Leiden, 2014/03/07, Intercity Seminar on work of P. Scholze
<u>Introduction, Bas Edixhoven, 20 minutes</u>.

<u>Goal</u> of this introduction: <u>motivate</u> you for the rest of this series of 3 days ICS: today and April 4: Scholze's IHES paper on "Perfectoid spaces", and April 11: Scholze himself on his preprint "Torsion...".

<u>Perfectoid spaces</u>: $\mathbb{Q}_p \rightsquigarrow \mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p = (\overline{\mathbb{Z}_p})^\wedge[1/p] = (\lim \overline{\mathbb{Z}}_p/p^n)[\frac{1}{p}]$, "but then extended to varieties over $\mathbb{Q}_p$". Almost.
<u>Example</u>: $\mathbb{Z}_p[x, x^{-1}][x^{1/p}, x^{1/p^2}, \ldots][p^{1/p}, p^{1/p^2}, \ldots]^\wedge[1/p]$, geometry.

$\mathbb{F}_p[[t]][x, x^{-1}][x^{1/p}, x^{1/p^2}, \ldots][t^{1/p}, t^{1/p^2}, \ldots]^\wedge[1/t]$, geometry.
Theory of perfectoid spaces: still under construction, but already various applications: 1. weight-monodromy conjecture;
2. geometry of Shimura varieties, "local Shimura varieties";
April 11 $\rightarrow$ 3. Langlands conjectures: Gal. repr. attached to torsion classes;
4. p-adic Hodge theory (this lies at the origin of "Perfectoid Sp", work of Faltings)(so-called "almost mathematics").
It is all very exciting!
About 3, then. I quote from "Torsion".

<u>Global Langlands-Clozel-Fontaine-Mazur conjecture</u>.
Let F be a number field, p a prime, $\iota: \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Then $\forall n \geq 1$
$\exists!$ bijection between
(i) the set of L-algebraic cuspidal automorphic repr. of $GL_n(\mathbb{A}_F)$
(ii) the set of isomorphism classes of irreducible continuous representations $Gal(\overline{F}/F) \to GL_n(\overline{\mathbb{Q}}_p)$ that are almost everywhere unramified, and de Rham at the places dividing p,
that matches Satake parameters with eigenvalues of Frobenius elements.

Scholze's thm (a special case) — more generally: $/p^m$, p-adic..

Let F be totally real or CM. Let $n \geq 1$, $K \subset Gl_n(\mathbb{A}_{F,f})$ a compact open subgroup.

Let $D := Gl_n(F \otimes_{\mathbb{Q}} \mathbb{R}) / \mathbb{R}^{\times}_{>0} \cdot K_\infty$, with $K_\infty \subset Gl_n(F \otimes_{\mathbb{Q}} \mathbb{R})^{\circ}$ max. compact subgr.

Let $X_K := Gl_n(F) \backslash (D \times Gl_n(\mathbb{A}_{F,f}) / K)$.

Let $i \geq 0$.

Consider $H^i(X_K, \mathbb{F}_p)$ as a module over the Hecke algebra

$$\mathbb{T}_K := \mathrm{End}_{\mathbb{F}_p[Gl_n(\mathbb{A}_{F,f})]}\left(\mathrm{Ind}_K^{Gl_n(\mathbb{A}_{F,f})} \mathbb{F}_p \right)$$

$= (K,K)$-inv. loc. const. funct. $Gl_n(\mathbb{A}_{F,f}) \to \mathbb{F}_p$ with compact support, + convolution.

Then for any system of Hecke eigenvalues at the unr. places for K occurring in $H^i(X_K, \overline{\mathbb{F}}_p)$, there is a continuous semisimple repr. $Gal(\overline{F}/F) \to Gl_n(\overline{\mathbb{F}}_p)$ such that Frobenius and Hecke eigenvalues match.

There is even hope that now GLL as above is now within reach! (for F totally real or CM, and the autom. repr. regular (no multiple HT-weights)).

On the pessimist side:

- we still do not know how to treat non-regular autom. repr. (HT weights with multiplicities)
- we cannot produce elliptic curves that should corr. to weight $(2,2)$ HMF over $\mathbb{Q}(\sqrt{d})$, $d>0$.