Etale cohomology seminar, 11 November 2014, Bas Edixhoven.

Lefschetz trace formula for Frobenius: why so general, examples, applications.

§1. Why do we have to work with such a general statement?

Thm (SP ~~03UF~~ 03V3) Let k be a finite field, X/k separated and of finite type, Λ a finite ring (not nec. commutative) with $\#\Lambda$ in $k^\times$, and K in $D_{ctf}(X,\Lambda)$. Then $R\Gamma_c(X_{\bar{k}}, K)$ is in $D_{perf}(\Lambda)$ and

$$\sum_{x\in X(k)} \mathrm{Tr}(\pi_X, K_{\bar{x}}) = \mathrm{Tr}(\pi_X^*, R\Gamma_c(X_{\bar{k}}, K)).$$

Last week, Qiyun discussed all notions occurring in this theorem. This gives an upper bound for the amount of technicalities that one has to suffer in order to have a LTF. The point is: this upper bound is also the lower bound. In other words: each technicality is either needed for having a statement (e.g., $\Gamma_c(\ )$, $D_{perf}(\Lambda)$), or for having a proof.

1.a. Why do we need non-comm. rings Λ?

Answer: ~~for~~ reduction from loc. constant sheaves of Λ-modules to constant ones.

Example. (See Steps 3–4 in the proof of Thm. 03UF of SP.)
Let X be a connected scheme, $n\in\mathbb{Z}_{\geq 0}$, $\mathcal{F}$ in $\mathrm{Sh}(X_{et}, \mathbb{F}_\ell)$ a loc. constant sheaf of n-dim. $\mathbb{F}_\ell$-vect. spaces.
Let $I := \underline{\mathrm{Isom}}_{X_{et}}(\underline{\mathbb{F}_\ell}^n, \mathcal{F})$, then I is a $\mathrm{Gl}_n(\mathbb{F}_\ell)$-torsor on X_{et}, therefore representable, i.e., I is a finite étale X-scheme.
Let Z be a connected component of I, $G\subset \mathrm{Gl}_n(\mathbb{F}_\ell)$ its stabiliser.
Then $Z \xrightarrow{f} X$ is a G-torsor, $f^{-1}\mathcal{F} = \underline{\mathbb{F}_\ell}^n$ on Z_{et}, and $\mathcal{F} = (f_*f^{-1}\mathcal{F})^G = (f_*\underline{\mathbb{F}_\ell}^n)^G = \underline{\mathrm{Hom}}(\underline{\mathbb{F}_\ell}, f_*\underline{\mathbb{F}_\ell}^n)$,

$$\mathcal{F} = (f_*f^{-1}\mathcal{F})_G = (f_*\underline{\mathbb{F}_\ell}^n)_G = \underline{\mathbb{F}_\ell} \otimes_{\underline{\mathbb{F}_\ell[G]}} f_*\underline{\mathbb{F}_\ell}^n.$$

Unfortunately, the map $\mathcal{F} = (f_* f^{-1}\mathcal{F})^G \longrightarrow (f_* f^{-1}\mathcal{F})_G = \mathcal{F}$ is multiplication by $\# G$, often divisible by ℓ.
That is why $\underline{\mathbb{F}_\ell[G]}$-modules on X_{et} have to be used systematically.

<u>Rem.</u> But one has the following useful construction that allows to prove finiteness properties, for example. Let $S \subset G$ be a Sylow ℓ-subgroup. Then $Z/S \xrightarrow{g} X$ has degree prime to ℓ and $g^{-1}\mathcal{F}$ has a filtration with successive quotients $\underline{\mathbb{F}_\ell}$.

So: by ~~"induction"~~ devissage $H^\circ((Z/S)_{et}, g^{-1}\mathcal{F})$ is good

$$\begin{aligned} &\parallel \\ &H^\circ(X_{et}, g_* g^{-1}\mathcal{F}) = H^\circ(X_{et}, \mathcal{F} \oplus \text{rest}) = \\ &= H^\circ(X_{et}, \mathcal{F}) \oplus H^\circ(X_{et}, \text{rest}). \end{aligned}$$

1.b. Why $R\Gamma_c$ and not $R\Gamma$?

Two things. $\underline{1}$. LTF says that $R\Gamma_c$ is the right notion for this, it is simply not true with $R\Gamma$ (unless X is proper).

$\underline{2}$. It is actually good to have $R\Gamma_c$ because it allows us to replace X by $U \amalg Y$ with $j: U \hookrightarrow X$ open with complement $i: Y \to X$: for $\mathcal{F}$ in $Sh(X_{et}, \Lambda)$ (X/k finite type and separated) we have

$$j_! j^{-1}\mathcal{F} \hookrightarrow \mathcal{F} \twoheadrightarrow i_* i^{-1}\mathcal{F}$$

giving a long exact sequence

$$\dots \to H^i_c(U_{\bar k}, j^{-1}\mathcal{F}) \to H^i_c(X_{\bar k}, \mathcal{F}) \to H^i_c(Y_{\bar k}, i^{-1}\mathcal{F}) \to H^{i+1}_c(U_{\bar k}, j^{-1}\mathcal{F}) \to \dots$$

and also, for K in $D(X_{et}, \Lambda)$, a distinguished triangle

$$R\Gamma_c(U_{\bar k}, j^{-1}K) \to R\Gamma_c(X_{\bar k}, K) \to R\Gamma_c(Y_{\bar k}, i^{-1}K) \to R\Gamma_c(U_{\bar k}, j^{-1}K)[1]$$

1c Why complexes of sheaves, and even of non-constant constructible sheaves: K in $D_{ctf}(X, \Lambda)$?

Because the proof of the LTF is by induction on the dimension of X, using

$$\begin{tikzcd} X \supset U \arrow[r, "f"] \arrow[d, hook, "j"'] & \mathbb{A}^1_k \\ \bar{U} \arrow[ur, "\bar{f}\ \text{proper}"'] & \end{tikzcd}$$

U open, non-empty.

The induction step is done by applying the <u>Proper base change theorem</u>.

<u>Thm</u> (SP 095S) Let $f: X \to Y$ in Sch be proper, $\mathcal{F}$ in $Ab(X_{et})$ torsion, let $g: Y' \to Y$ in Sch,

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then the natural morphisms

$g^{-1} R^i f_* \mathcal{F} \to R^i f'_* (g')^{-1} \mathcal{F}$ are isomorphisms.

So, even if one ~~th~~ wants to prove LTF for $\mathbb{Z}/n\mathbb{Z}$ on X, one gets to deal with $Rf_! \mathcal{F} = R\bar{f}_*(j_! \mathcal{F})$ on $\mathbb{A}^1_k$.

One also needs that constructibility and finite tor dimension is preserved by $Rf_!$. I do not see this in SP.

<u>Thm</u> (SGA 4.5, IV. 6.2) Let $f: X \to Y$ in Sch, with X, Y noetherian, separated, of finite type, $\mathcal{F}$ constructible on X_{et}. Then $Rf_! \mathcal{F}$ is constructible.

Let us then do the induction step: we assume that LTF holds for all varieties of dimension smaller than $\dim(X)$, and <u>for dim 1</u>.

$$\begin{array}{c} \mathcal{U} \\ f\downarrow \\ \mathbb{A}^1_k \end{array} \qquad \sum_{x\in \mathcal{U}(k)} \mathrm{Tr}\left(\pi_{\mathcal{U}}, K_{\bar{x}}\right) = \sum_{y\in \mathbb{A}^1(k)} \sum_{\substack{x\in \mathcal{U}(k) \\ f(x)=y}} \mathrm{Tr}\left(\pi_{\mathcal{U}}, K_{\bar{x}}\right)$$

On the other hand:

$R\Gamma_c(\mathcal{U}_{\bar{k}}, K) = R\Gamma_c(\mathbb{A}^1_{\bar{k}}, Rf_! K)$. Note that $(Rf_!)K$ is in $D^b_c(\mathbb{A}^1_k,\Lambda)$ by the result of SGA 4.5 just mentioned, and that the $(Rf_! K)_{\bar{y}} = R\Gamma_c(\mathcal{U}_{\bar{y}}, K)$ (proper base change!) are perfect by LFT for $\mathcal{U}_y$, y any closed pt. of $\mathbb{A}^1_k$, hence $Rf_! K$ is in $D_{ctf}(\mathbb{A}^1_k, \Lambda)$.

$$\text{Then: } \mathrm{Tr}\left(\pi^*_{\mathcal{U}}, R\Gamma_c(\mathcal{U}_{\bar{k}}, K)\right) = \mathrm{Tr}\left(\pi_{\mathbb{A}^1_k}, R\Gamma_c(\mathbb{A}^1_{\bar{k}}, Rf_! K)\right) \overset{\text{LFT dim 1}}{=}$$

$$= \sum_{y\in \mathbb{A}^1(k)} \mathrm{Tr}\left(\pi_{\mathbb{A}^1_k}, (Rf_! K)_{\bar{y}}\right)$$

$$= \sum_{y\in \mathbb{A}^1(k)} \mathrm{Tr}\left(\pi_{\mathbb{A}^1_k}, R\Gamma_c(\mathcal{U}_{\bar{y}}, K)\right) \quad \text{(by proper base change)}$$

$$= \sum_{y\in \mathbb{A}^1(k)} \sum_{x\in \mathcal{U}_y(k)} \mathrm{Tr}\left(\underset{\overset{\shortparallel}{\pi_{\mathcal{U}}}}{\pi_{\mathcal{U}_y}}, K_{\bar{x}}\right) \quad \text{(by LTF for each } \mathcal{U}_y\text{)} \qquad \boxtimes$$

So: the case $\dim(X)=1$ is essential. See SP 03UF.

1.d. Frobenius. For $X/\mathbb{F}_p$ any scheme let $F_X : X \to X$ be the <u>absolute p-Frobenius</u> : id_X on X as top. space, and $\forall\, U \subset X$ open, $F_X^\#(U) : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$, $f \mapsto f^p$.

If you prefer: on affine schemes $\mathrm{Spec}\,A$ it is Spec(pth power map).

Then $\forall\, f : X \to Y$ in $\mathrm{Sch}/\mathbb{F}_p$:

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y \end{tikzcd}$$

commutes, and is Cartesian if f is étale, bec. of SP 03SN. proof of + necess.

For k finite with $q = p^f$ elements and X/k we have the <u>changes</u>

<u>abs. q-Frobenius</u> $\pi_X := F_X^f$,

$$\begin{tikzcd} X \arrow[rr, "\pi_X"] \arrow[dr] & & X \arrow[dl] \\ & \mathrm{Spec}\,k & \end{tikzcd} \quad ; \quad \begin{tikzcd} X \arrow[r, "F_X"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\,k \arrow[r, "F_{\mathrm{Spec}\,k}"] & \mathrm{Spec}\,k \end{tikzcd}.$$

CSP calls this geometric Frob, which I dislike: geometric is over $\bar{k}$!)

For $X/\mathbb{F}_p$ and $\mathcal{F}$ in $\mathrm{Sh}(X_{et})$ we have a natural F_X-morphism (SP 008J) : $\mathcal{F} \xleftarrow{F_X^*} \mathcal{F}$ $\quad \mathcal{F}(U) \xleftarrow{id} \mathcal{F}(U)$

$$X \xrightarrow{F_X} X \qquad \begin{tikzcd} U \arrow[r, "F_U"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "f"] \\ X \arrow[r, "F_X"'] & X \end{tikzcd}$$

For $\mathcal{F}$ in $\mathrm{Ab}(X_{et})$:

$$\forall i : \quad F_X^* : H^i(X_{et}, \mathcal{F}) \to H^i(X_{et}, \mathcal{F}) \text{ is identity!}$$

(proof: use the definition!) (SP 03SN)

Let k finite, $q = p^f = \#k$, and X in Sch/k. Let $k \to \bar{k}$ be an alg. clos. Then we have $X_{\bar{k}} := X \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,\bar{k}$, $(\pi_X)_{\bar{k}} = \pi_X \times id : X_{\bar{k}} \to X_{\bar{k}}$, the <u>geometric q-Frobenius</u>.

Note: $\pi_{(X_{\bar{k}})} = \pi_X \times \pi_{\mathrm{Spec}\,\bar{k}} = (\pi_X)_{\bar{k}} \circ (id_X \times \pi_{\mathrm{Spec}\,\bar{k}}) =$

$$= (id_X \times \pi_{\mathrm{Spec}\,\bar{k}}) \circ (\pi_X)_{\bar{k}}.$$

We are interested in the action of $(\pi_X)_{\bar{k}}^*$ on the $H^i(X_{\bar{k},et}, \mathcal{F}_{\bar{k}})$.

To define that action, we need a $(\pi_X)_{\bar{k}}$-map: $\mathcal{F}_{\bar{k}} \xleftarrow{(\pi_X^*)_{\bar{k}}} \mathcal{F}_{\bar{k}}$

$$X_{\bar{k}} \xrightarrow{(\pi_X)_{\bar{k}}} X_{\bar{k}}.$$

This works in general (how general?) (for continuous morphisms of sites??)

Let $X \xrightarrow{f} Y$ in Sch, $\mathcal{G} \xleftarrow{\varphi} \mathcal{F}$ an f map, $\mathcal{F}$ in $Sh(Y_{et})$,

$\mathcal{G} \xleftarrow[\varphi]{} \mathcal{F}$ $\qquad$ $\mathcal{G}$ in $Sh(X_{et})$.

Let
$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \cancel{X}' \arrow[u, "g'"] \arrow[r, "f'"'] & Y' \arrow[u, "g"'] \end{tikzcd}$$
be commutative. Then φ induces an f'-map $(g')^{-1}\mathcal{G} \xleftarrow{\varphi'} g^{-1}\mathcal{F}$.

Namely:

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y & \\ & f^{-1}V \arrow[ul] \arrow[rr] & & V \arrow[ul, "\text{étale}"'] \\ X' \arrow[uu, "g'"] \arrow[rr, "f'"] & & Y' \arrow[uu, "g"'] & \\ f'^{-1}U \arrow[u] \arrow[uur, dashed, "\exists!"] \arrow[rr] \arrow[r, phantom, "\square"] & & U \arrow[u, "\text{étale}"'] \arrow[uur, bend right] & \end{tikzcd}$$

$\mathcal{G}(f^{-1}V) \xleftarrow{\varphi} \mathcal{F}(V)$, take colimit, sheafify.

So this explains that we have, in the situation of LTF:

$$(*) \qquad \begin{tikzcd} K_{\bar k} & K_{\bar k} \arrow[l, "(\pi_X^*)_{\bar k}"'] \\ X_{\bar k} \arrow[r, "(\pi_X)_{\bar k}"'] & X_{\bar k} \end{tikzcd}$$

$R\Gamma_c(X_{\bar k}, K_{\bar k}) \xleftarrow{(\pi_X^*)_{\bar k}} R\Gamma_c(X_{\bar k}, K_{\bar k})$

in $D_{perf}(\Lambda)$,

hence: $\mathrm{Tr}\left((\pi_X^*)_{\bar k}, R\Gamma_c(X_{\bar k}, K_{\bar k})\right)$

the RHS of LTF.

Let us now look at the LHS of LTF.

~~Claim~~ Def.!: It is obtained by restricting $(*)$ to the fixed point locus of $(\pi_X)_{\bar k}$!

LTF is true by definition of $\dim(X) \leq 0$.

To prove it for $\dim(X)$ arbitrary, it is sufficient to do so for $\dim(X) = 1$.