

Etale cohomology seminar, 11 November 2014, Bas Edixhoven.

Lefschetz trace formula for Frobenius; why so general, examples, applications.

§1. Why do we have to work with such a general statement?

Thm (SP ^{03V3} 034F) let k be a finite field, X/k separated and of finite type, Λ a finite ring (not nec. commutative) with $\#\Lambda$ in k^\times , and K in $D_{\text{ctf}}(X, \Lambda)$. Then $R\Gamma_c(X_k, K)$ is in $D_{\text{perf}}(\Lambda)$ and

$$\sum_{x \in X(k)} \text{Tr}(\pi_x, K_{\bar{x}}) = \text{Tr}(\pi_X^*, R\Gamma_c(X_k, K)).$$

Last week, Qiyun discussed all notions occurring in this theorem. This gives an upper bound for the amount of technicalities that one has to suffer in order to have a LTF. The point is: this upper bound is also the lower bound. In other words: each technicality is either needed for having a statement (e.g., $D_{\text{perf}}(\Lambda)$), or for having a proof.

1.e. Why do we need non-comm. rings Λ ?

Answer: for reduction from loc. constant Λ -modules (sheaves of Λ -modules) to constant ones.

Example. (See Steps 3-4 in the proof of Thm. 034F of SP.)

Let X be a connected scheme, $n \in \mathbb{Z}_{\geq 0}$, \mathcal{F} in $\text{Sh}(X_{\text{et}}, \mathbb{F}_\ell)$ a loc. constant sheaf of n -dim. \mathbb{F}_ℓ -vect. spaces.

Let $I := \text{Isom}_{X_{\text{et}}}(\mathbb{F}_\ell^n, \mathcal{F})$, then I is a $\text{GL}_n(\mathbb{F}_\ell)$ -torsor on X_{et} , therefore representable, i.e., I is a finite etale X -scheme.

Let Z be a connected component of I , $G \subset \text{GL}_n(\mathbb{F}_\ell)$ its stabiliser.

Then $Z \xrightarrow{f} X$ is a G -torsor, $f^{-1}\mathcal{F} = \mathbb{F}_\ell^n$ on Z_{et} , and $\mathcal{F} = (f_* f^{-1}\mathcal{F})^G = (f_* \mathbb{F}_\ell^n)^G = \text{Hom}(\mathbb{F}_\ell, f_* \mathbb{F}_\ell^n)$,

$$\mathcal{F} = (f_* f^{-1}\mathcal{F})_G = (f_* \mathbb{F}_\ell^n)_G = \mathbb{F}_\ell \otimes_{\mathbb{F}_\ell[G]} f_* \mathbb{F}_\ell^n.$$

Unfortunately, the map $F = (f_* f^{-1} F)^G \rightarrow (f_* f^{-1} F)_G = F$ is multiplication by $\#G$, often divisible by l .

That is why $\mathbb{F}_l[G]$ -modules on $X_{\text{ét}}$ have to be used systematically.

Rem. But one has the following useful construction that allows to prove finiteness properties, for example. Let $S \subset G$ be a Sylow l -subgroup. Then $Z/S \xrightarrow{g} X$ has degree prime to l and $g^* F$ has a filtration with successive quotients \mathbb{F}_l .

So: by "induction" $H^*(Z/S)_{\text{ét}}, g^* F$ is good
descent

$$\begin{aligned} H^*(X_{\text{ét}}, g_* g^* F) &= H^*(X_{\text{ét}}, F \oplus \text{rest}) = \\ &= H^*(X_{\text{ét}}, F) \oplus H^*(X_{\text{ét}}, \text{rest}). \end{aligned}$$

1.6. Why $R\Gamma_c$ and not $R\Gamma$?

Two things. 1. LTF says that $R\Gamma_c$ is the right action for this, it is simply not true with $R\Gamma$ (unless X is proper).

2. It is actually good to have $R\Gamma_c$ because it allows us to replace X by $U \sqcup Y$ with $j: U \hookrightarrow X$ open with complement $i: Y \rightarrow X$: for F in $\text{Sh}(X_{\text{ét}}, \Lambda)$ (X/k finite type and separated) we have

$$j_! j^{-1} F \hookrightarrow F \twoheadrightarrow i_* i^{-1} F$$

giving a long exact sequence

$$\dots \rightarrow H_c^i(U_{\mathbb{F}_k}, j^{-1} F) \rightarrow H_c^i(X_{\mathbb{F}_k}, F) \rightarrow H_c^i(Y_{\mathbb{F}_k}, i^{-1} F) \rightarrow H_c^{i+1}(U_{\mathbb{F}_k}, j^{-1} F) \rightarrow \dots$$

and also, for K in $\mathcal{D}(X_{\text{ét}}, \Lambda)$, a distinguished triangle

$$R\Gamma_c(U_{\mathbb{F}_k}, j^{-1} K) \rightarrow R\Gamma_c(X_{\mathbb{F}_k}, K) \rightarrow R\Gamma_c(Y_{\mathbb{F}_k}, i^{-1} K) \rightarrow R\Gamma_c(U_{\mathbb{F}_k}, j^{-1} K)[1]$$

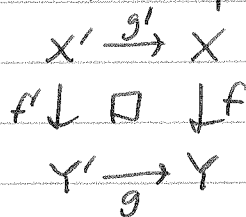
1c Why complexes of sheaves, and even of constructible sheaves:
 K in $D_{ctf}(X, \Lambda)$? non-constant

Because the proof of the LTF is by induction on the dimension of X , using $X \supseteq U \xrightarrow{f} A'_k$, U open, non-empty.



The induction step is done by applying the Proper base change theorem.

Thm (SP 095S) Let $f: X \rightarrow Y$ in Sch be proper, F in $Ab(X_{et})$ torsion, let $g: Y' \rightarrow Y$ in Sch ,



Then the natural morphisms

$$g'_! R^i f_* F \rightarrow R^i f'_* (g')^* F \text{ are isomorphisms.}$$

So, even if one ~~star~~ wants to prove LTF for $\mathbb{Z}/n\mathbb{Z}$ on X , one gets to deal with $Rf_! F = R\bar{f}_*(j_! F)$ on A'_k .

One also needs that constructibility and finite tor dimension is preserved by $Rf_!$. I do not see this in SP. with X, Y noetherian,

Thm (SGA 4.5, IV.6.2) Let $f: X \rightarrow Y$ in Sch , separated, of finite type, F constructible on X_{et} . Then $Rf_! F$ is constructible.

Let us then do the induction step: we assume that LTF holds for all varieties of dimension smaller than $\dim(X)$, and for dim 1.

$$\begin{array}{c} U \\ \downarrow f \\ A'_k \end{array} \quad \sum_{x \in U(k)} \text{Tr}(\pi_x, K_{\bar{x}}) = \sum_{y \in A'(k)} \sum_{\substack{x \in U(k) \\ f(x) = y}} \text{Tr}(\pi_x, K_{\bar{x}})$$

On the other hand:

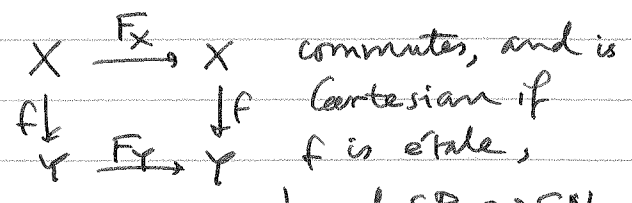
$R\Gamma_c(U_k, K) = R\Gamma_c(A'_k, Rf_! K)$. Note that $(Rf_! K)$ is in $D_c^b(A'_k, \Lambda)$ by the result of SGA 4.5 just mentioned, and that the $(Rf_! K)_y = R\Gamma_c(U_y, K)$ (proper base change!) are perfect by LFT for U_y , y any closed pt. of A'_k , hence $Rf_! K$ is in $D_{\text{ctf}}(A'_k, \Lambda)$.

$$\begin{aligned} \text{Then: } \text{Tr}(\pi_U^*, R\Gamma_c(U_k, K)) &= \text{Tr}(\pi_{A'_k}, R\Gamma_c(A'_k, Rf_! K)) \stackrel{\text{LFT dim 1}}{=} \\ &= \sum_{y \in A'(k)} \text{Tr}(\pi_{A'_k}, (Rf_! K)_y) \\ &= \sum_{y \in A'(k)} \text{Tr}(\pi_{A'_k}, R\Gamma_c(U_y, K)) \quad (\text{by proper base change}) \\ &= \sum_{y \in A'(k)} \sum_{x \in U_y(k)} \text{Tr}(\pi_x, K_{\bar{x}}) \quad (\text{by LFT for each } U_y) \quad \square \end{aligned}$$

So: the case $\dim(X) = 1$ is essential. See SP 03UF.

1.d. Frobenius. For X/\mathbb{F}_p any scheme let $F_X: X \rightarrow X$ be the absolute p -Frobenius: id_X on X as top. space, and $\forall U \subset X$ open, $F_X^\#(U): \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(U), f \mapsto f^p$.

If you prefer: on affine schemes $\text{Spec } A$ it is $\text{Spec}(p\text{th power map})$.
Then $\forall f: X \rightarrow Y$ in Sch/\mathbb{F}_p :

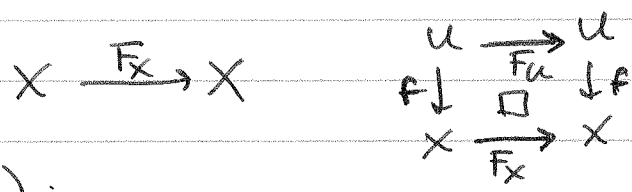


bec. of SP 03SN .
proof of + neces. changes.

For k finite with $q = p^f$ elements and X/k we have the abs. q -Frobenius $\pi_X := F_X^f, X \xrightarrow{\pi_X} X, X \xrightarrow{F_X} X$
CSP calls this geometric Frob, which I dislike: geometric is over t !



For X/\mathbb{F}_p and F in $\text{Sh}(X_{\text{ét}})$ we have a natural F_X -morphism (SP 00 8J): $F \xleftarrow{F_X^*} F, F(U) \xleftarrow{\text{id}} F(U)$



For F in $\text{Ab}(X_{\text{ét}})$:

$$\forall i: F_X^*: H^i(X_{\text{ét}}, F) \rightarrow H^i(X_{\text{ét}}, F) \text{ \& \# identity!}$$

(proof: use the definition!) (SP 03SN)

Let k finite, $q = p^f = \#k$, and X in Sch/k . Let $k \rightarrow t$ be an alg. clsr. Then we have $X_t := X \times_{\text{Spec } k} \text{Spec } t, (\pi_X)_t = \pi_X \times \text{id}: X_t \rightarrow X_t$, the geometric q -Frobenius.

$$\begin{aligned} \text{Note: } \pi_{(X_t)} &= \pi_X \times \pi_{\text{Spec } t} = (\pi_X)_t \circ (\text{id}_X \times \pi_{\text{Spec } t}) = \\ &= (\text{id}_X \times \pi_{\text{Spec } t}) \circ (\pi_X)_t. \end{aligned}$$

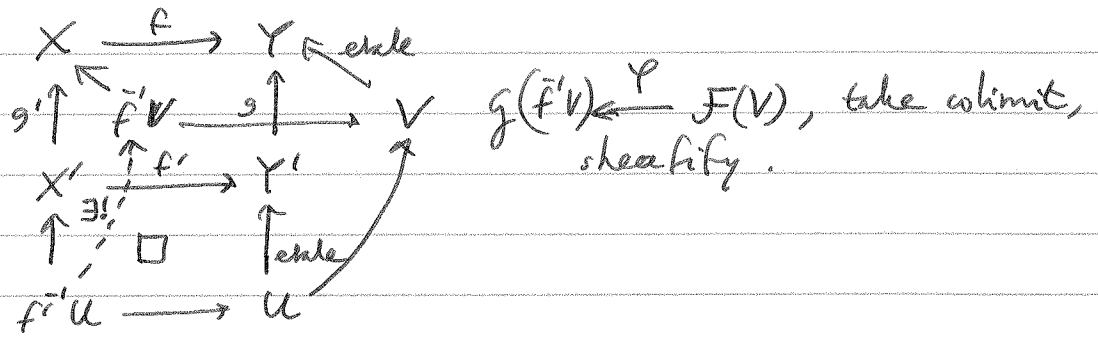
We are interested in the action of $(\pi_X)_t^*$ on the $H^i(X_{t, \text{ét}}, F_t)$. To define that action, we need a $(\pi_X)_t$ -map: $F_t \xleftarrow{(\pi_X)_t^*} F_t, X_t \xrightarrow{(\pi_X)_t^*} X_t$.

This works in general (how general?) (for continuous morphisms of sites??)

Let $X \xrightarrow{f} Y$ in Sch, $g \xleftarrow{\varphi} F$ an f -map, F in $Sh(Y_{\text{ét}})$,
 $g \xleftarrow{\varphi} F$ g in $Sh(X_{\text{ét}})$.

Let $X \xrightarrow{f} Y$ be commutative. Then φ induces
 $g' \uparrow \quad \uparrow g$ an f' -map ~~φ~~
 $X' \xrightarrow{f'} Y'$ $(g')^{-1}g \xleftarrow{\varphi'} g'^{-1}F$.

Namely:



So this explains that we have, in the situation of LTF:

$$\begin{array}{ccc}
 K_{\mathbb{k}} \xleftarrow{(\pi_X^*)_{\mathbb{k}}} K_{\mathbb{k}} & R\Gamma_C(X_{\mathbb{k}}, K_{\mathbb{k}}) \xleftarrow{(\pi_X^*)_{\mathbb{k}}} R\Gamma_C(X_{\mathbb{k}}, K_{\mathbb{k}}) & \\
 \textcircled{*} & & \text{in } \mathcal{D}_{\text{perf}}(\Lambda), \\
 X_{\mathbb{k}} \xrightarrow{(\pi_X)_{\mathbb{k}}} X_{\mathbb{k}} & & \\
 & \text{hence: } T_{\Gamma}((\pi_X^*)_{\mathbb{k}}, R\Gamma_C(X_{\mathbb{k}}, K_{\mathbb{k}})) & \\
 & \text{the RHS of LTF.} &
 \end{array}$$

Let us now look at the LHS of LTF.

~~Claim~~: It is obtained by restricting $\textcircled{*}$ to the fixed point locus

Def! of $(\pi_X)_{\mathbb{k}}$!

LTF is true by definition of $\dim(X) \leq 0$.

To prove it for $\dim(X)$ arbitrary, it is sufficient to do so for $\dim(X) = 1$.