_{i_0,\dots,\hat{i_j},\dots,i_{p+1}}|U_{i_0,\dots,i_{p+1}}.$$

With these choices: $d\delta + \delta d = 0$.
Form the total complex $\mathrm{Tot}^n((\mathcal{C}^\cdot\Omega^\cdot)X) := \bigoplus_{p+q=n} (\mathcal{C}^p\Omega^q)X$, $d = d + \delta$.

Then $H^\bullet_{dR}(X) := H^\bullet(\mathrm{Tot}^\cdot((\mathcal{C}^\cdot\Omega^\cdot)X), d) =: \mathbb{H}^\cdot(X, \Omega^\cdot_X)$

hypercohomology.

This is independent of choices.

Thm. Let $X/\mathbb{C}$ smooth separated, quasicompact. $H^\cdot_{dR}(X) = H^\cdot(X^{an}, \mathbb{C}_X)$. For X projective this is "easy".
(comparison of 2 spectral sequences).

Nice consequence: also the Hodge filtration on $H^\cdot(X^{an}, \mathbb{C})$ (for X proper) can be defined algebraically.
For $i\in\mathbb{Z}$: $\Omega^{\geq i}_X \subset \Omega^\cdot_X$ is a decreasing filtration of $\Omega^\cdot_X$.

This gives: $\mathcal{C}^\cdot\Omega^{\geq i} \to \mathcal{C}^\cdot\Omega^\cdot$, $\mathrm{Tot}^\cdot(\mathcal{C}^\cdot\Omega^{\geq i}) \to \mathrm{Tot}^\cdot(\mathcal{C}^\cdot\Omega^\cdot)$,

$$\mathbb{H}^\cdot(X, \Omega^{\geq i}) \to \mathbb{H}^\cdot(X, \Omega^\cdot) = H^\cdot_{dR}(X) \hookleftarrow \mathrm{Fil}^i(H^\cdot_{dR}(X))$$

Thm. (Hodge decomp.) Let $X/\mathbb{C}$ projective, smooth.

~~These~~ We have $\mathrm{Fil}^p H^n_{dR}(X) =: \mathrm{Fil}^p H^n(X^{an},\mathbb{C})$.

Let $H^{p,\cancel{q}\,n-p}(X^{an}) := \mathrm{Fil}^p H^n(X^{an},\mathbb{C}) \cap \overline{\mathrm{Fil}^{n-p} H^n(X^{an},\mathbb{C})} \subset H^n(X^{an},\mathbb{C})$

then $H^n(X^{an},\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X^{an})$, and $H^{p,q}(X^{an}) = H^q(X,\Omega^p)$.

The proof of this is difficult (no algebraic proof is known) ~~(no proof known that doesn't~~

But ~~it~~ the ~~statement~~ following consequence is ~~equivalent to that an~~ algebraic, ~~statement~~ degeneration at $E_1^{\bullet}$ of Hodge-de Rham spectral sequence and has an algebraic proof.

$\forall p \in \mathbb{Z}$; we have $0 \to \Omega^{>p} \to \Omega^{\bullet} \to \Omega^p[-p] \to 0$

and all sequences

$$0 \to \mathbb{H}^i(X,\Omega^{>p}) \to \mathbb{H}^i(X,\Omega^{\geq p}) \to H^{i-p}(X,\Omega^p) \to 0$$

are exact.

Example/exercise. Let X be a projective nonsing. connected curve.

$X = U_0 \cup U_1$.

$$\begin{array}{cccccc}
f_0 - f_1 & \mathcal{O}(U_{0,1}) & \xrightarrow{d} & \Omega^1(U_{0,1}) & -\omega_0+\omega_1 \\
\uparrow & \uparrow & & \uparrow & \uparrow \\
(f_0,f_1) & \mathcal{O}(U_0)\oplus\mathcal{O}(U_1) & \xrightarrow{d} & \Omega^1(U_0)\oplus\Omega^1(U_1) & (\omega_0,\omega_1)
\end{array}$$

Filter the double complex by columns; gives ~~filtration~~ a s.e.s. of total complexes:

$$\big(0 \to \Omega^1(U_0)\oplus\Omega^1(U_1) \to \Omega^1(U_{0,1})\big) \rightarrowtail \mathrm{Tot}^\bullet(C^\bullet\Omega^\bullet) \twoheadrightarrow \big(\mathcal{O}(U_0)\oplus\mathcal{O}(U_1) \to \mathcal{O}(U_{0,1}) \to 0\big)$$

hence a long ex. seq:

$$0 \to H^0_{dR}(X) \to \mathbb{C} \xrightarrow{0} \Omega^1(X) \to H^1_{dR}(X) \to H^1(X,\mathcal{O}_X) \xrightarrow{0} H^1(X,\Omega^1) \to H^1_{dR}(X) \to 0$$

Filter by rows: ... .