"Shafarevich" seminar. Leiden, 2013/05/21; 45 minutes.

Families, VHS, Gauss-Manin connection.

Let $S/\mathbb{C}$ be a nonsing. alg. var., $f\colon X \to S$ smooth and projective.
Then $f^{an}\colon X^{an} \to S^{an}$ is, topologically a fiber bundle: locally on S^{an} it is a product: $\forall s \in S^{an}\ \exists\ U$ open neighb., $\exists\ X_s^{an} \times U \xrightarrow{\sim} f^{an,-1}U$ (over U).

Hence the $(R^n f_*^{an})\mathbb{Z}_{X^{an}} = \left(U \mapsto H^n(f^{an,-1}U, \mathbb{Z})\right)^{\#}$ are locally constant sheaves of $\mathbb{Z}$-modules on S^{an}: $\forall s \in S^{an}\ \exists$ ~~[illegible]~~ open neighb. U s.t. ~~[illegible]~~ $\mathcal{F}(U) \to \mathcal{F}_s$ is an isom, and $(\mathcal{F}_s)_U \to \mathcal{F}|_U$ isom.
(if S connected: repr. of π_1.)

We have: ~~$\mathcal{O}_{S^{an}} \otimes_{\mathbb{C}_{S^{an}}} (R^n f_*^{an})\mathbb{C}_{X^{an}}$~~ $\mathbb{C}_{S^{an}} \rightarrowtail \mathcal{O}_{S^{an}}$, $\underbrace{\mathbb{C}_{X^{an}} = f^{-1}\mathbb{C}_{S^{an}}}_{\text{loc. const.}} \hookrightarrow \underbrace{f^{-1}\mathcal{O}_{S^{an}}}_{\text{loc. const.}}$

$$(R^n f_*^{an})\,\mathbb{C}_{X^{an}} \longrightarrow (R^n f_*)(f^{-1}\mathcal{O}_{S^{an}}) = (R^n f_*^{an})(\Omega^{\bullet}_{X^{an}/S^{an}}) \overset{\text{def}}{=} H^n_{dR}(X^{an}/S^{an})$$

on fibres.

locally: $\left(H^n(X_s^{an}, \mathbb{C})\right)_U \longrightarrow \mathcal{O}_{S^{an}}|_U \otimes_{\mathbb{C}} H^n_{dR}(X_s^{an})$

So:
$$\mathcal{O}_{S^{an}} \otimes_{\mathbb{C}_{S^{an}}} \underbrace{(R^n f_*^{an})\,\mathbb{C}_{X^{an}}}_{\substack{\text{loc. const. sheaf}\\ \text{of } \mathbb{C}\text{-vect. spaces.}}} \xrightarrow{\ \sim\ } \underbrace{H^n_{dR}(X^{an}/S^{an})}_{\text{loc. free } \mathcal{O}_{S^{an}}\text{-module}} = \left(H^n_{dR}(X/S)\right)^{an}.$$

So $\mathcal{E} := H^n_{dR}(X/S)$ is a loc. free $\mathcal{O}_S$-module of finite rank, and it has the following extra structure: $\mathcal{E}^{an}$ is naturally of the form $\mathcal{O}_{S^{an}} \otimes_{\mathbb{C}_{S^{an}}} \mathcal{F}$, with $\mathcal{F}$ a loc. const. sheaf of $\mathbb{C}$-vect. spaces.

Question Can this extra structure on $\mathcal{E}$ be described algebraically?

Yes: the Gauss-Manin connection.

We have:
$$f \otimes s \longmapsto (df) \otimes s$$
$$0 \to \mathcal{F} \to \mathcal{O}_{S^{an}} \otimes_{\mathbb{C}_{S^{an}}} \mathcal{F} \xrightarrow{\ \nabla\ } \Omega^1_{S^{an}} \otimes_{\mathcal{O}_{S^{an}}} \mathcal{O}_{S^{an}} \otimes_{\mathbb{C}_{S^{an}}} \mathcal{F}$$
$$\mathcal{E} \longrightarrow \Omega^1_{S^{an}} \otimes_{\mathcal{O}_{S^{an}}} \mathcal{E}$$

Think of local triv. of $\mathcal{F}$:
$\mathcal{O}_{S^{an}}^r \longrightarrow (\Omega^1_{S^{an}})^r$
$(f_1, \ldots, f_r) \mapsto (df_1, \ldots, df_r)$

∇ is a connection on $(\mathcal{E}, X \to S)$: $\nabla(f.s) = f.(\nabla s) + (df) \otimes s$

∇ is integrable: $\nabla^2 = 0$; $\quad \mathcal{E} \xrightarrow{\nabla} \Omega^1 \otimes \mathcal{E} \xrightarrow{\nabla} \Omega^2 \otimes \mathcal{E} \to \cdots$

∇^2 is $\mathcal{O}_X$-linear, curvature of ∇.

For ξ a holom. vector field on S^{an}. $\xi: \Omega^1_{S^{an}} \to \mathcal{O}_{S^{an}}$, $\mathcal{O}_{S^{an}}$-linear,

we get $\nabla_\xi : \mathcal{E} \xrightarrow{\nabla} \Omega^1_{S^{an}} \otimes_{\mathcal{O}_{S^{an}}} \mathcal{E} \xrightarrow{\xi \otimes id} \mathcal{E}$ is a derivation on $\mathcal{E}$:

(1st order diff. op.)

$(\nabla_\xi)(f.s) = f.\nabla_\xi s + (\xi f).s.$ (phycisist language: covariant derivative).

Thm. $\exists!$ connection $\nabla: H^\bullet_{dR}(X/S) \longrightarrow \Omega^1_S \otimes_{\mathcal{O}_S} H^\bullet_{dR}(X/S)$ inducing the one over X^{an} above.

Algebraic construction: see Katz-Oda for details, or Voisin.

Let $A := \Omega^\bullet_X = \Lambda^\bullet_{\mathcal{O}_X} \Omega^1_X$, sheaf of differential graded comm. alg. on X

$B := \Omega^\bullet_{X/S} = \Lambda^\bullet_{\mathcal{O}_X} \Omega^1_{X/S}$, ______________________.

$I :=$ the diff. ideal in A generated by $f^*\Omega^1_S$: $f^*\Omega^1_S \rightarrowtail \Omega^1_X \twoheadrightarrow \Omega^1_{X/S}$.

Then $B = A/I$.

We filter A by powers of I: $A \supset I \supset I^2 \supset \cdots$ ($I^n = 0$ if $n > \dim S$).

We have the s.e.s. on X: $I/I^2 \rightarrowtail A/I^2 \twoheadrightarrow B$,

giving:
$$\begin{array}{ccc} (\mathbb{R}^n f_*) B & \longrightarrow & (\mathbb{R}^{n+1} f_*)(I/I^2) \\ \| & & \| \leftarrow \text{I will explain why, a bit.} \\ H^n_{dR}(X/S) & \xrightarrow{\nabla} & \Omega^1_S \otimes_{\mathcal{O}_S} H^n_{dR}(X/S) \end{array}$$

Locally: assume S affine, $t_1, \dots, t_d \in \mathcal{O}(S)$ "coordinates": $S \xrightarrow{t} \mathbb{A}^d_{\mathbb{C}}$ étale.

let $U \subset X$ affine open, $x_1, \dots, x_r \in \mathcal{O}(U)$ relative coordinates: $U \xrightarrow{x} \mathbb{A}^r_S$ étale.

Then $\Omega^q_U = \Lambda^q_{\mathcal{O}_U} \Omega^1_U$ has $\mathcal{O}_U$-basis $(dt_{i_1} \cdots dt_{i_{q_1}} dx_{j_1} \cdots dx_{j_{q_2}})_{q_1+q_2=q}$

$1 \le i_1 < \cdots < i_{q_1} \le d$

$1 \le j_1 < \cdots < j_{q_2} \le r.$

$I \cap \Omega^q_U$: $q_1 \geq 1$

$I^2 \cap \Omega^q_U$: $q_1 \geq 2$

$\dfrac{I \cap \Omega^q_U}{I^2 \cap \Omega^q_U}$: $q_1 = 1$: $(dt_i\, dx_{j_1} \cdots dx_{j_{q-1}})_{1 \le i \le d,\ 1 \le j_1 < \cdots < j_{q-1} \le r}$ — also a basis for $f^*\Omega^1_S \otimes_{\mathcal{O}_U} \Omega^{q-1}_{U/S}$

Relation with Kodaira-Spencer morphism / class

$KS(f) \in \left(\Omega^1_S \otimes_{\mathcal{O}_S} (R^1 f_*)(T_{X/S})\right)(S)$. (recall from lecture 7 in the seminar).

$$
\begin{tikzcd}
H^n_{dR}(X/S) \arrow[r, "\nabla_{X/S}"] & \Omega^1_S \otimes_{\mathcal{O}_S} H^n_{dR}(X/S) \\
\mathrm{Fil}^p \arrow[u, phantom, "\cup"] \arrow[r, "\nabla_{X/S}"] \arrow[d, two heads] & \Omega^1_S \otimes_{\mathcal{O}_S} \mathrm{Fil}^{p-1} \arrow[u, phantom, "\cup"] \arrow[d, two heads] \\
\mathrm{Gr}^p \arrow[r, "\mathrm{Gr}^{-1}\nabla_{X/S} \text{ is } \mathcal{O}_S\text{-linear}"] \arrow[d, equal] & \Omega^1_S \otimes_{\mathcal{O}_S} \mathrm{Gr}^{p-1} \arrow[d, equal] \\
(R^{n-p} f_*)\Omega^p_{X/S} \arrow[r] & \Omega^1_S \otimes_{\mathcal{O}_S} (R^{n-p+1} f_*)\Omega^{p-1}_{X/S}
\end{tikzcd}
$$

(Griffiths transversality, direct consequence of the alg. constr. of ∇)

induced by $KS(f)$.

Using this, one can study the question: $\nabla_{X/S} = 0 \overset{?}{\Longrightarrow} KS(f) = 0$.

We have that for curves, abelian schemes, for example.

<u>Example</u>: Let X/S be a <u>curve</u>, $n=1$, $p=1$.

Then $\mathrm{Gr}^{-1}\nabla_{X/S} : f_* \Omega^1_{X/S} \longrightarrow \Omega^1_S \otimes_{\mathcal{O}_S} (R^1 f_*)\mathcal{O}_X$

$\mathrm{Gr}^{-1}\nabla_{X/S} \in \left((f_* \Omega^1_{X/S})^\vee \otimes_{\mathcal{O}_S} (f_* \Omega^1_{X/S})^\vee \otimes_{\mathcal{O}_S} \Omega^1_S\right)(S)$ (use Serre duality)

$$KS(f) \in \left(((R^1 f_*) T_{X/S}) \otimes_{\mathcal{O}_S} \Omega^1_S\right)(S) = \left(\left(f_*(\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^1_{X/S})\right)^\vee \otimes_{\mathcal{O}_S} \Omega^1_S\right)(S)$$

So $KS(f) \longmapsto \mathrm{Gr}^{-1}\nabla_{X/S}$ via $(f_* \Omega^1_{X/S}) \otimes_{\mathcal{O}_S} (f_* \Omega^1_{X/S}) \longrightarrow f_*(\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^1_{X/S})$, and we know that this is surjective, hence the dual is <u>injective</u>.