Etale cohomology seminar, 18 November 2014, Bas Edixhoven (45 minutes) 1.

<u>Lefschetz Trace Formula: an example.</u>

We will treat: $k \supset \mathbb{F}_p$, $\# k = q = p^r$, X/k a smooth proj. curve, geom. connected, $n \geq 1$ prime to p, $\Lambda = \mathbb{Z}/n\mathbb{Z}$, $K = \mu_{n,X}$ in degree 0.
But first we recall a few things about F_X^* on sheaves on X_{et}, and about how one could get confused.

μ_n because Giulio showed us the $H^i(X_{\bar{k},et}, \mu_n)$.

1. <u>Back to F_X-maps between sheaves.</u>

Let $X/\mathbb{F}_p$ any $\mathbb{F}_p$-scheme. Let $n \geq 1$ with $p \nmid n$.
Last week I defined, for $\mathcal{F}$ in $Sh(X_{et})$:

$$\mathcal{F} \xleftarrow{F_X^*} \mathcal{F}$$

$$\begin{tikzcd} X \arrow[r, "F_X"] & X \\ U \arrow[u, "f"] \arrow[r, "F_U"'] \arrow[ur, phantom, "\square"] & U \arrow[u, "f \text{ etale}"'] \end{tikzcd} \qquad \mathcal{F}(U) \xleftarrow{id} \mathcal{F}(U)$$

for X/k, k as above

Recall $\pi_X = F_X^r$, abs. q-Frobenius.
We must consider the pullback of via $\operatorname{Spec} \bar{k} \to \operatorname{Spec} k$, for (π_X, π_X^*).
I explained that on page 6, but it becomes easier to understand for $\mathcal{F}$ in $Sh(X_{et})$ that are representable, e.g. $\mathbb{Z}/n\mathbb{Z}$ and $\mu_{n,X}$.

We have $\mu_{n,\mathbb{F}_p} \to \operatorname{Spec}(\mathbb{F}_p)$ finite etale $(\operatorname{Spec} \mathbb{F}_p[x]/(x^n-1))$

$(\mathbb{Z}/n\mathbb{Z})_{\mathbb{F}_p} \nearrow$ $\quad (\operatorname{Spec}(\mathbb{F}_p^{\mathbb{Z}/n\mathbb{Z}}))$

By definition: $(\mathbb{Z}/n\mathbb{Z})_X = (\mathbb{Z}/n\mathbb{Z})_{\mathbb{F}_p} \times_{\operatorname{Spec} \mathbb{F}_p} X$, $\mu_{n,X} = \mu_{n,\mathbb{F}_p} \times_{\operatorname{Spec} \mathbb{F}_p} X$.

For $\mathcal{F} \to X$ etale,

$$\begin{tikzcd} & \mathcal{F} \arrow[d] \\ U \arrow[ur, dashed, "s"] \arrow[r, "\text{etale}"'] & X \end{tikzcd}$$

$\mathcal{F}(U) = \{\text{such } s\}$, sheaf on X_{et}.

Then we have:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "F_{\mathcal{F}}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{F} \arrow[d] \\ X \arrow[r, "F_X"'] & X \end{tikzcd}$$

, and pullback of local sections on $U \to X$ gives $\mathcal{F}(U) \xleftarrow{id} \mathcal{F}(U) : F_X^*(U)$

Important source of confusion: for $\mu_{n,X}$ we are used to something else! Namely:

$$\begin{tikzcd} \mu_{n,X}(U) \arrow[d] \arrow[r, phantom, "\subset"] & \mathcal{O}(U)^\times \arrow[d] & z \arrow[d, mapsto] \\ \mu_{n,X}(U) \arrow[r, phantom, "\subset"] & \mathcal{O}(U)^\times & z^p \end{tikzcd}$$

$$\boxed{F_X^{\#}(U) = p \cdot F_X^*(U) \text{ on } \mu_{n,X}}$$

2. The LHS of LTF.

k as before, $p \nmid n$, $\Lambda = \mathbb{Z}/n\mathbb{Z}$, $K = \mu_{n,X}$ deg. 0, X/k of finite type.

$$\begin{tikzcd} \mu_{n,X} \arrow[r, "\pi_{\mu_{n,X}}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mu_{n,X} \arrow[d] \\ X \arrow[r, "\pi_X"] & X \end{tikzcd}$$

$$\pi_{\mu_{n,X}} : \mu_{n,X} = \mu_{n,k} \times_{\operatorname{Spec} k} X \xrightarrow{\pi_{\mu_{n,k}} \times \pi_X} \mu_{n,k} \times_{\operatorname{Spec} k} X$$

Pull back to $\bar{k}$:

$$\begin{tikzcd} \mu_{n,X_{\bar{k}}} \arrow[r, "(\pi_{\mu_n})_{\bar{k}} \times (\pi_X)_{\bar{k}}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mu_{n,X_{\bar{k}}} \arrow[d] \\ X_{\bar{k}} \arrow[r, "(\pi_X)_{\bar{k}}"'] & X_{\bar{k}} \end{tikzcd}$$

morph. of $\bar{k}$-schemes.

$$\begin{tikzcd} (\pi_X)_{\bar{k}} : X(\bar{k}) \arrow[r] \arrow[d, phantom, "\cap"] & X(\bar{k}) \arrow[d, phantom, "\cap"] \\ \mathbb{P}^m(\bar{k}) \arrow[r] & \mathbb{P}^m(\bar{k}) \end{tikzcd}$$

$(a_0 : \dots : a_m) \longmapsto (a_0^q : \dots : a_m^q)$

Fixed points of $(\pi_X)_{\bar{k}}$: $X(k)$.

For $\bar{x} : \operatorname{Spec}(\bar{k}) \longrightarrow X_{\bar{k}} \to \operatorname{Spec}(\bar{k})$: $\underbrace{(\mu_{n,X_{\bar{k}}})_{\bar{x}}}_{\text{stalk}} = \underbrace{\bar{x}^{-1}\mu_{n,X_{\bar{k}}}}_{\text{pullback of sheaves via } \bar{x}} = \mu_n(\bar{k})$

bec. of representability: pullback is fib. pr., then actions.

What does $(\pi_X^*)_{\bar{k}}$ on this?

$$\begin{tikzcd} \mu_n(\bar{k}) \times X(\bar{k}) = \mu_{n,X}(\bar{k}) \arrow[r, "\pi_{\mu_{n,X}}"] \arrow[d] & \mu_{n,X}(\bar{k}) = \mu_n(\bar{k}) \times X(\bar{k}) \arrow[d] \\ X(\bar{k}) \arrow[r, "\pi_X"] & X(\bar{k}) \end{tikzcd}$$

$$(z, \bar{x}) \longmapsto (z^q, \pi_X(\bar{x}))$$

So, the pullback of $(z^q, \pi_X(\bar{x}))$ is $(z, \bar{x})$.

So: for $\bar{x} \in X(k)$, and $z \in \mu_n(\bar{k})$:

$$(\pi_X^*)_{\bar{k}} : \mu_n(\bar{k}) \longrightarrow \mu_n(\bar{k}), \quad z \longmapsto z^{1/q}.$$

So: LHS of LTF is $\frac{1}{q} \cdot \# X(k)$.

3. <u>The RHS of LTF.</u> Situation as before, but now we assume that X/k is a smooth projective geom. conn. curve.

RHS : $\mathrm{Tr}\left((\pi_X^*)_{\bar k}, \underbrace{R\Gamma(X_{\bar k, et}, (\mu_{n,X})_{\bar k})}\right)$. is in $D_{perf}(\mathbb{Z}/n\mathbb{Z})$, hence isom. to a bounded complex of free $\mathbb{Z}/n\mathbb{Z}$-modules.

Giulio: $H^0 = \mu_n(\bar k)$ free rk. 1

$H^1 = \mathrm{Pic}(X_{\bar k})[n]$ free, $2g$

$H^2 = \mathbb{Z}/n\mathbb{Z}$ free, 1.

Question: what does $(\pi_X^*)_{\bar k}$ do on these?

Recall that we have: $\pi_X^{\#} = q \cdot \pi_X^*$ as π_X-maps, and $\pi_X^{\#}$ is the usual Picard functoriality.

On $H^0 = \mu_n(\bar k)$: $(\pi_X^{\#})_{\bar k}$: id, bec. it is a $\bar k$-morphism. $(\pi_X)_{\bar k}$

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\quad \deg \nearrow \mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z} \\ \text{On } H^1 \text{ and } H^2: \ H^1 \hookrightarrow \mathrm{Pic}(X_{\bar k}) \xrightarrow{n\cdot} \mathrm{Pic}(X_{\bar k}) \twoheadrightarrow H^2 \end{array}$$

To see the action of $(\pi_X^{\#})_{\bar k}$, $\mathrm{Div}(X_{\bar k}) \twoheadrightarrow \mathrm{Pic}(X_{\bar k})$

Make explicit on $\mathbb{A}^1_{\bar k}$: D, degree: $\cdot q$

$(\pi_{\mathbb{A}^1}^{\#})_{\bar k}$: $a \mapsto q \cdot a^{1/q}$ $\qquad (\pi_X)_{\bar k}^{-1} D$

$\pi_{\mathrm{Spec}\, \bar k}^*$: $a \mapsto a^q$ — together: $a \mapsto q.a$, as it should $\pi^*_{(X_{\bar k})} = \mathrm{id}$.

Conclusion: RHS $= \frac{1}{q} \cdot \mathrm{Tr}$ of $(\pi_X^{\#})_{\bar k}$ pullback of geom. q-Frob on $X_{\bar k}$.

$$= \frac{1}{q}\left(1 - \underbrace{\mathrm{Tr}\left((\pi_X^{\#})_{\bar k}, \mathrm{Pic}(X_{\bar k})[n]\right)}_{\textstyle \parallel} + q\right)$$

$$\mathrm{Tr}\left(\underbrace{(\pi^*_{\mathrm{Spec}\,\bar k})^{-1}}_{\text{= inv. of } q\text{-Frob. of } \bar k}, \mathrm{Pic}(X_{\bar k})[n]\right)$$