Leiden, 2013/10/25. Bas Edixhoven

The André–Oort conjecture for A_g, under GRH, after Pila–Tsimerman.

Reference: arxiv, Pila–Tsimerman, "Ax–Lindemann for A_g".

Let $g \in \mathbb{Z}_{\geq 0}$.

$$A_g = \{ (A,\lambda) : A \text{ complex ab. var. dim. } g,\ \lambda : A \to A^t \text{ princ. pol.} \} / \cong .$$

Note: for such (A,λ), $H_1(A,\mathbb{Z}) \cong \mathbb{Z}^{2g}$, $\lambda \sim$ perfect alternating bil. form on $H_1(A,\mathbb{Z})$, and $(H_1(A,\mathbb{Z}),\lambda) \cong (\mathbb{Z}^{2g},\psi)$, $\psi(x,y) = x^t \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot y$, isom. unique up to $Sp_{2g}(\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^{2g},\psi)$.

Also: $A = H_1(A,\mathbb{Z})_{\mathbb{R}} / H_1(A,\mathbb{Z})$, and $\mathbb{C} \xrightarrow{h} \mathrm{End}_{\mathbb{R}}(H_1(A,\mathbb{Z})_{\mathbb{R}})$,
and $\forall z \in \mathbb{C}$, $\forall x,y \in H_1(A,\mathbb{Z})_{\mathbb{R}}$: $\lambda(zx,zy) = z\bar{z}\cdot\lambda(x,y)$, $\lambda(ix,x) \geq 0$.
hence: $h : \mathbb{C}^\times \to GSp(H_1(A,\mathbb{Z})_{\mathbb{R}}, \lambda)$. "$h$ is a HS. of type $\{(-1,0),(0,-1)\}$ s.t. λ is a polarisation". All possible h's: 1 orbit under the action of $GSp(H_1(A,\mathbb{Z})_{\mathbb{R}},\lambda)$. We take as base point: $(\mathbb{C}/\mathbb{Z}(i))^g$, $V_{\mathbb{Z}} = \mathbb{Z}^{2g}$, $\psi : \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
So: $A_g = \Gamma \backslash X^+$, $X^+ \subset \mathrm{Hom}(\mathbb{C}^\times, GSp_{2g}(\mathbb{R}))$ the $Sp_{2g}(\mathbb{R})$-orbit of $h_0 : a+bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $\Gamma = Sp_{2g}(\mathbb{Z})$. $X^+ = Sp_{2g}(\mathbb{R})/\text{stab of } h_0 = U_g(\mathbb{R})$

More familiar, perhaps: $X^+ \xrightarrow{\sim} \mathbb{H}_g^+ = \{\tau \in M_{g,g}(\mathbb{C}) : \tau^t = \tau,\ \mathrm{Im}(\tau) \text{ pos. def.}\}$

$$h \longmapsto \tau \text{ s.t. } V_h^{0,-1} = \{(\tau y, y) : y \in \mathbb{C}^g\} \subset \mathbb{C}^{2g}.$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot h_0 \longmapsto (ai+b)(ci+d)^{-1}.$$

"$\leq$", $\mathbb{R}$-points!

Note: $\mathbb{H}_g^+ \subset M_{g,g}^+(\mathbb{C})$ is open, semi-algebraic.
$$M_{g,g}^+(\mathbb{C}) = \mathbb{R}^{g^2+g} = \mathbb{A}^{g^2+g}(\mathbb{R})$$ — this will be used!

For $h \in X^+$: $MT(h)$ = smallest $H \subset GSp_{2g,\mathbb{Q}}$ (alg. subgrp) s.t.
$$\begin{tikzcd} & H(\mathbb{R}) \arrow[d, hook] \\ \mathbb{C}^\times \arrow[ur] \arrow[r, "h"] & GSp_{2g}(\mathbb{R}) \end{tikzcd}$$

Special subvarieties of A_g.

For $H \subset GSp_{2g}$ an alg. subgr., $\{x \in X^+ : x(\mathbb{C}^\times) \subset H(\mathbb{R})\}$ is a finite union of $H(\mathbb{R})^+$-orbits, the images in A_g of such X_H^+ are closed irred. alg. subv, called special.
$\forall x \in X^+$: $q(MT(x)(\mathbb{R})\cdot x)$ is the smallest special subv. of A_g containing $q(x)$.
$$q : X^+ \to A_g$$
$\{q(x)\}$ special $\iff MT(x)$ is a torus $\iff x$ is a CM point.

<u>A-O conjecture for A_g</u>: Let $\Sigma \subset A_g$ be a set of special points. Then each irreducible component of Σ^{Zar} is a special subv. of A_g.

<u>Thm</u> (Pila–Tsimerman). GRH for CM-fields $\Longrightarrow$ A-O for A_g.
Inconditional for $g \leq 6$.

An important intermediate step in their proof is the following theorem, called Ax-Lindemann(-Weierstrass). But to state it, we need the notion of weakly special subvariety of A_g. (Notation: $X^+ \xrightarrow{q} A_g$.), and of alg. subv. of $X^+ = \mathbb{H}_g^+$.

<u>Def</u>. Let $V \subset A_g$ be an irred. Zar. closed . Then V is <u>weakly special</u> if $\exists\ x \in X^+$, a decomposition $MT(x)^{ad} = M_1 \times M_2$, such that $V = q(M_1(\mathbb{R})^+ \cdot x)$.
(Moonen gave other characterisations: totally geodesic, "linear"...)
<u>Example</u> • for $g = g_1 + g_2$, $A_{g_1} \times A_{g_2} \hookrightarrow A_g$, $\forall\ y \in A_{g_2}$, $A_{g_1} \times \{y\}$ is weakly sp.
• Points are weakly special.

(*)

<u>Def</u>. Let $Y \subset \mathbb{H}_g \subset \mathbb{C}^{(g^2+g)/2}$. Then Y is irreducible algebraic if Y is an analytic irreducible component of $Y^{Zar} \cap \mathbb{H}_g$.

<u>Ax-Lindemann Thm</u> (Pila–Tsimerman). Let $V \subset A_g$ be Zariski closed, and $W \subset q^{-1}V \subset \mathbb{H}_g$ <u>maximal</u> irreducible algebraic. Then $q(W)$ is weakly spec.

Background, motivation for the name:

Lindemann (≈ 1885): $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ $\mathbb{Q}$-lin. indep. $\Rightarrow e^{\alpha_1}, \dots, e^{\alpha_n}$ alg. indep.

Ax (1971): let $\pi : \mathbb{C}^n \to (\mathbb{C}^\times)^n$, $(z_1, \dots, z_n) \mapsto (e^{z_1}, \dots, e^{z_n})$, let $V \subset \mathbb{C}^{\times, n}$ Zariski closed. Let $W \subset \pi^{-1}V$ be maximal irreducible algebraic.
Then $\exists\ z \in \mathbb{C}^n$, $\exists\ U \subset \mathbb{Q}^n$ sub-$\mathbb{Q}$-vect. sp. s.t. $W = z + U_{\mathbb{C}}$.

<u>Remarks</u> 1. Let $V \subset A_g$ be an irred. closed subv., $x \in X^+$ s.t. $q(x) \in V$.
Let $H \subset Sp_{2g,\mathbb{Q}}$ be the connected alg. monodr. gr. of (V, x).
Then $q(H(\mathbb{R})^+ \cdot x)$ is the smallest weakly special containing V.

2. special = image of a Shim. morph. of Shim. var's.
w. sp. = also fibres of ______ .

3. w. sp + special point $\Longrightarrow$ special.

On the proof of Ax-Lindemann.

Induction on $\dim V$. We may assume $\dim W > 0$.
Let $V \subset A_g$ be irred. Zar. closed, $W \subset \bar{q}^{-1}V$ maximal irred. algebraic.
We may assume V minimal: replace V by $(qW)^{Zar}$.
Then to prove: $qW = V$.
Let $w \in W$, $H \subset Sp_{2g,\mathbb{Q}}$ the connected algebr. monodr. gr. of (V, w)

$$\begin{tikzcd}
 & \Gamma_Y \circlearrowright \text{ irred. anal. comp.} & & H(\mathbb{Z})^+ \circlearrowright & \circlearrowright \Gamma = Sp_{2g}(\mathbb{Z}) \\
W \subset & Y \subset & \bar{q}^{-1}V \arrow[d] \subset & H(\mathbb{R})^+ \cdot w \arrow[d, "\bar{q}"] \subset & \mathbb{H}_g^+ \arrow[d, "q"] \\
 & & V \subset & q(H(\mathbb{R})^+ \cdot w) \subset & A_g
\end{tikzcd}$$

To show: $W = H(\mathbb{R})^+ \cdot w$, so: W is stable under many elements of $H(\mathbb{R})^+$.
My method in such cases was to find a non-discrete group, using Hecke corr.
Pila's method: use his thm. about heights of rat'l points on analytic var's.

$$X := \{ \gamma \in H(\mathbb{R})^+ : \dim\big((\gamma \cdot W) \cap Y \cap \underset{\text{for } H(\mathbb{Z})^+}{\text{fund. dom.}}\big) = \dim W \}$$

This is a definable subset of $H(\mathbb{R})^+ \subset \mathbb{R}^{4g^2}$, in the "o-minimal structure $\mathbb{R}_{an,exp}$", by Peterzil-Starchenko.

Let X^{alg} := union of all pos. dim. connected semi-alg. $Z \subset X$
Pila-Wilkie: $\forall \varepsilon > 0\ \exists c$ s.t. $\forall T > 0$ $\#\{x \in X(\mathbb{Q}) : H(x) < T,\ x \notin X^{alg}(\mathbb{Q})\} < c \cdot T^{\varepsilon}$.

On the other hand: $\exists \varepsilon > 0, \exists c > 0$ s.t. $\forall T > 0$ $\#\{x \in X(\mathbb{Z}) : H(x) < T\} > c \cdot T^{\varepsilon}$.
(ideas by Ullmo-Yafaev) these come from Γ_Y!

Conclusion: $\exists Z \subset X$ semi alg, connected, pos. dim., s.t. $W \subset Z \cdot W \subset Y$,
and by maximality of W: $Z \cdot W = W$. Etc.

On the proof of André–Oort (under GRH).

Let $V \subset A_g$ be irred. and Zar. closed, s.t. $\Sigma := \{x \in V : x \text{ special}\}$ is Zar. dense.
To prove: V is special.
Induction on $\dim V$. May assume $\dim V > 0$. Hence $\#\Sigma = \infty$.
The field of def. of $V \subset A_g$ is finite $/\mathbb{Q}$, bec. all $x \in \Sigma$ are in $A_g(\overline{\mathbb{Q}})$.
Tsimerman proved (under GRH): $\exists\, \varepsilon > 0, \exists\, c > 0$ s.t. $\forall\, x \in A_g(\overline{\mathbb{Q}})$ special,
$\#(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x) \geq c \cdot |\mathrm{discr}(\mathrm{center}(\mathrm{End}(A_x)))|^{\varepsilon}$.

Pila-Wilkie applied to $q^{-1}V \cap$ fund. domain for $\mathrm{Sp}_{2g}(\mathbb{Z})$, inv. images of special points, + Ax-Lindemann! : for almost all $x \in \Sigma$, $\exists$ special $W \subset V$ with $x \in W$, $\dim W > 0$.

Hence: the union of the pos. dim. special $W \subset V$ is Zar. dense in V,
hence the union of the pos. dim. weakly special too!
Now 3 lemmas.
Lemma 1. $\bigcup \{W \subset V : W \text{ weakly special}, \dim W > 0\} =: Z$ is a countable union of algebraic closed subv. of V, indexed by $H \subset \mathrm{Sp}_{2g,\mathbb{Q}}$ semisimple alg. subgroups.
Lemma 2. $Z \subset V$ is definable in $\mathbb{R}_{an,exp}$.
Proof: Only finitely many $H_{\mathbb{R}} \subset \mathrm{Sp}_{2g,\mathbb{R}}$ up to conj. to consider.
Lemma 3. Z is a finite union, only finitely many H's $\subset \mathrm{Sp}_{2g,\mathbb{Q}}$ necessary.

Conclusion: $\exists\, (H, X_H^+) \subset (\mathrm{GSp}_{2g}, \mathcal{H}_g^+)$ sub Shim. datum conn., $H^{ad} = H_1 \times H_2$, $X_H^+ = X_1 \times X_2$
an alg. subv. $V' \subset \Gamma_2 \backslash X_2^+$ s.t.
$V = (\Gamma_1 \backslash X_1^+) \times V'$.
Then use the induction hypothesis on V'!

give some details: this uses Ax-Lindemann again!

$$Z = q\Big(\bigcup_{\substack{F \subset \mathrm{Sp}_{2g,\mathbb{R}} \text{ up to conjugacy} \\ \gamma \in \mathrm{Sp}_{2g}(\mathbb{R})}} \big(\gamma F(\mathbb{R})^+_{\gamma^{-1}}\text{-orbits contained in } q^{-1}V\big)\Big)$$

because each maximal such orbit is weakly special by Ax-Lindemann.

Further developments:

- Ax-Lindemann for arbitrary Shimura varieties (of ab. type?) by Klingler-Ullmo-Yafaev, arxiv, "recent".

- This will imply André-Oort (under GRH) very probably but some work with heights of CM-points in X^+ still needs to be done.

- ~~The whole thing~~ Ax-Lindemann for mixed Shim. varieties : Ziyang Gao! very recent on arxiv.