Leiden, 2014/03/07, ICS on P.S.

Almost mathematics, 1 hour, Bas Edixhoven.

References: §4 of "Perfectoid"
Almost ring theory, book by Gabber-Ramero, ch. 2-3.

I. Introduction.

Goal: to explain what Thm 4.16 (f.ét. in "almost" cat.) and Thm. 4.17 (f.ét and deformation theory) of "Perfectoid" mean, and to say something about proofs in special cases.

Recall the situation: K a perfectoid field, we want equivalences:

$$K_{fet} \cong O^a_{K,fet} \cong (O^a_K/\pi)_{fet} = (O^a_{K^\flat}/\pi^\flat)_{fet} \cong O^a_{K^\flat,fet} \cong K^\flat_{fet}$$

a special case of deformation theory of finite étale covers,

a special case of Faltings's "almost purity", to be generalised in Thm. 7.9 of "Perfectoid". (and proved by "tilting methods").
This case is proved at the end of §5 of "Perfectoid".

Origin of it all: Tate's Driebergen 1967 article on p-div. groups.
Faltings's "p-adic HT" 1988 and "Almost ét. ext" 2002.

Thm. (Tate) Let $A/\mathbb{Z}_p$ be an ab. scheme, $i \geq 0$. Then $\exists$ natural $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-equiv. isom. of $\mathbb{C}_p$-vect. sp:

$$\mathbb{C}_p \otimes_{\mathbb{Z}_p} H^i(A_{\overline{\mathbb{Q}}_p, et}, \mathbb{Z}_p) \xrightarrow{\sim} \bigoplus_{\substack{j+k=i \\ j,k \geq 0}} H^j(A_{\mathbb{Q}_p}, \Omega^k_{A_{\mathbb{Q}_p}/\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(-k).$$

Tate's philosophy: $\overline{\mathbb{Q}}_p$ is too big to understand, but
$\mathbb{Q}_p \subset \mathbb{Q}_p(\zeta_{p^\infty}) \subsetneq \overline{\mathbb{Q}}_p$ almost unramified. Then analysis.

Faltings: make it more geometrical, higher dimensions.

II. "Almost modules" Ch 2. of Gabber–Ramero.

<u>Assumptions</u>: V is a ring, $\mathfrak{m} \subset V$ an ideal with $\mathfrak{m}^2 = \mathfrak{m}$ such that $\tilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$ is a flat V-module. (these cond. are stable under ring maps $V \to W$).

<u>Exercise</u>. Check that for K perfectoid, $(O_K, \overset{\mathfrak{m}}{\text{max. id.}})$ have these properties. In fact $\mathfrak{m}$ is O_K-flat (use: $\mathfrak{m}$ is a directed colimit of free O_K-modules).

<u>Example</u> $\begin{cases} V/\mathfrak{m} \overset{a}{=} 0 \\ O_K/p \text{ is } \underline{\text{not}} \text{ almost } 0. \end{cases}$

<u>Def.</u> Let M be in V-mod.

M is <u>almost zero</u> if $\mathfrak{m} M = 0$.

For $f: M \to N$ in V-mod: f is an <u>almost isomorphism</u> if $\ker(f)$ and $\operatorname{coker}(f)$ are almost zero.

<u>Remarks</u>. 1. If $M \overset{a}{=} 0$ then $\mathfrak{m} \otimes_V M = \mathfrak{m}^2 \otimes_V M = 0$ $\quad x \cdot y \otimes m = x \otimes \underset{\overset{\|}{0}}{ym}$

2. If $\mathfrak{m}$ is flat$/V$, then $\mathfrak{m} \otimes_V (\mathfrak{m} \hookrightarrow V \twoheadrightarrow V/\mathfrak{m})$ is exact, hence $\tilde{\mathfrak{m}} \xrightarrow{\sim} \mathfrak{m}$.

3. For $f: M \to N$, $\quad f$ almost isom. $\iff$ $\mathrm{id} \otimes f: \tilde{\mathfrak{m}} \otimes_V M \to \tilde{\mathfrak{m}} \otimes_V N$ is an isomorphism.

<u>Proof</u>: $(\Rightarrow)$

$$\begin{tikzcd}
\ker(f) \arrow[r, hook] & M \arrow[r, "f"] \arrow[dr, two heads] & N \arrow[r, two heads] & \operatorname{coker}(f) \\
 & & f(M) \arrow[u, hook] &
\end{tikzcd}$$

$$\begin{tikzcd}
\underset{\overset{\|}{0}}{}(\mathfrak{m} \otimes_V \ker f) \arrow[r] & \mathfrak{m} \otimes_V M \arrow[r, "\mathrm{id} \otimes f"] \arrow[d, "\sim"] & \mathfrak{m} \otimes_V N \arrow[r] & \overset{0}{\overset{\|}{\mathfrak{m} \otimes_V \operatorname{coker}(f)}} \\
 & \mathfrak{m} \otimes_V fM \arrow[ur, "g"] & & \\
 & \ker(g) \arrow[u] & &
\end{tikzcd}$$

$\ker(g) \Leftarrow \operatorname{Tor}_1^V(\mathfrak{m}, \operatorname{coker} f) \overset{a}{=} 0$: it is a V-module via $V/\mathfrak{m}$.

So: $\underset{\overset{\|}{0}}{\mathfrak{m} \otimes_V \ker(g)} \to \tilde{\mathfrak{m}} \otimes_V M \overset{\sim}{\twoheadrightarrow} \tilde{\mathfrak{m}} \otimes_V N$ exact.

$\Leftarrow$: Now assume $\tilde{m}\otimes_V M \xrightarrow[\sim]{\mathrm{id}\otimes f} \tilde{m}\otimes_V N$. Use $m \overset{a}{\xrightarrow{\sim}} V$, hence $\tilde{m} \overset{a}{\xrightarrow{\sim}} V$,

So
$$\begin{tikzcd}
\tilde{m}\otimes_V M \arrow[r, "\sim"] \arrow[d, "\wr a"] & \tilde{m}\otimes_V N \arrow[d, "\wr a"] \\
M \arrow[r, "f"] & N
\end{tikzcd} \quad \boxtimes$$

$$\boxed{\text{u. } m\cdot \tilde{m}\otimes_V M = \tilde{m}\otimes_V M.}$$

<u>Def.</u> $\Sigma :=$ the full subcat. of V-mod of almost zero V-modules.

Then Σ closed under: ker, coker, im, $\underset{\text{finite}}{\oplus}$, and <u>extensions</u> ($m^2 = m$), hence a <u>thick</u> or <u>Serre</u> subcat.

Hence there is a standard "localisation procedure", for inverting almost isomorphisms. (Details: Weibel, Intr. to hom. alg., Exercise 10.3.2.)

$$V\text{-mod} \longrightarrow V\text{-mod}/\Sigma =: \overset{\text{almost}}{V^a}\text{-mod} \quad \text{"almost modules".}$$

note: $\nexists$ ring V^a (if $V = O_K$, m = max. id).

notation: $M \longmapsto M^a$

$ob(V^a\text{-mod}) = ob(V\text{-mod})$.

morphisms: $X^a \to Y^a$:
$$\begin{tikzcd}
& X_1 \arrow[dl, "s"', "\wr a"] \arrow[dr, "f"] & \\
X \arrow[rr, "f s^{-1}"'] & & Y
\end{tikzcd}$$

modulo equivalence:
$$\begin{tikzcd}
& X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\
X & X_3 \arrow[l, "s_3"'] \arrow[r, "f_3"] \arrow[u] \arrow[d] & Y \\
& X_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

<u>Example</u>: f.g. $\mathbb{Z}$-mod / torsion $= \mathbb{Q}$-mod, f.d.

$R\text{-mod}/\{M : S^{-1}M = 0\} = S^{-1}R\text{-mod}$.

[X a noeth. scheme, $U \subset X$ open, $Z := X - U$.
$Coh(X)/\{Coh\, X,\ \mathrm{supp} \subset Z\} = Coh(U)$]

<u>In our case it is easier:</u>

initial alm. isom. /M

$$\begin{tikzcd}
\tilde{m}\otimes_V M \arrow[d, "\wr a"] \arrow[dr] & \\
M & N
\end{tikzcd}$$

$\mathrm{Hom}_{V^a\text{-mod}}(M^a, N^a) = \mathrm{Hom}_{V\text{-mod}}(\tilde{m}\otimes_V M, N)$

<u>is a V-module</u>, no nontriv. m-torsion

$(m\cdot \tilde{m}\otimes_V M = \tilde{m}\otimes_V M)$

<u>Philosophy</u>: $\mathcal{O}_K\text{-mod} \to \mathcal{O}_K^a\text{-mod} \to K\text{-mod}$
objects and properties $/K$ extend to $\mathcal{O}_K^a$-mod, alg etc.
Easy to prove if $K \supset \mathbb{F}_p$, using Frob: $K \xrightarrow{\sim} K$.

We go on. (Def.)

$$\begin{tikzcd} V^a\text{-mod} \times V^a\text{-mod} \arrow[r, "\mathrm{Hom}(-,-)"] \arrow[rr, bend right, "\mathrm{alHom}(-,-)"'] & V\text{-mod} \arrow[r] & V^a\text{-mod} \end{tikzcd}$$

$$\begin{tikzcd} V\text{-mod} \times V\text{-mod} \arrow[r, "-\otimes_V -"] \arrow[d] & V\text{-mod} \arrow[d] \\ V^a\text{-mod} \times V^a\text{-mod} \arrow[r, dashed, "-\otimes-"] & V^a\text{-mod} \end{tikzcd}$$

<u>Prop.</u> 4.5 ("Perfectoid"): $\mathcal{O}_K^a$-mod is an abelian $\otimes$-cat., s.t. $M \mapsto M^a$ preserves ker, coker, $\otimes$, and for L, M, N in $\mathcal{O}_K^a$-mod:
$$\mathrm{Hom}(L, \mathrm{alHom}(M,N)) = \mathrm{Hom}(L \otimes M, N).$$
(Also true as in G–R.)

<u>Def.</u> An $\mathcal{O}_K^a$-algebra is a pair (A, μ), A in $\mathcal{O}_K^a$-mod, $\mu: A \otimes A \to A$, s.t. usual conditions in $\mathcal{O}_K^a$-mod. $\mathcal{O}_K^a$-alg: the category.
We have a functor: $\mathcal{O}_K\text{-alg} \to \mathcal{O}_K^a\text{-alg}$. $R \mapsto R^a$.
For A in $\mathcal{O}_K^a$-alg, we have A-mod: (M, a), M in $\mathcal{O}_K^a$-mod, $a: A \to \mathrm{alHom}(M,M)$.

<u>Prop.</u> 4.6 ("Perfectoid"). $M \mapsto M^a$, $\mathcal{O}_K\text{-mod} \to \mathcal{O}_K^a\text{-mod}$ has a right adjoint
$$N \mapsto N_* := \mathrm{Hom}_{\mathcal{O}_K^a\text{-mod}}(\mathcal{O}_K^a, N).$$

$\forall N$: $(N_*)^a \to N$ is an isom.
$\forall M$: $(M^a)_* = \mathrm{Hom}(m, M)$.

(Compare with: $U \overset{j}{\hookrightarrow} X$, $\begin{tikzcd} Ab(X) \arrow[r, "j^{-1}"] & Ab(U) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$) (But in case of coh. $\mathcal{O}$-modules you do not expect $j_!$.)

For A in O_K^a-alg: A_* is an O_K-algebra, $(A_*)^a = A$.
For M in A-mod: M_* is an A_*-module, $(M_*)^a = M$. A-mod: ab. $\otimes$-cat.
Can define A-alg. for A in O_K^a-alg.

Then comm. alg. in A-mod, for A in O_K^a-alg.
M in A-mod is flat $\overset{def}{\Longleftrightarrow}$ $M \otimes_A - : A\text{-mod} \to A\text{-mod}$ is exact

Prop. For R in O_K-alg, M in R-mod:
M^a is R^a-flat $\Longleftrightarrow$ $\forall i > 0$, $\forall$ R-module N, $\mathrm{Tor}_i^R(M,N) \overset{a}{=} 0$.

Similarly: almost projective: $\mathrm{alHom}_A(M,-)$ is exact. The categorical notion $\mathrm{Hom}_{A\text{-mod}}(M,-)$ is not good.
almost fin. generated (uniformly ....).

Def. Let A be in O_K^a-alg, B in A-alg, $\mu : B \otimes_A B \to B$ multiplication.
(i) $A \to B$ is unramified if $\exists\, e \in (B \otimes_A B)_*$ s.t. $e^2 = e$, $\mu(e) = 1$,
and $\forall x \in (\ker \mu)_*$: $xe = 0$. (e = the diagonal idempotent, the diagonal is an open immersion).
(ii) $A \to B$ is étale if it is unramified and flat.
(iii) $A \to B$ is finite étale if it is étale and almost fin. pres.
The category: $A_{fét}$. (Equiv: (thm) B is alm. proj. + alm. fin. gen. and trace map $B \to B^*$ is an isom. of A-mod.)

Thm. Let $A \in O_K^a$-alg, flat, and π-adic. complete: $A \to \lim A/\pi^n$ isom.

Then $A_{fét} \longrightarrow (A/\pi)_{fét}$, $B \longmapsto B/\pi$ is an equivalence.
Proof: follow the usual proof, categorify it .... ($\approx$ 10 pages in G-R).

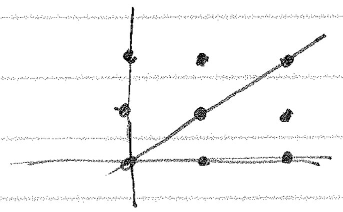
Let $\mathbb{Q}_p \to K$ ($\in \overline{\mathbb{Q}_p}$) be a finite tot. ram. extension, $n := \dim_{\mathbb{Q}_p} K$.

$\mathbb{Z}_p \to O_K$: unif. π, $v_p(\pi) = \frac{1}{n}$, $f := f^{\pi}_{\mathbb{Q}_p} = x^n + pa_{n-1}x^{n-1} + \dots + pa_0$, $a_0 \in \mathbb{Z}_p^{\times}$.

$= (x - r_1)\cdots(x - r_n)$, $r_i \in \overline{\mathbb{Z}_p} \cap \overline{\mathbb{Q}_p}$ distinct!

<u>Denominator of the idempotent in $K \otimes_{\mathbb{Q}_p} K$</u>

$K \otimes_{\mathbb{Q}_p} K = \mathbb{Q}_p[x,y]/(f(x), f(y))$.



$\{(r_i, r_j) : 1 \le i,j \le n\}$

$f(x) - f(y) = (x-y)\cdot g(x,y)$ in $\mathbb{Z}_p[x,y]$

$\forall\, i,j$: $\underbrace{f(r_i) - f(r_j)}_{0} = (r_i - r_j)\cdot g(r_i, r_j)$

So: $V(f(x), f(y), g(x,y))$ = off-diag. part, $g(x,y)$ is a unit on diagonal: its value on diagonal is $f'(x) = f'(y)$.

Diagonal idempotent of $K \otimes_{\mathbb{Q}_p} K$: $\dfrac{g(x,y)}{f'(x)}$.

$f'(\pi) = f'(x) \in \left(\mathbb{Q}_p[x]/(f(x))\right)^{\times}$

Conclusion: $N_{K/\mathbb{Q}_p}(f'(\pi))$ is a denominator in $\mathbb{Z}_p$ of the diag. idem

More geometrically: $\Omega^1_{O_K/\mathbb{Z}_p} = O_K \cdot dx \,/\, O_K f'(\pi)\cdot dx$
(intrinsically).

So: $O_K \cdot f'(\pi) = \mathrm{Ann}_{O_K}(\Omega^1_{O_K/\mathbb{Z}_p})$.

<u>Now the almost unramifiedness.</u> For $m \ge 1$: $A_m := \mathbb{Z}_p[p^{1/m}] = \mathbb{Z}_p[z]/(z^m - p)$.

$B_m := (A_m \otimes_{A_1} O_K)^{\sim}$ (normalisation) $(A_1 = \mathbb{Z}_p)$.

We approximate B_m as follows (assume $m > n$): write $p_m = p^{1/m}$

$m = q\cdot n + r$, $0 \le r < n$.

$f = x^n + p_m^m a_{n-1} x^{n-1} + \dots + p_m^m a_1 x + p_m^m a_0$, $x = p_m^q \cdot x_m$ $\quad \pi_m := \pi / p_m^q$

$= p_m^{qn} x_1^n + p_m^{m+q(n-1)} a_{n-1} x_1^{n-1} + \dots + p_m^{m+n} a_n x_1 + p_m^m a_0$ $\quad v(\pi_m) = \frac{1}{n} - \frac{q}{m} =$

$= p_m^{qn} \cdot \big(\underbrace{x_1^n + p_m^{m-n} a_{n-1} x_1^{n-1} + \dots + p_m^{m-qn} a_0}_{f_m(x_1)}\big)$ $\quad = \frac{r}{nm} < \frac{1}{m}$

$v(f_m'(\pi_m)) \le v(n\cdot \pi_m^{n-1}) < v(n) + \overbrace{\frac{n-1}{m}}^{\to 0}$

I want $p \nmid n$. Add a factor x.