# Some elliptic curves from the real world

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Universiteit Leiden

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mathematics colloquium Luxembourg
and the conference
Frontiers in Serre's modularity conjecture"

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These notes can be downloaded from my homepage (talks/. . . ).

# Real Elliptic curves in Weierstrass form

For $a, b \in \mathbb{R}$ with $4a^3 + 27b^2 \neq 0$:

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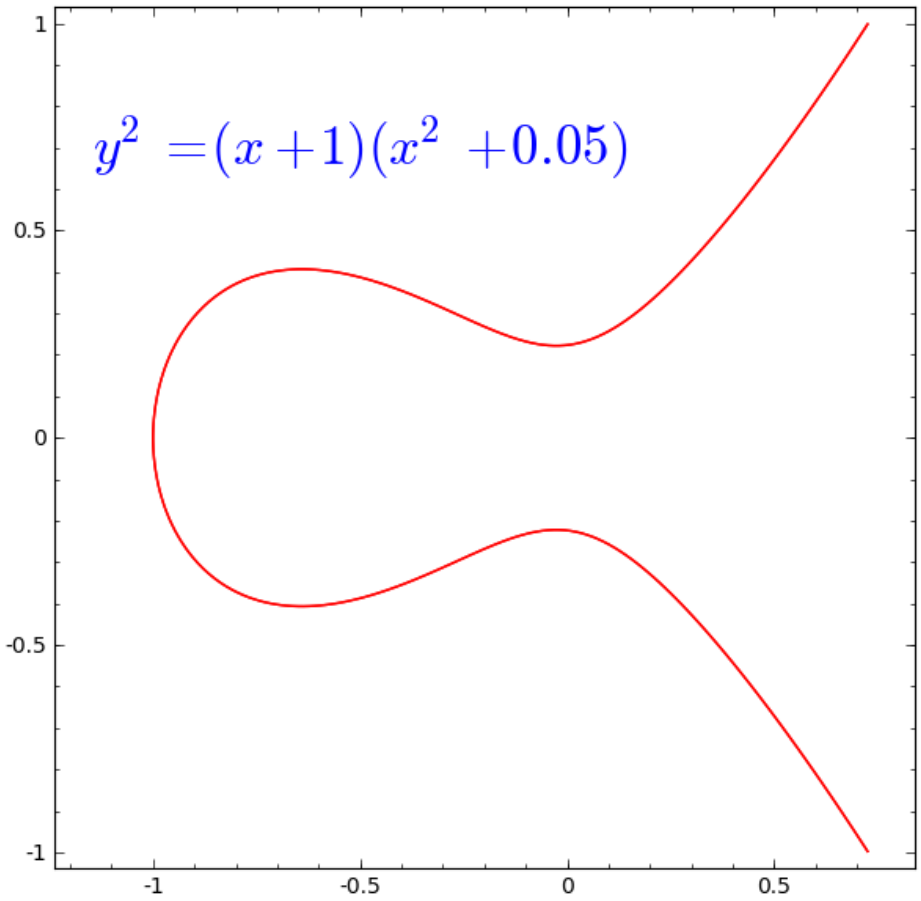


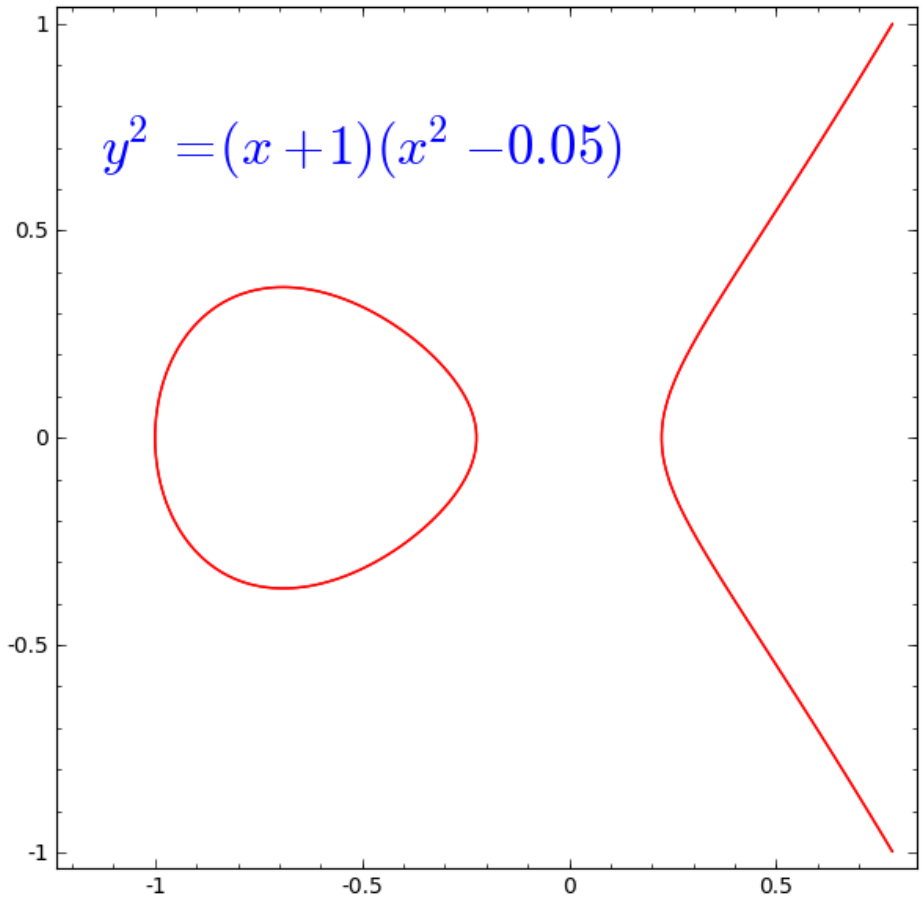

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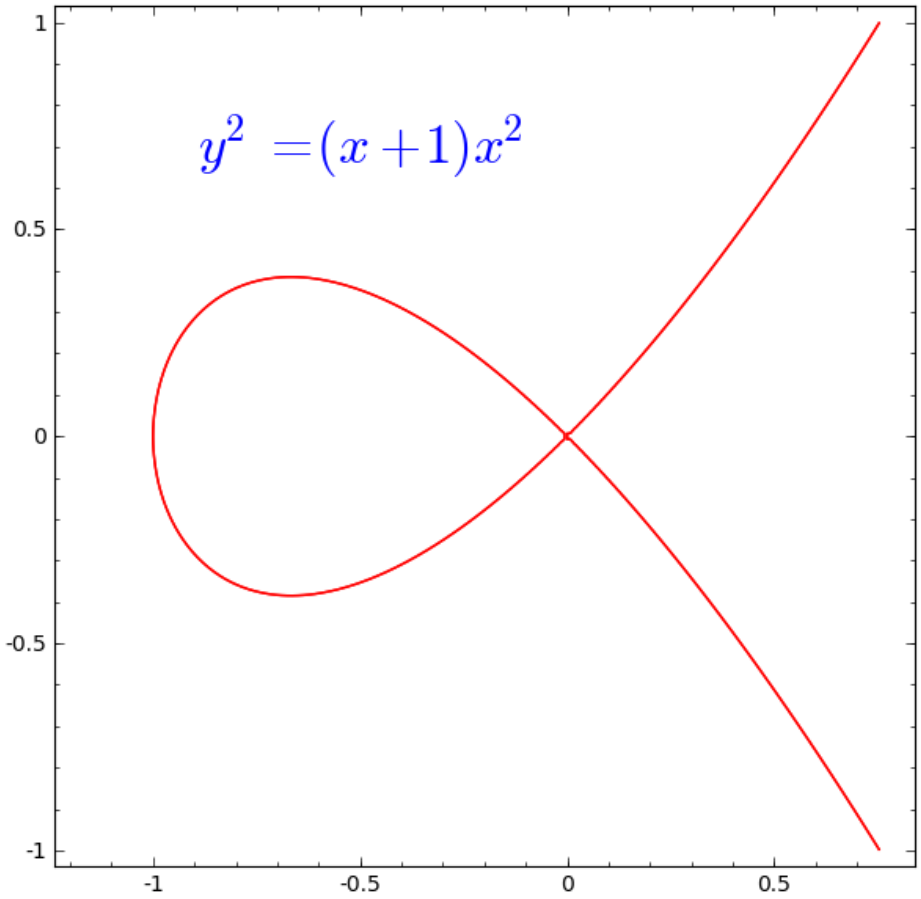


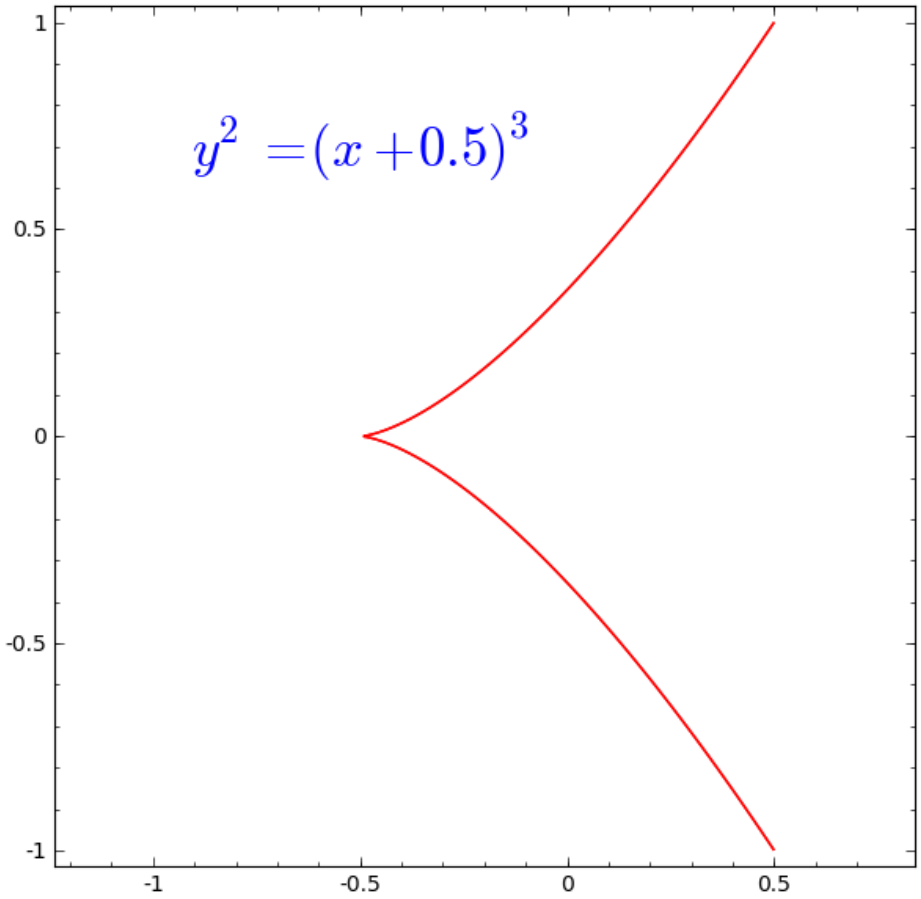

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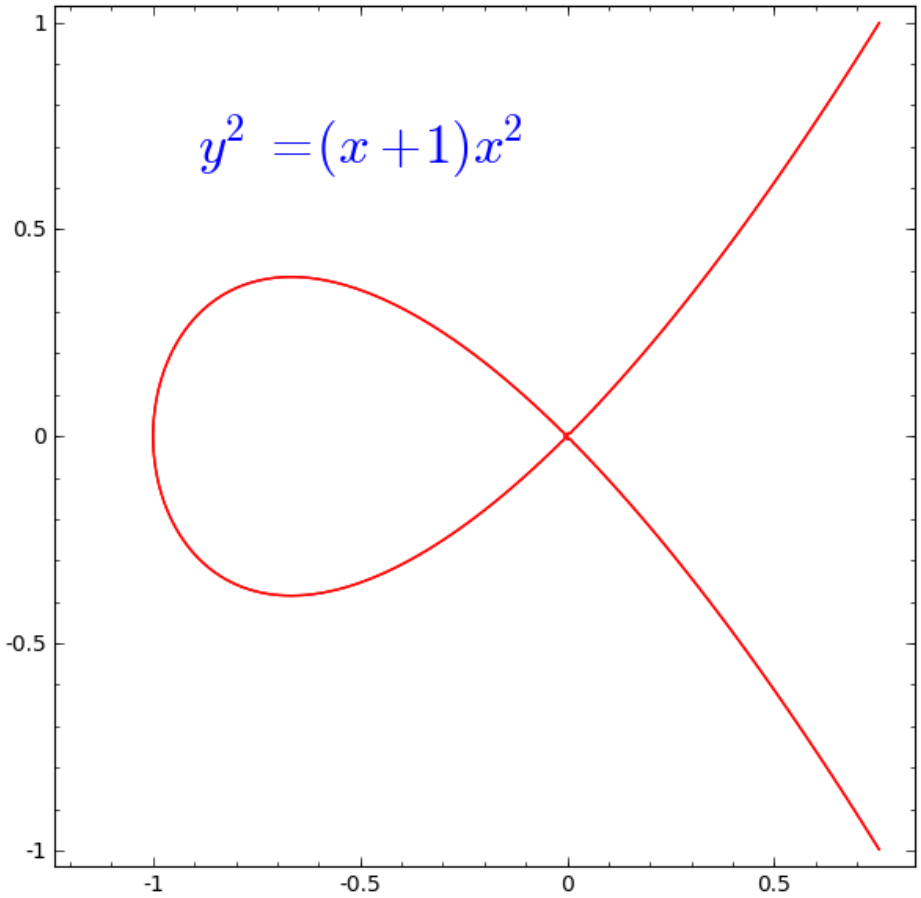


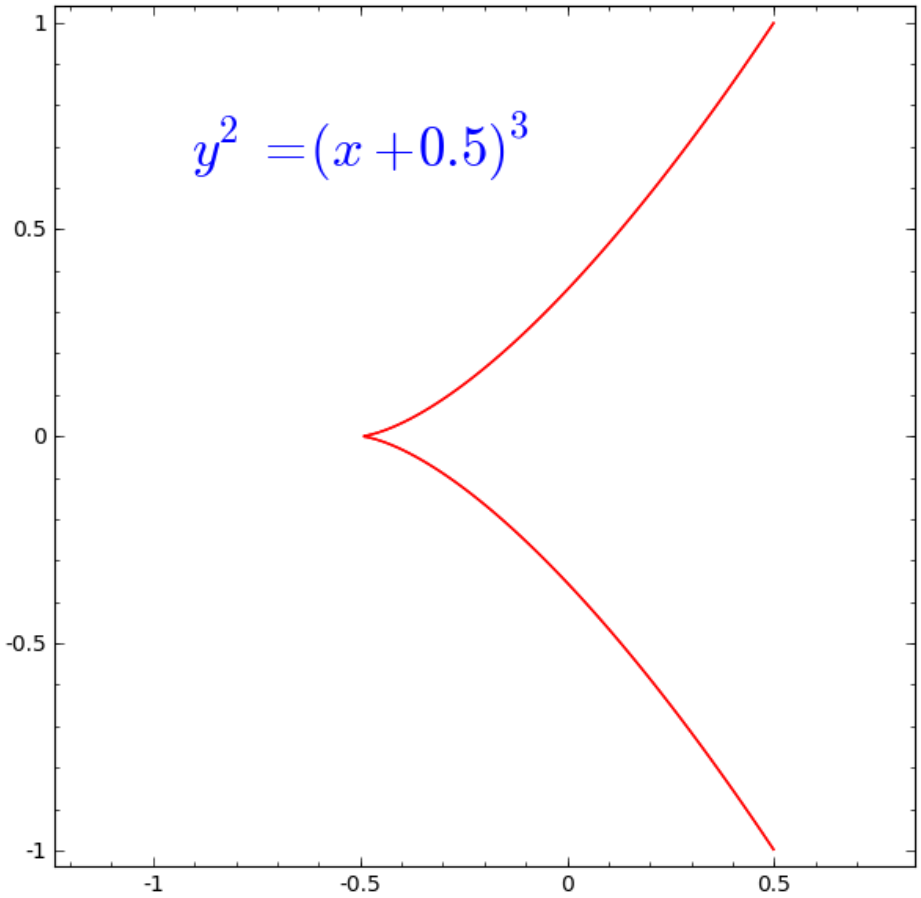


All non-singular degree 3 curves in $\mathbb{R}^2$ can be brought in Weierstrass form by projective transformations.

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Instead of $\mathbb{C}$ we can use any algebraically closed field, and, in fact, any ring (commutative, with 1).

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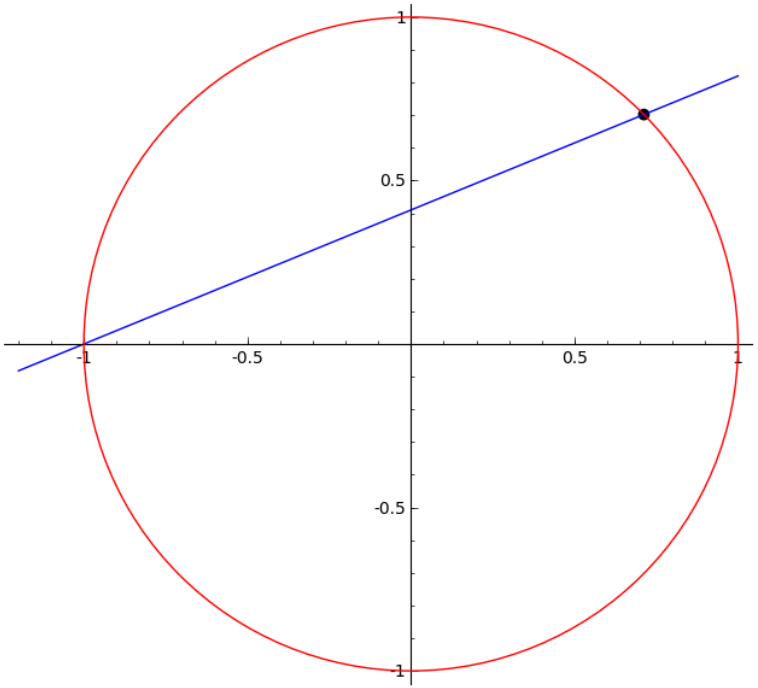

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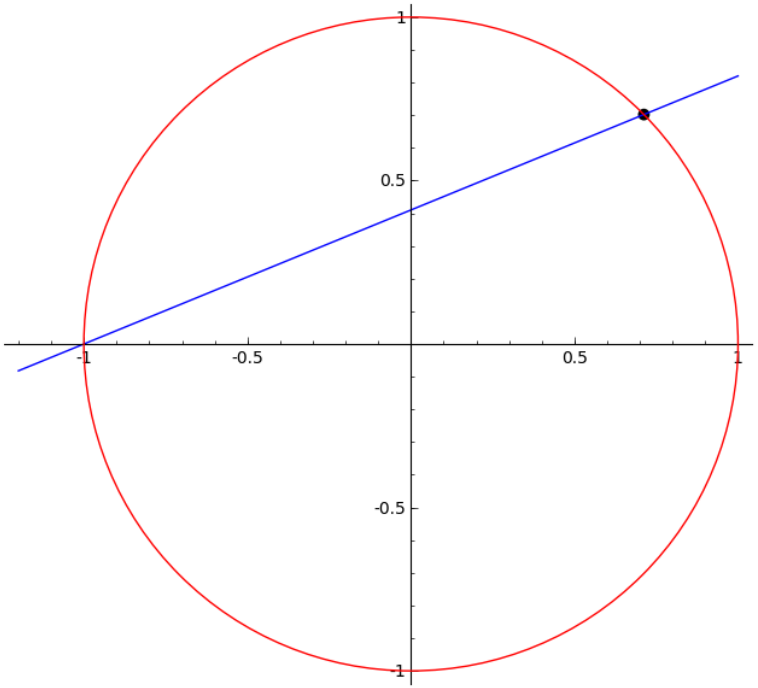


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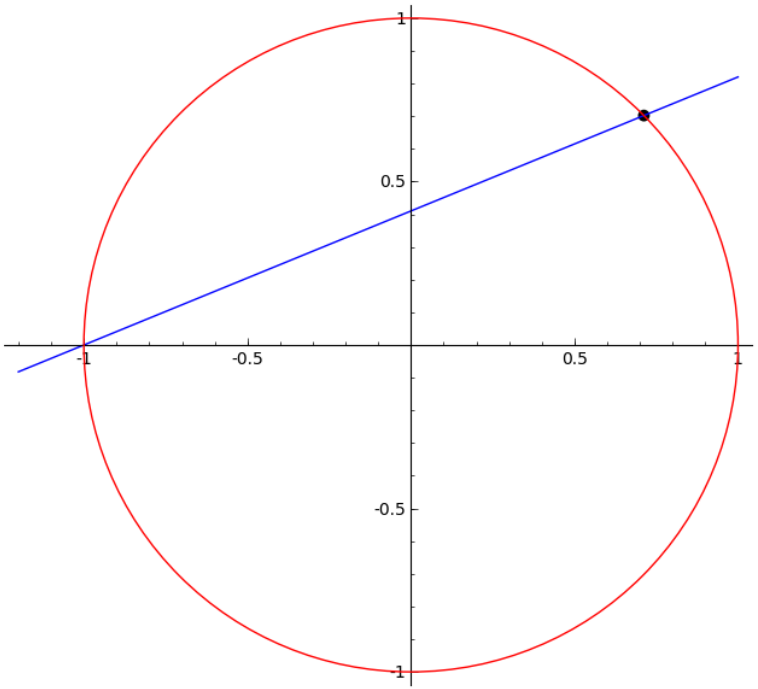


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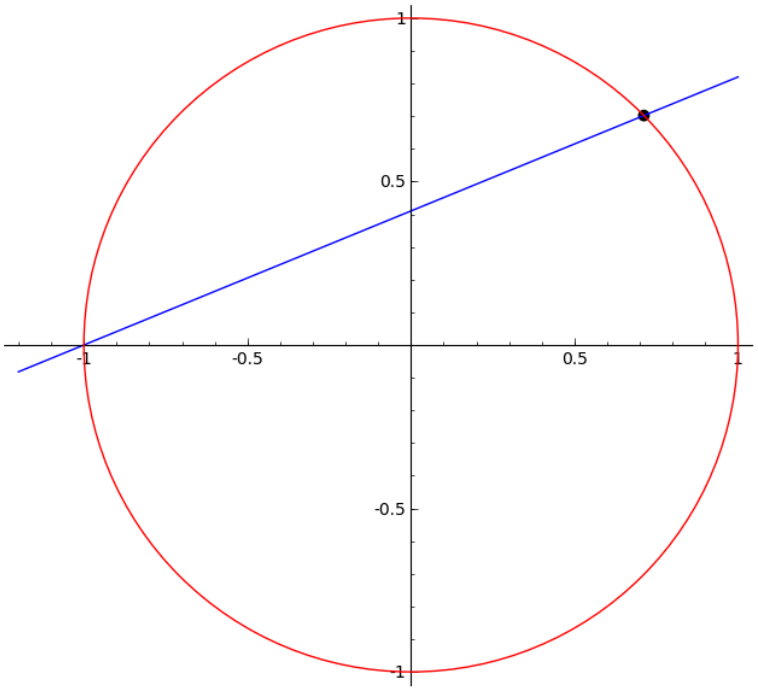


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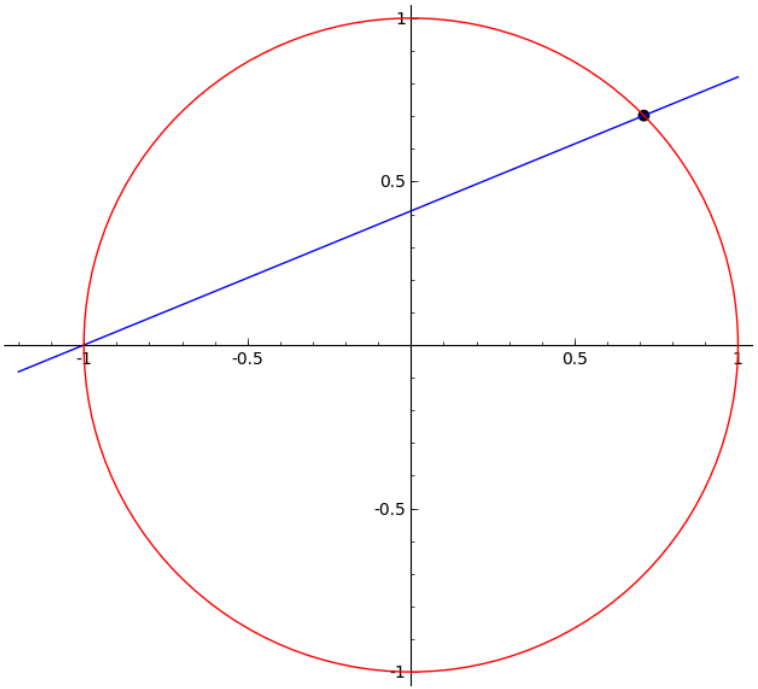


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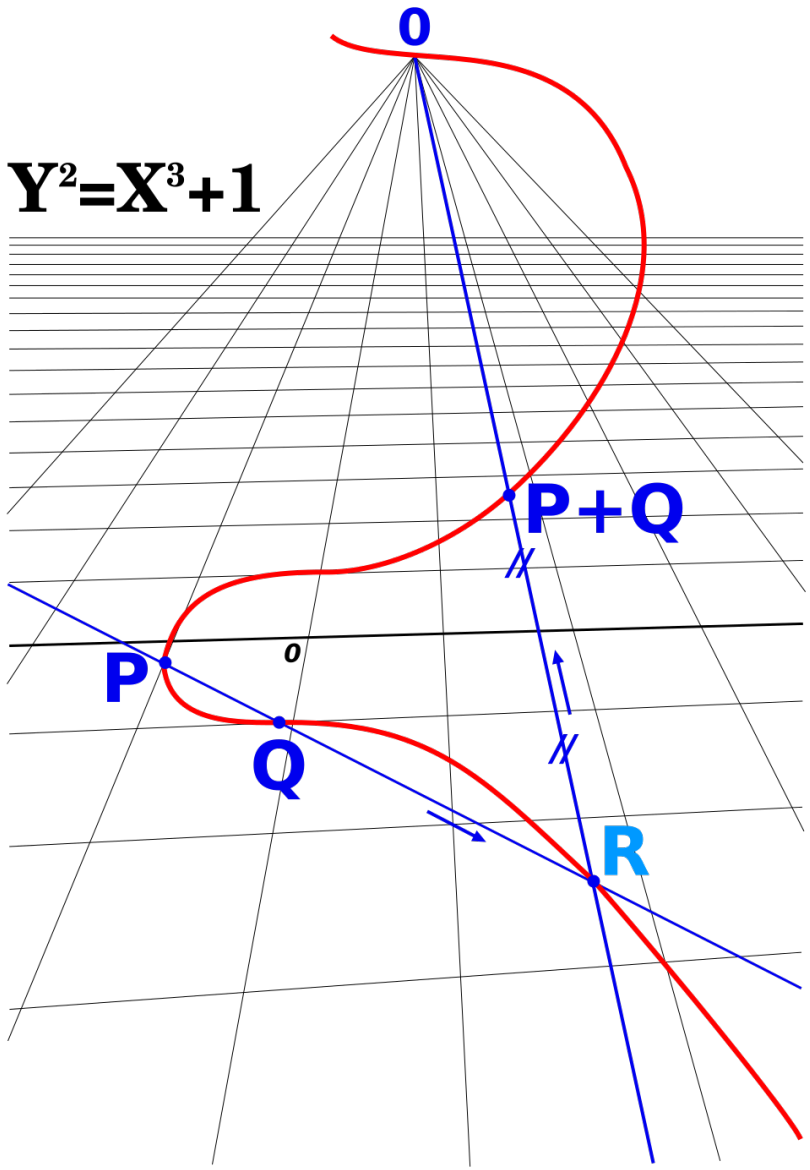
Fact: elliptic curves cannot be parametrised, not even locally. Lines intersect them in 1 or 3 points, if we count with multiplicity and use the "projective plane".

Parallel lines intersect on the horizon: one extra point for each direction in $\mathbb{R}^2$.



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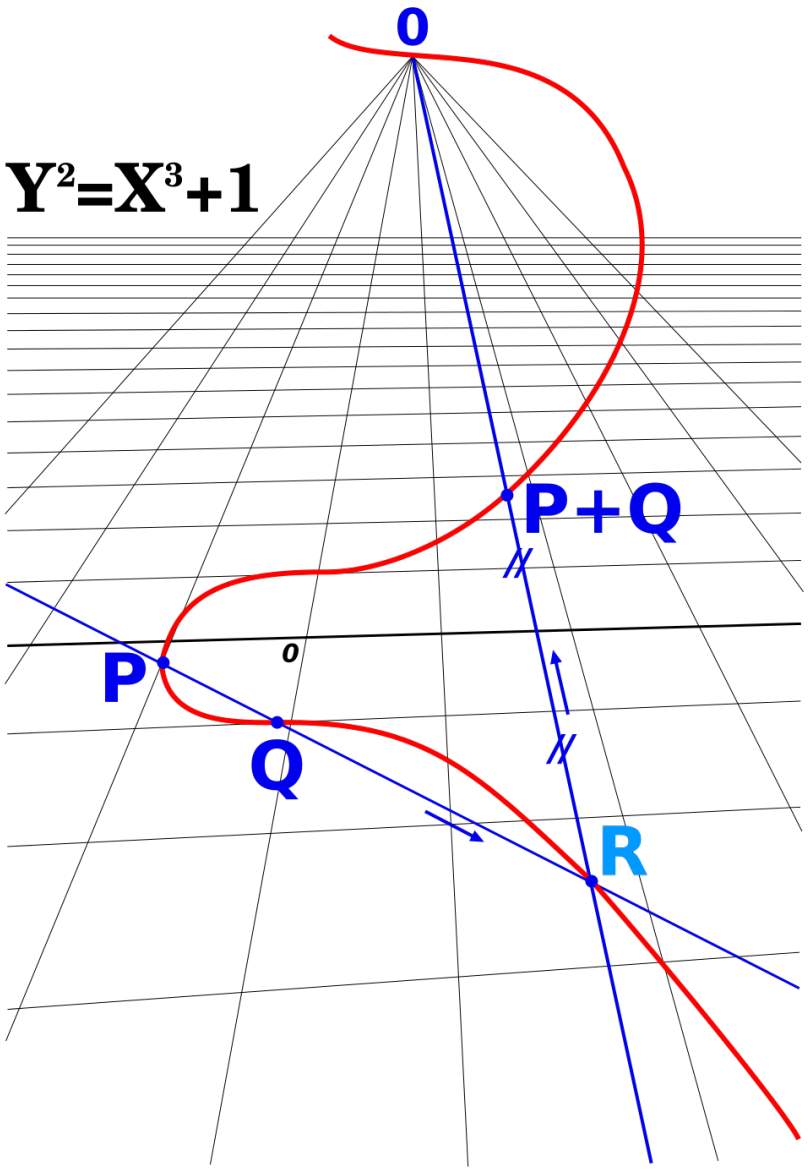

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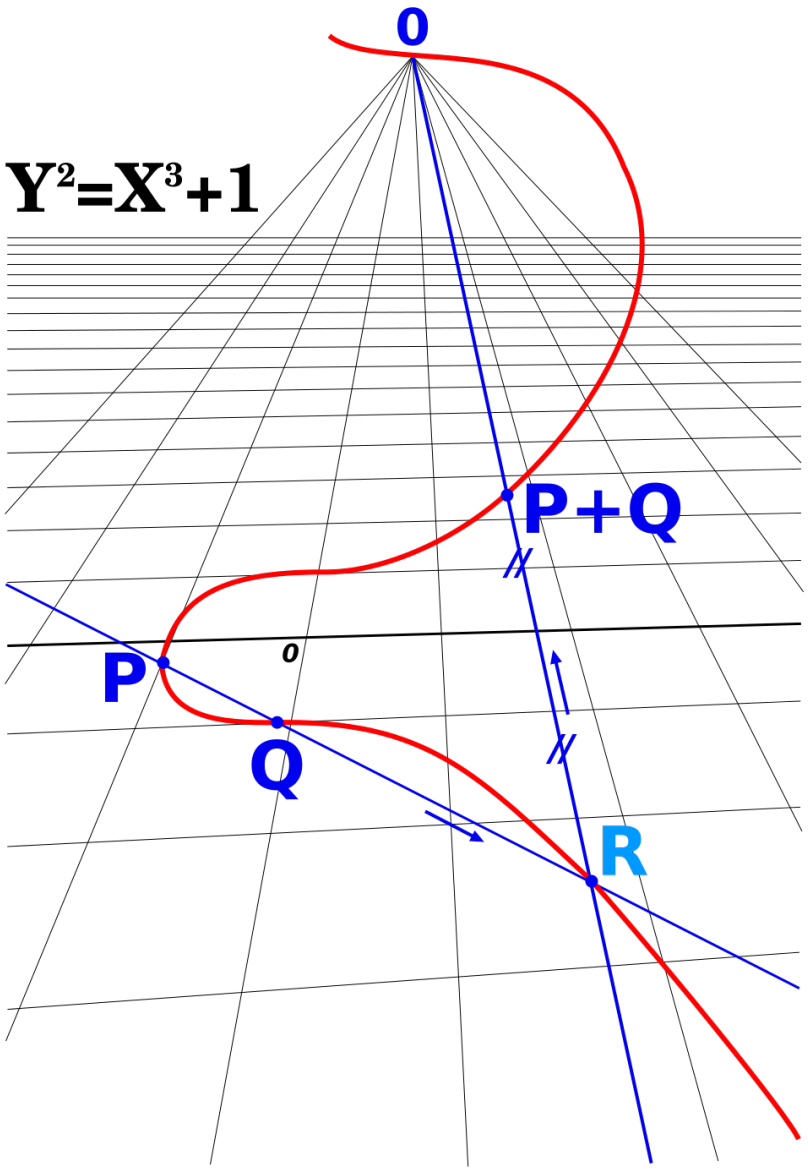

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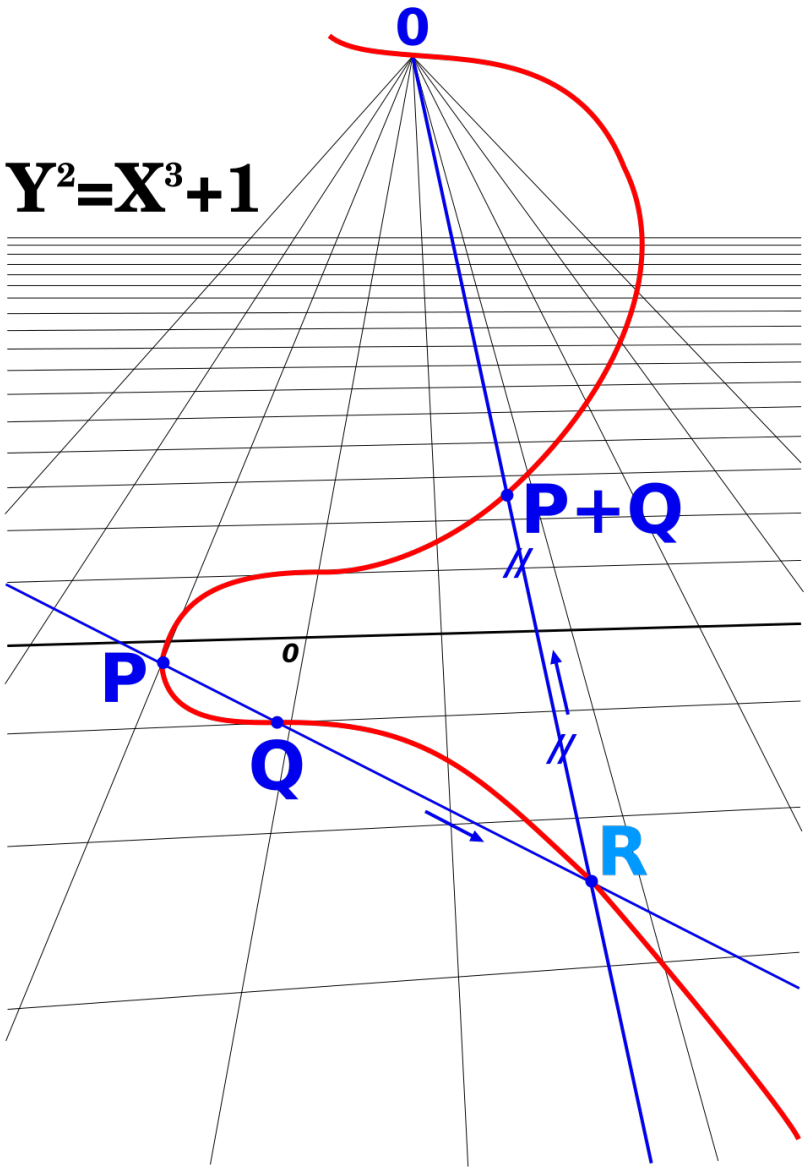

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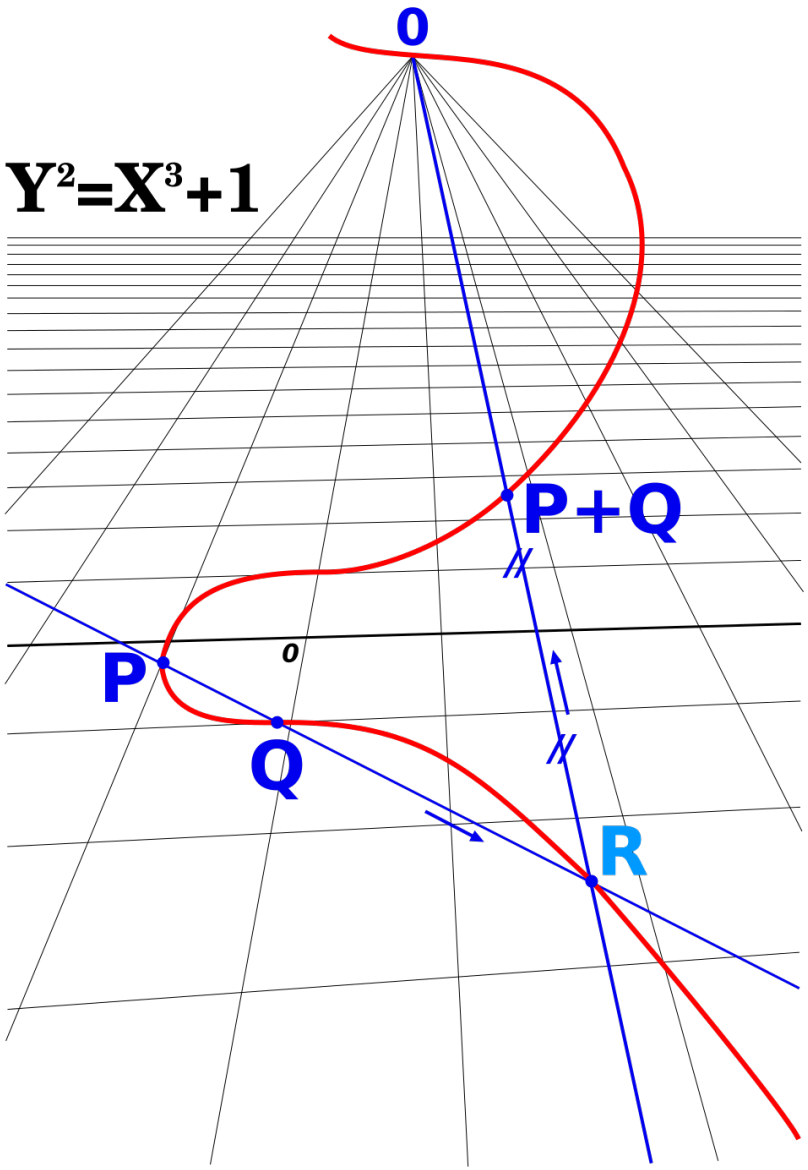
(Picture made by: Jean Brette.)



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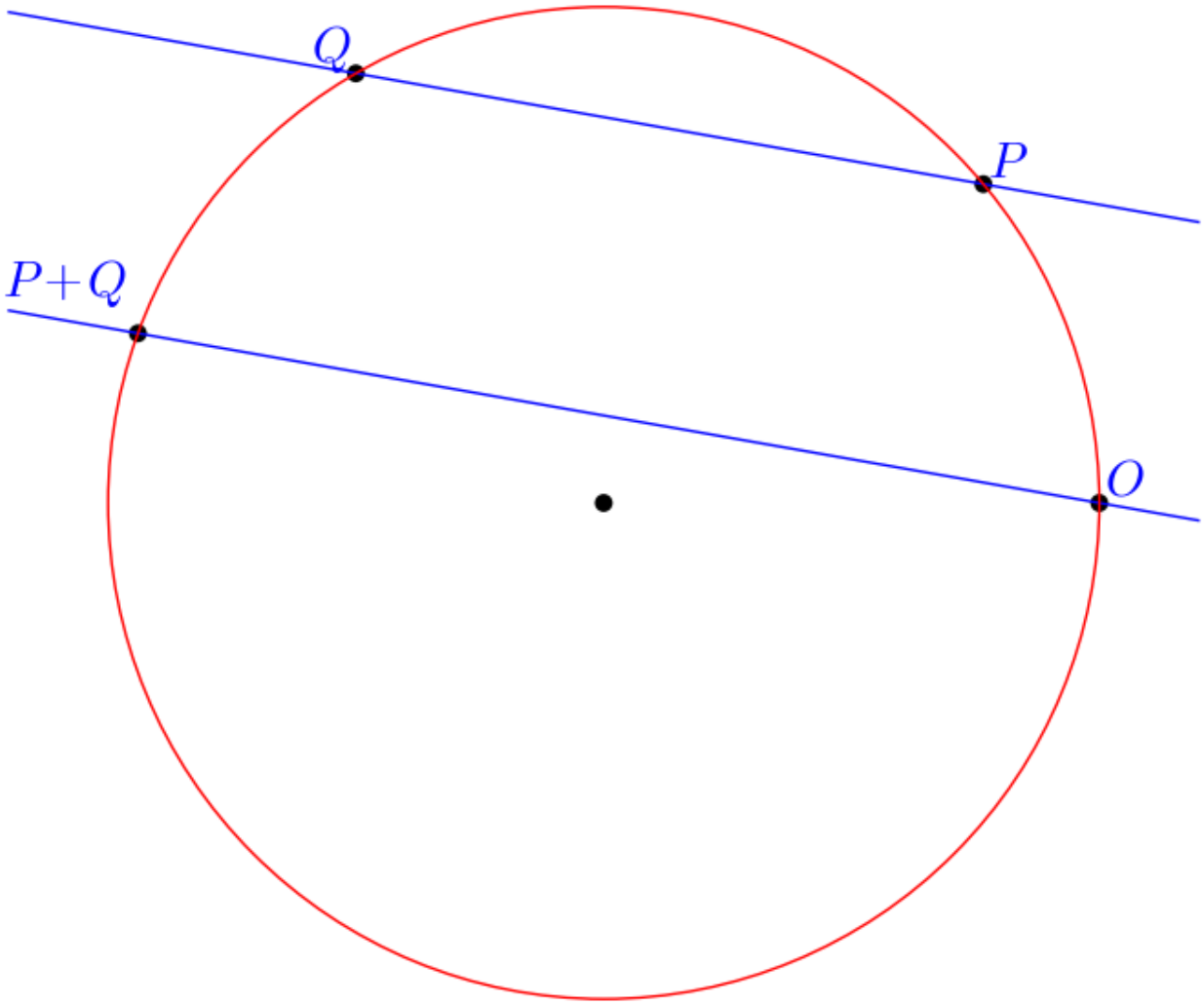

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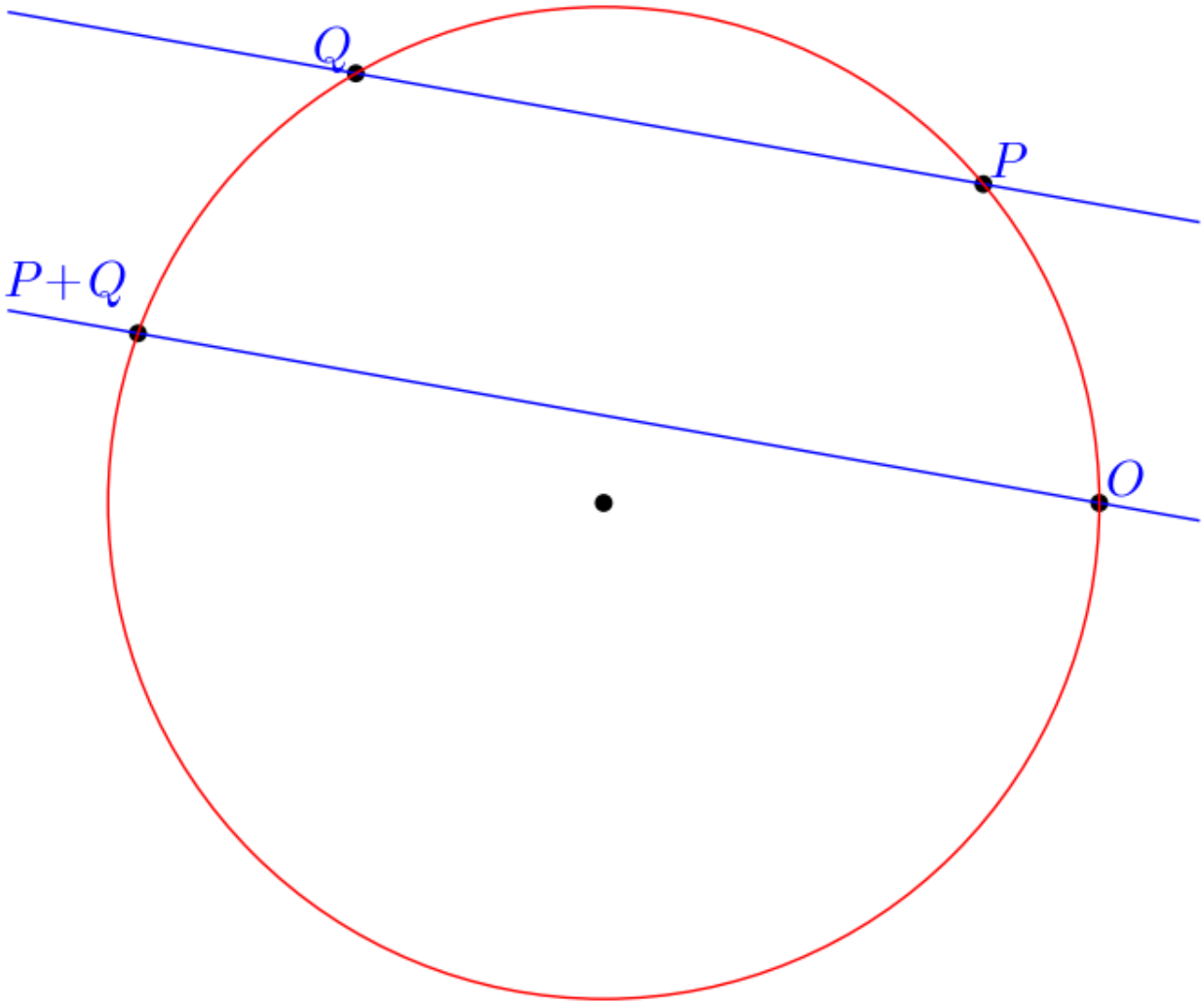

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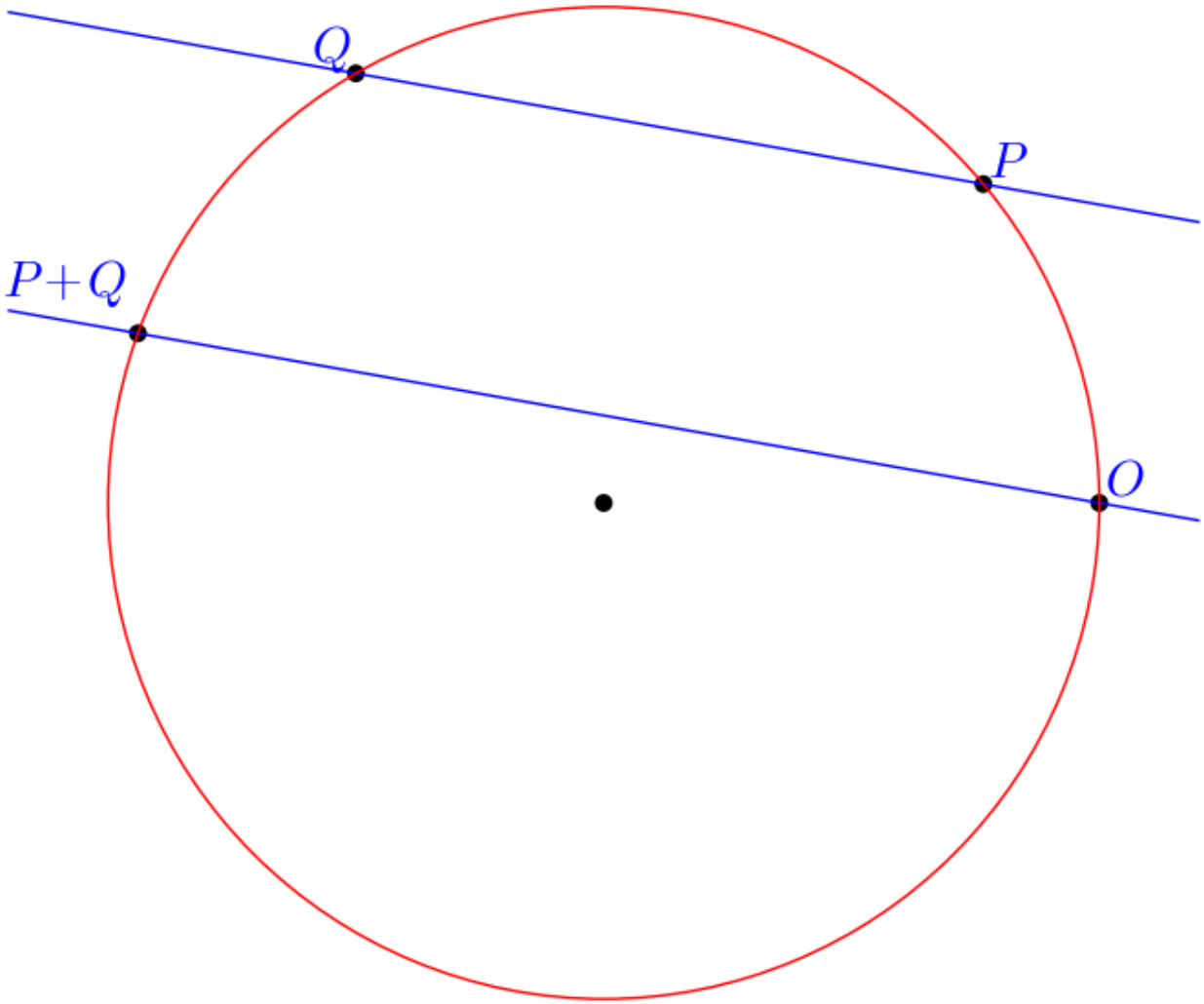

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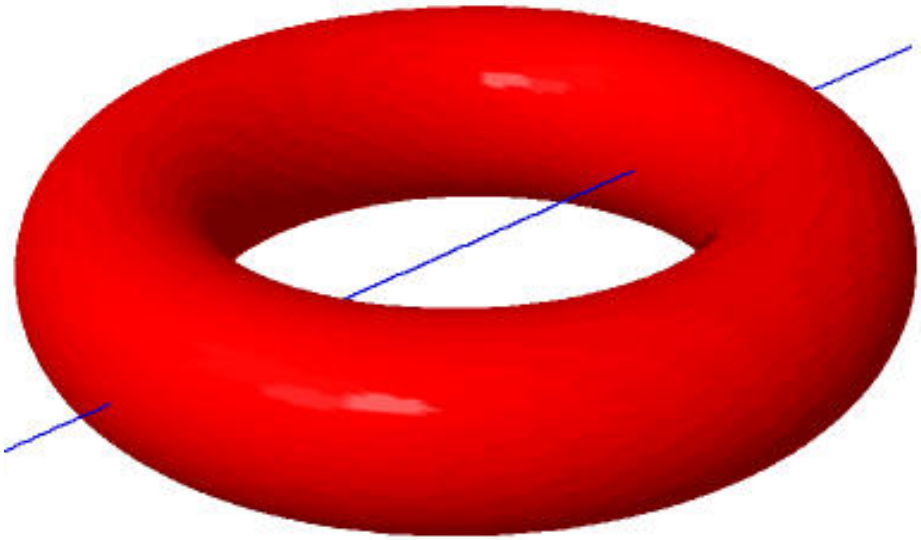
*Analysis.* Weierstrass functions. Let $L \subset \mathbb{C}$ be a lattice. Then

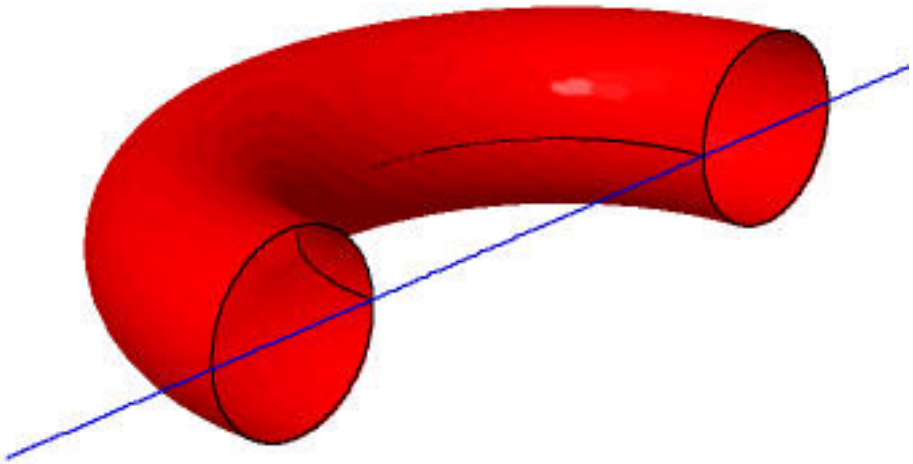
$$P\colon \mathbb{C} - L \to \mathbb{C}, \quad z \mapsto \frac{1}{z^2} + \sum_{\lambda \in L - \{0\}} \left( \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

is $L$-periodic, and $z \mapsto (P(z), P'(z))$ gives $\mathbb{C}/L \to E_L(\mathbb{C})$.

# Elliptic curves as double cover of $\mathbb{P}^1$

*Riemann surfaces.* $E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, $(x, y) \mapsto x$ is a 2 to 1 map with 4 ramification points. It is the quotient map for a rotation about 180°.





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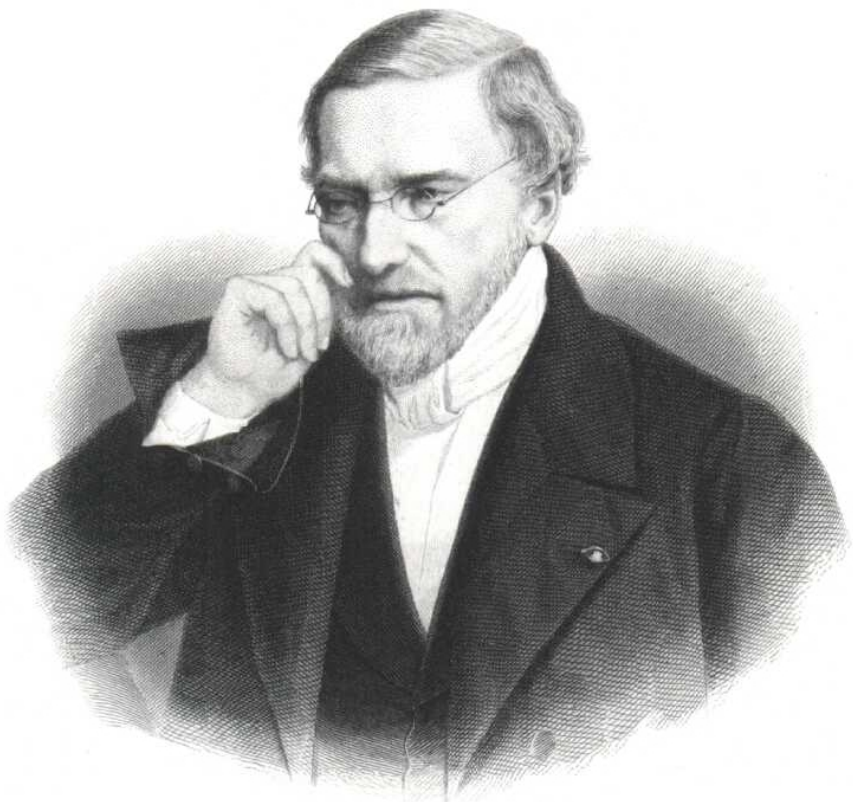
These are affine transformations: $P \mapsto a(P) + B$, with $a$ in $\mathrm{Aut}(E, O)$ and $B$ in $E$.

# Poncelet

Jean-Victor Poncelet (1788–1867) was a French engineer and mathematician who served most notably as the commandant general of the École polytechnique. He is considered a reviver of projective geometry, and his work "Traité des propriétés projectives des figures" is considered the first definitive paper on the subject since Gérard Desargues' work on it in the 17th century.



Source: wikipedia.

# Poncelet's closure theorem, 1822

*Let $C$ and $D$ be two plane conics. If it is possible to find, for a given $n > 2$, one $n$-sided polygon that is simultaneously inscribed in $C$ and circumscribed around $D$, then it is possible to find infinitely many of them.*

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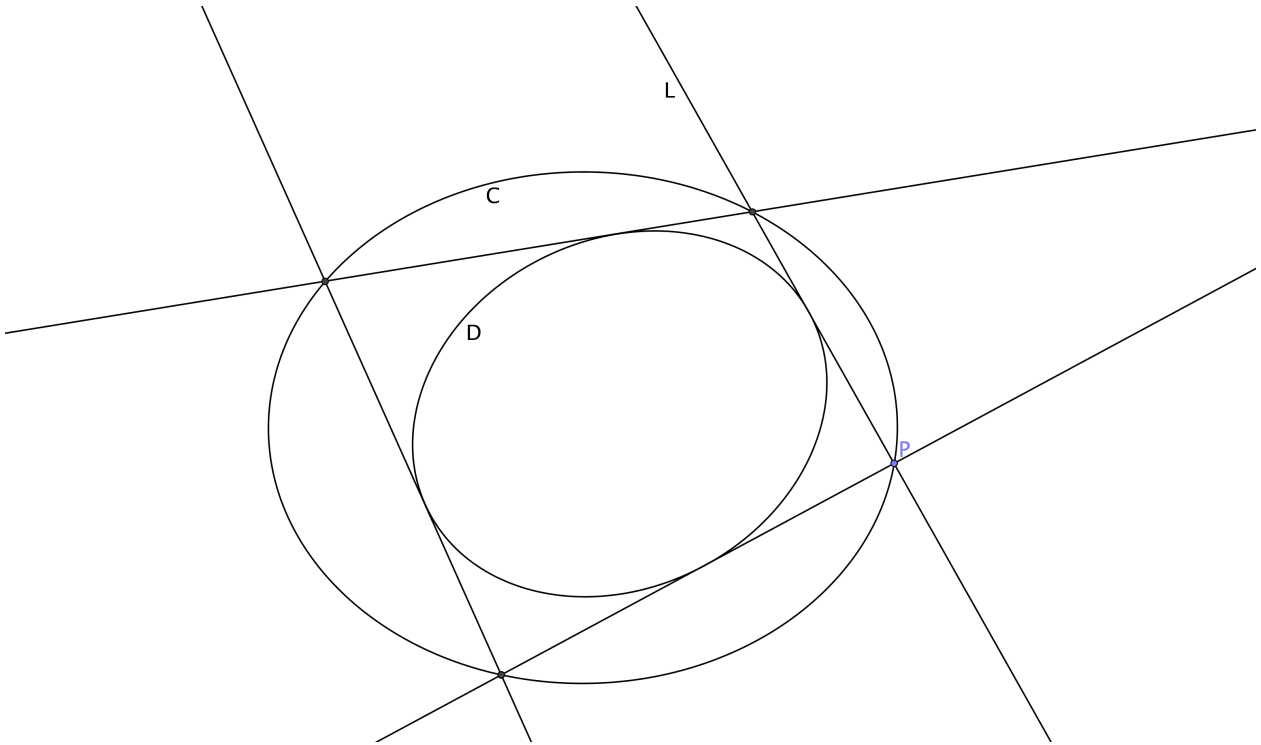
In fact, the construction "moves": one can move the starting point on *C* anywhere.



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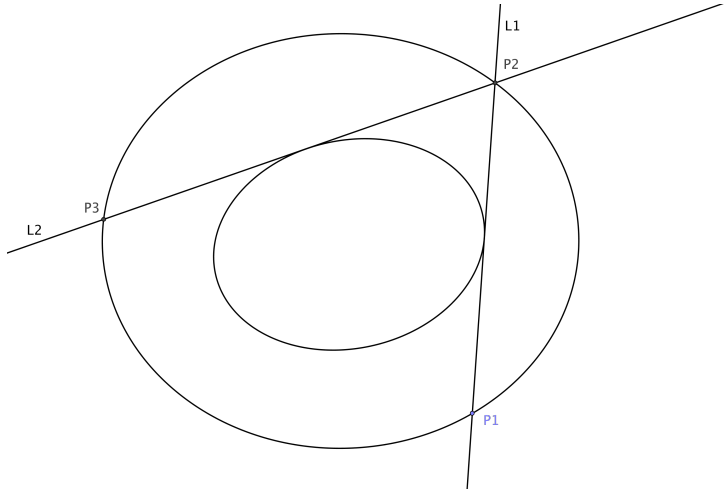

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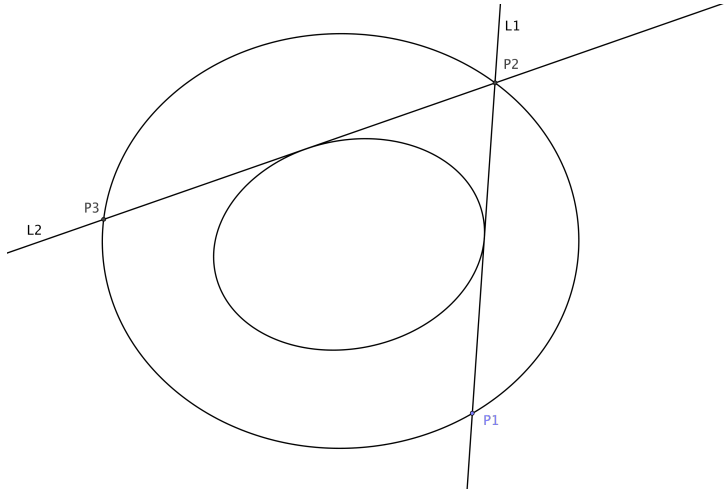


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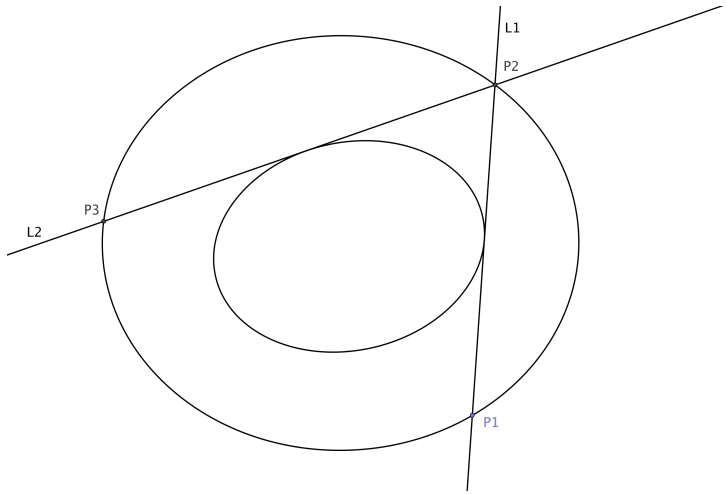


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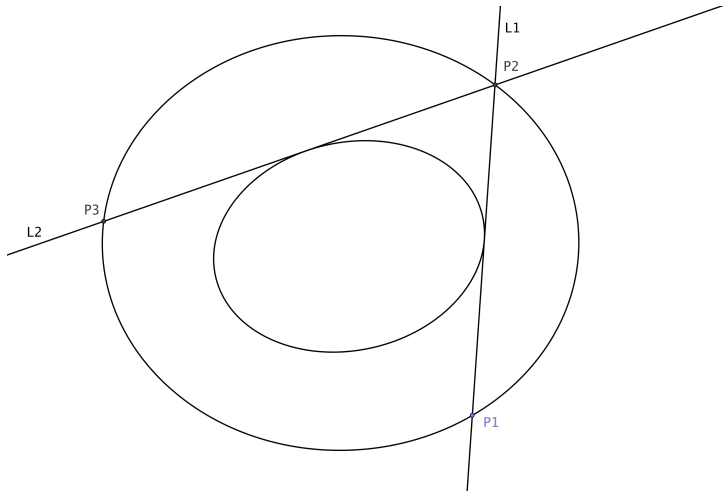


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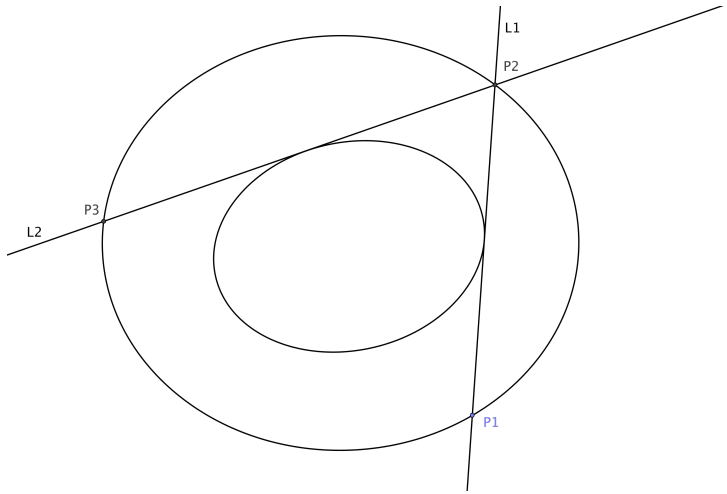


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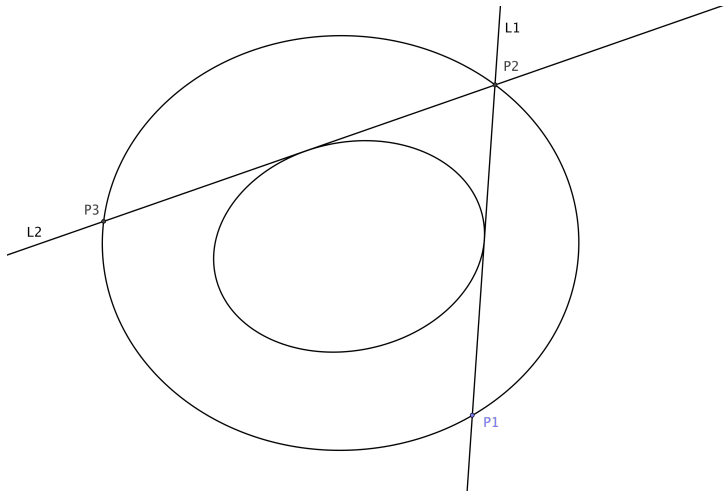


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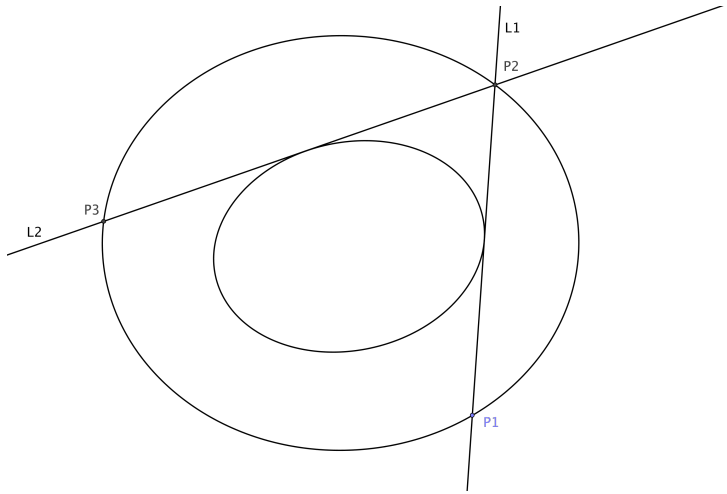


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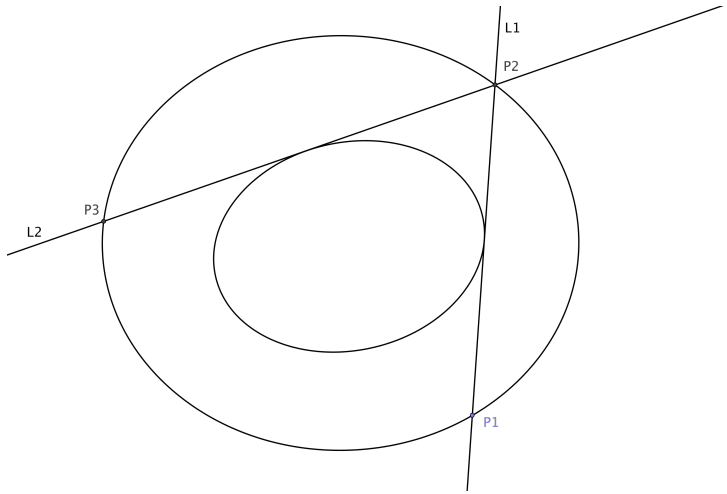


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3. Duistermaat has written a whole book on such dynamical systems: *Discrete Integrable Systems, QRT maps and Elliptic Surfaces.*

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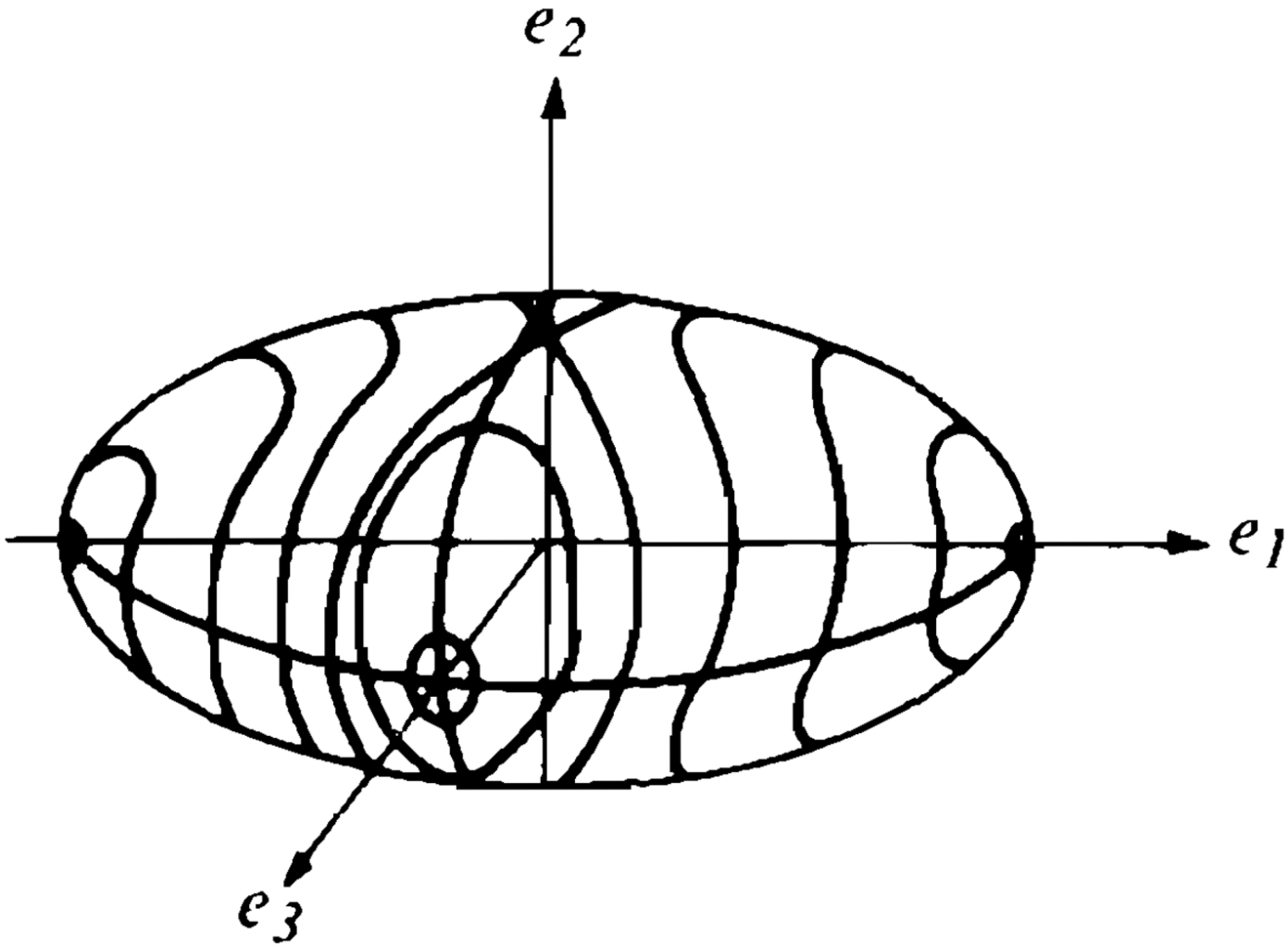
So the movement is over the intersection of the level surfaces: elliptic curves! Indeed, complex projectively, the first quadric is $\mathbb{P}^1 \times \mathbb{P}^1$, and the intersection with the second quadric is a $(2,2)$-curve! So, same situation as with Poncelet.

# Intersections of 2 ellipsoids, one fixed




Picture from V.A. Arnold's book "Math. Methods of Class. Mech."

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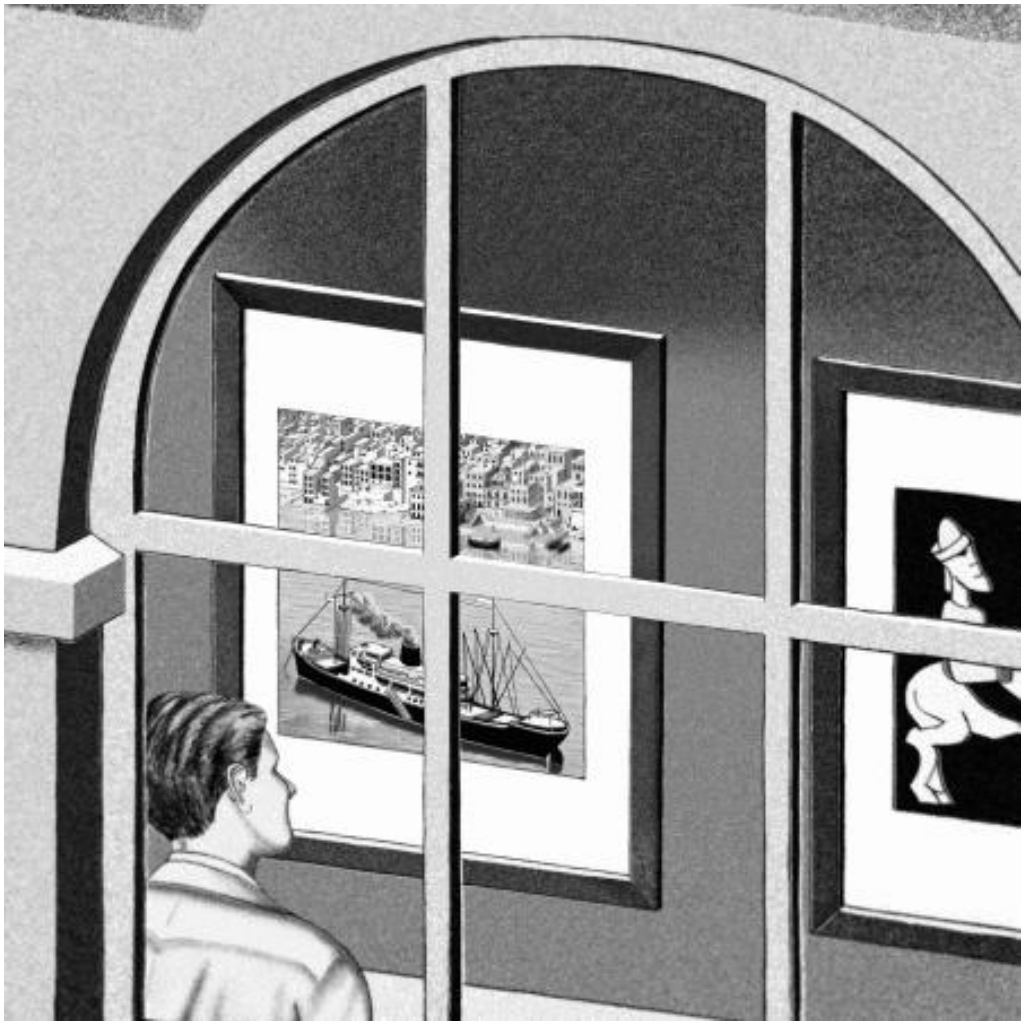
- Explicit solutions for rotation of a rigid body involve Weierstrass functions, or other functions parametrising elliptic curves. That flow is also not algebraic.
- To prove that the vector field is translation invariant, it suffices to see that on the projective complex elliptic curves it has no poles. A simple but annoying computation, and it makes the outcome look like a miracle. Conversely, if one knew that the translations by the flow are algebraic, then one could deduce (without computation) that the vector field (on the complex projective curves) has no poles.

# Escher

Source: `http://escherdroste.math.leidenuniv.nl`

# Droste-Escher

Escher's print gallery is a transformed Droste picture: the "straight picture" contains a copy of itself, scaled by $q := 1/256$.



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So, on $\mathbb{C}$ we have the picture $\tilde{f}\colon \mathbb{C} \to X$, invariant under the lattice $\mathbb{Z}\cdot 2\pi i + \mathbb{Z}\cdot \log(q) = \mathbb{Z}\cdot i\cdot 6.283\ldots + \mathbb{Z}\cdot 5.545\ldots$.

# The lattice-invariant picture on $\mathbb{C}$

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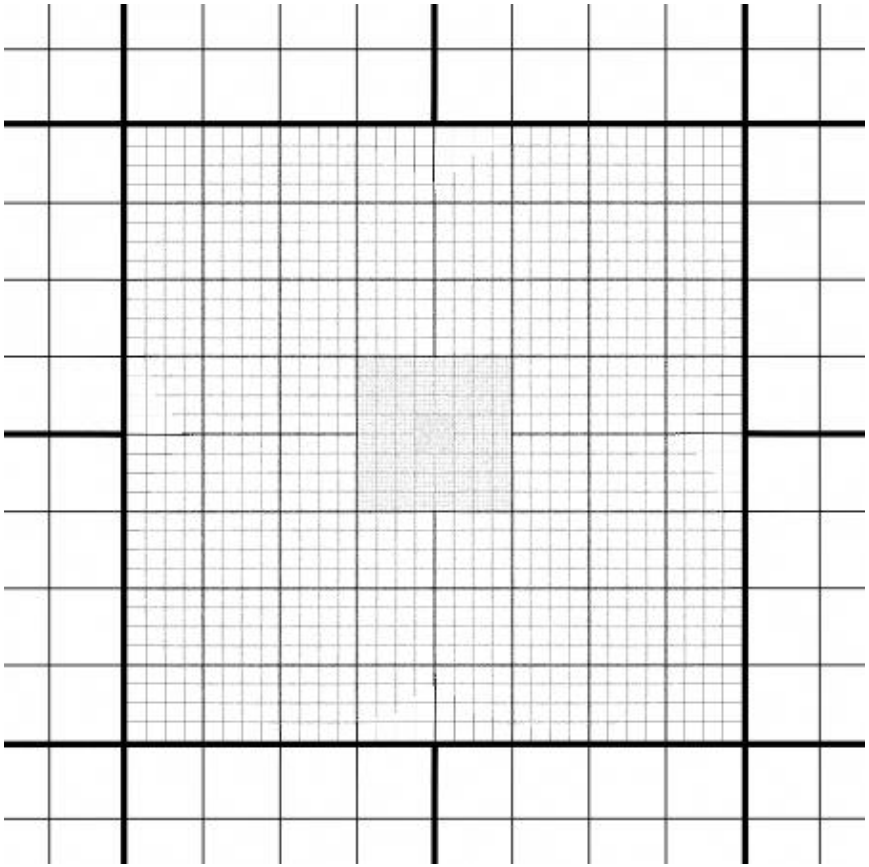
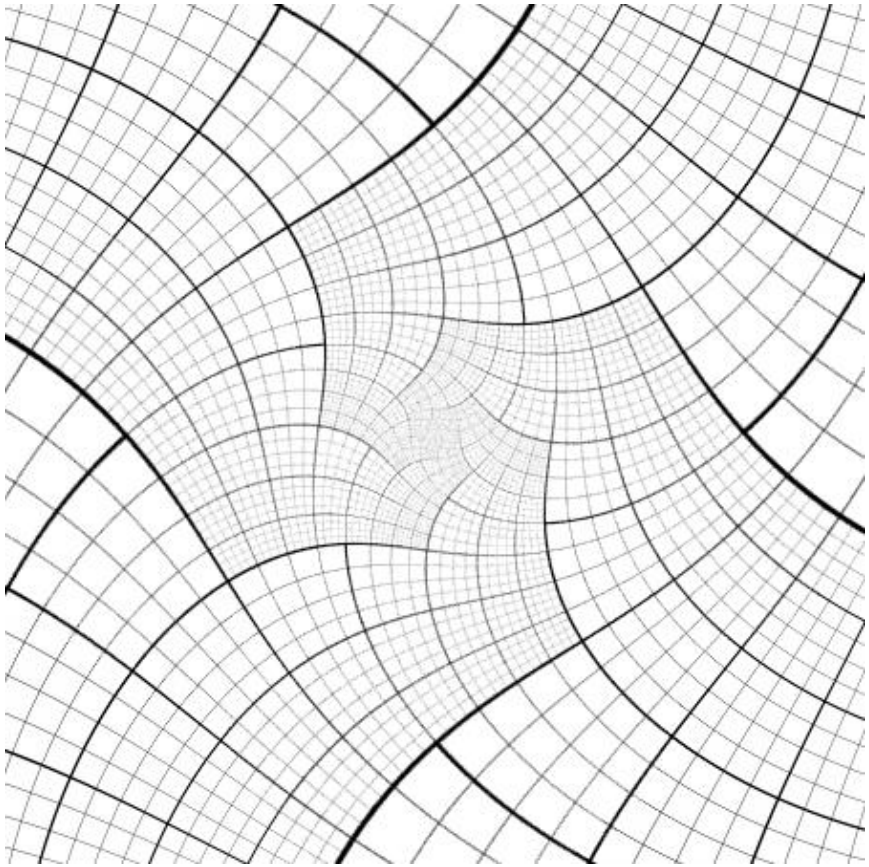
# The Droste and Escher grids





See the animation

`http://escherdroste.math.leidenuniv.nl/index.php?menu=animation&sub=bmpeg&a=1&b=1`

# Elliptic curves over finite fields $\mathbb{F}_p$

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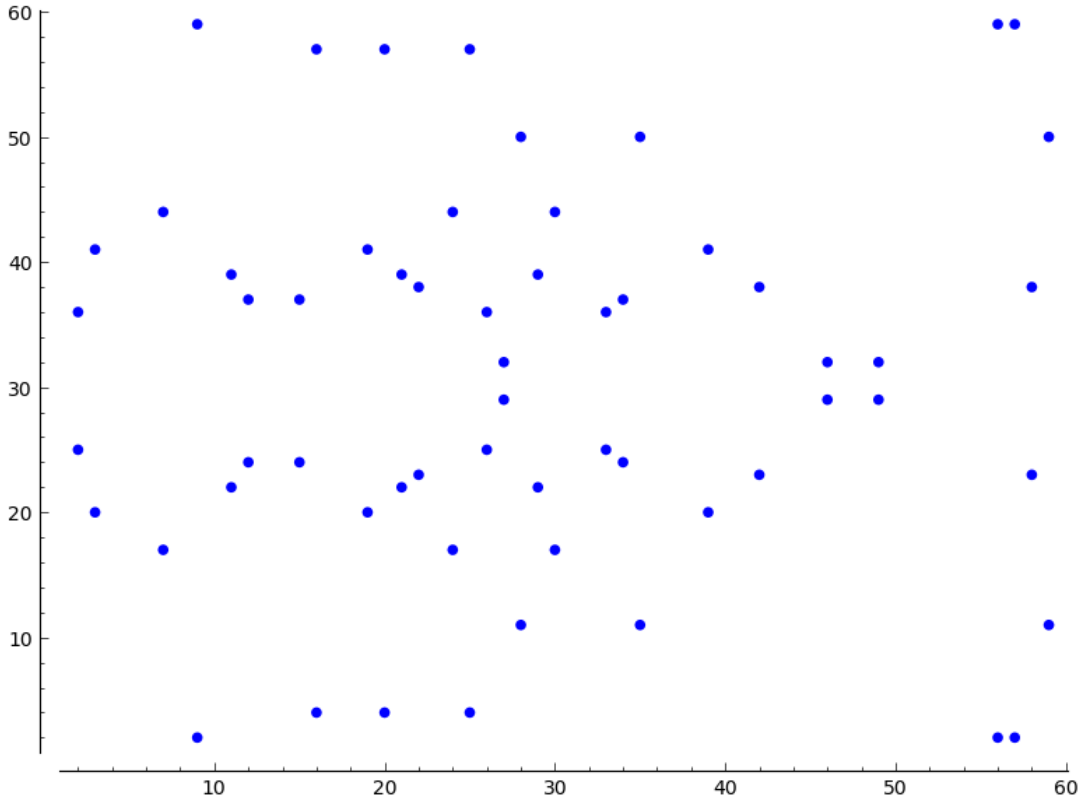


$$\cup \{O\}$$

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This number $n$ is also prime, and that is important.

# Richard Serra's "torqued ellipse", Guggenheim, Bilbao

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So he rolls a plane around the ellipses, or rolls his wheel on a sheet of lead.

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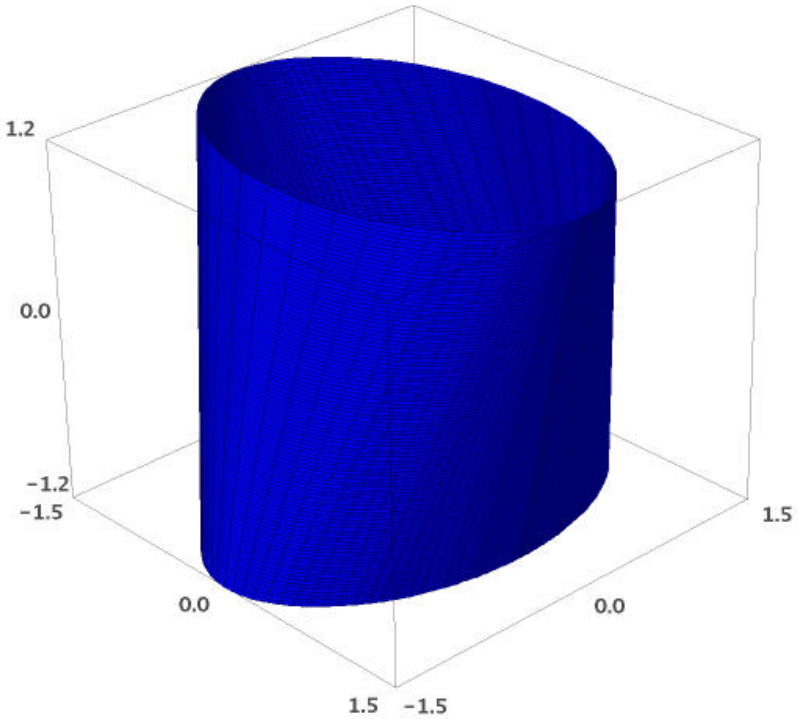
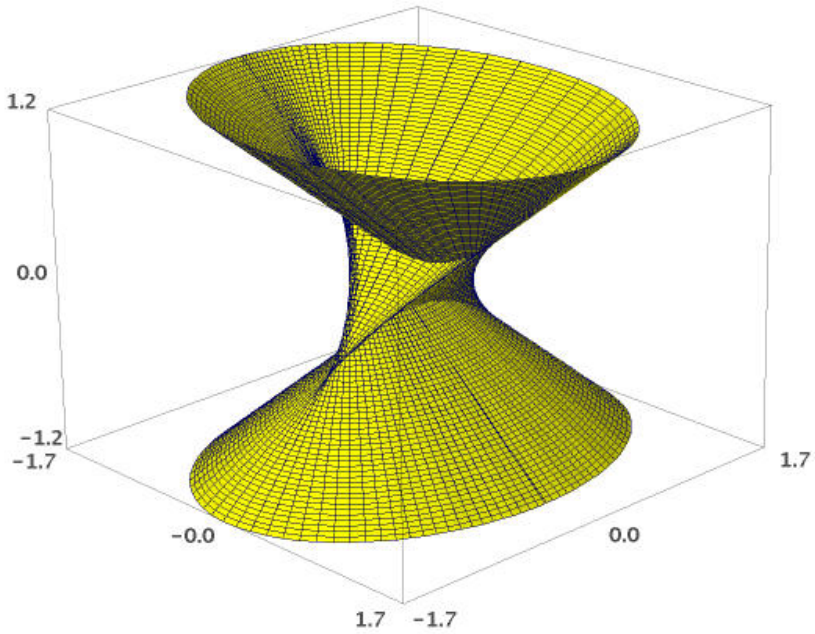
These two halves cannot be separated algebraically, Serra's surface has a siamese twin.

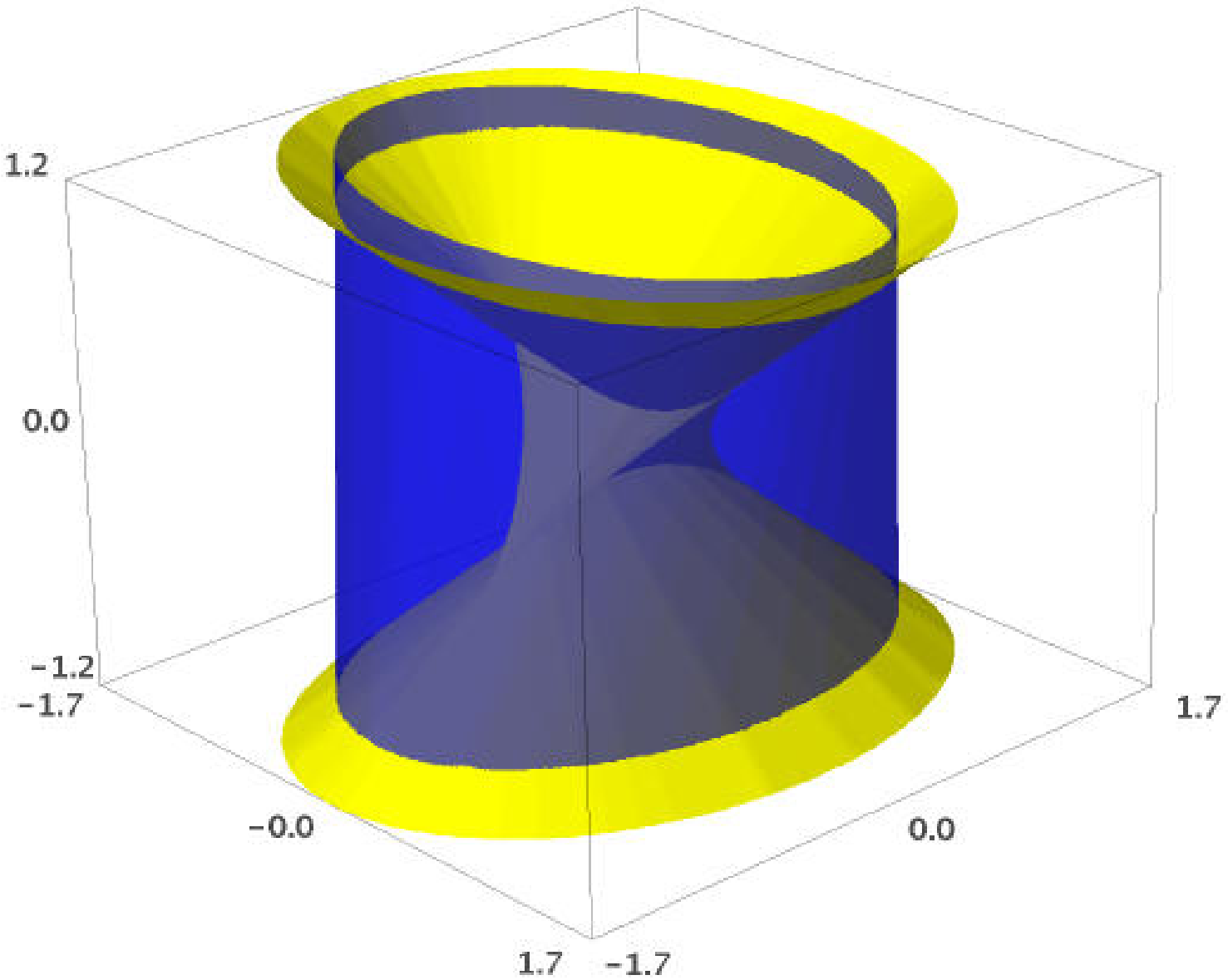
# Siamese twins, apart




Pictures and computations by sage.

# Siamese twins, together



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*Proof*: use complex projective geometry. $L := H_1 \cap H_2$.

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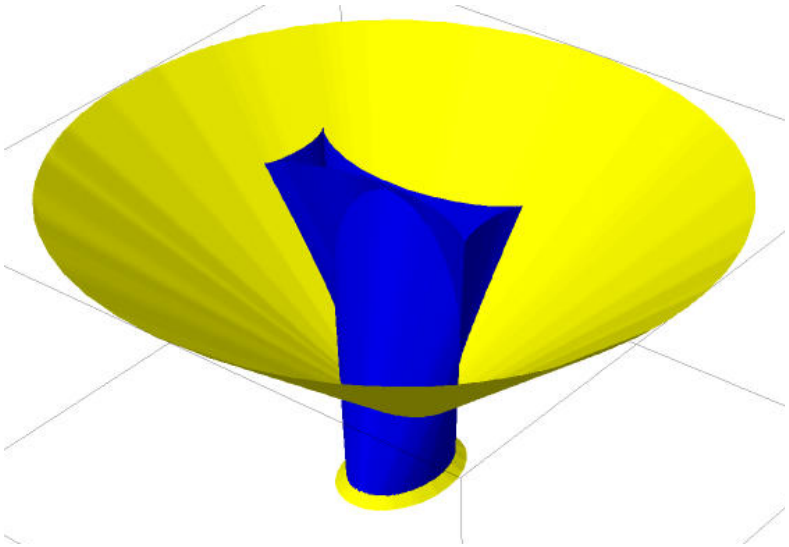
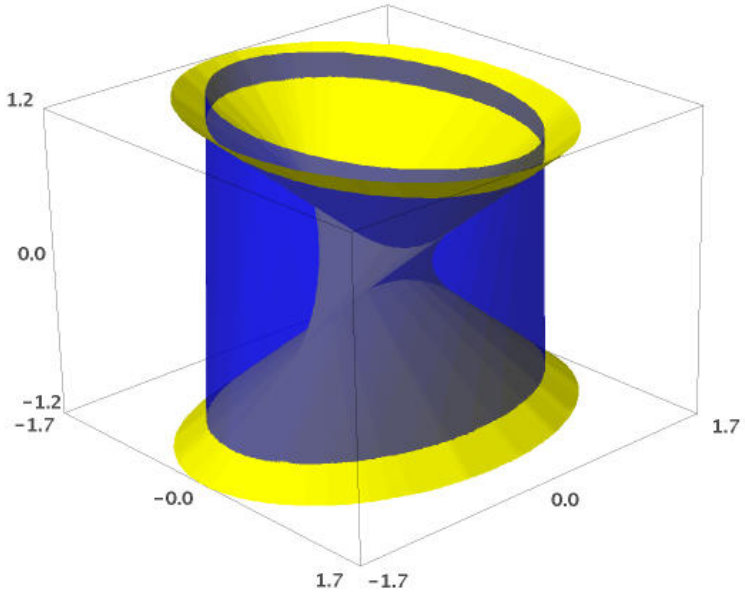
Serra's surface $S$ is parametrised by a $\mathbb{P}^1$-bundle over $E$, in fact by $\mathbb{P}^1 \times E$.
In technical terms: the normalisation is $\mathbb{P}^1 \times E$.

# Some more pictures, and an automorphism



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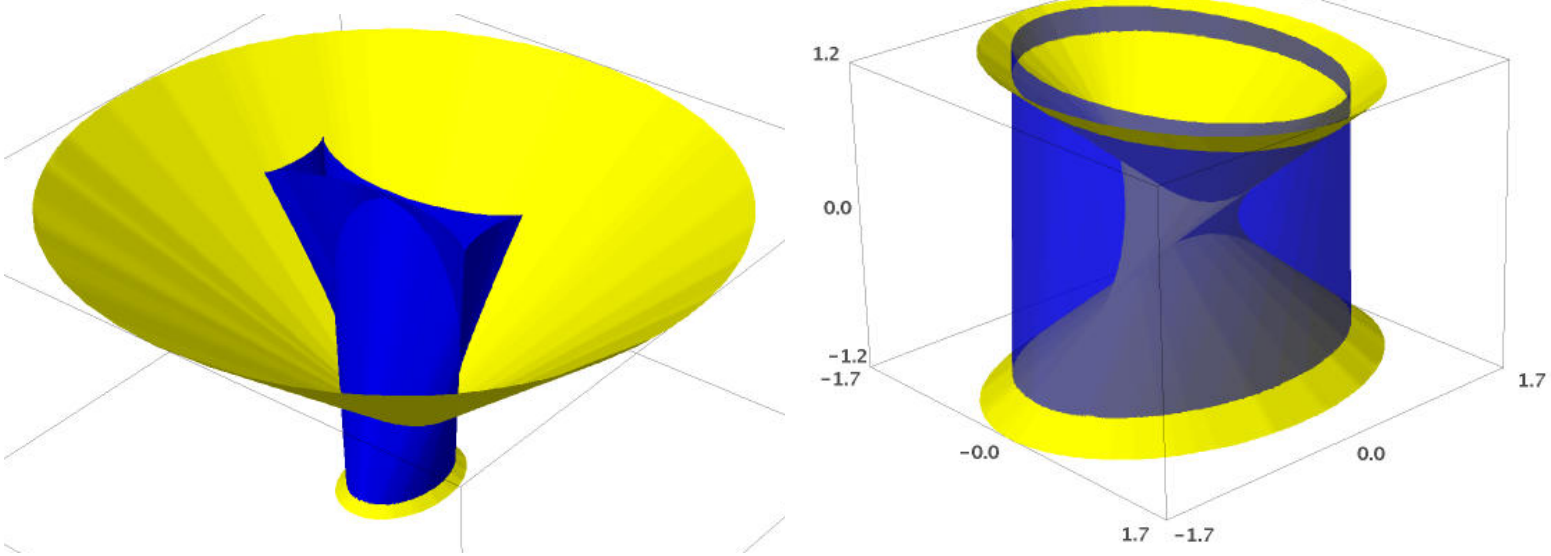


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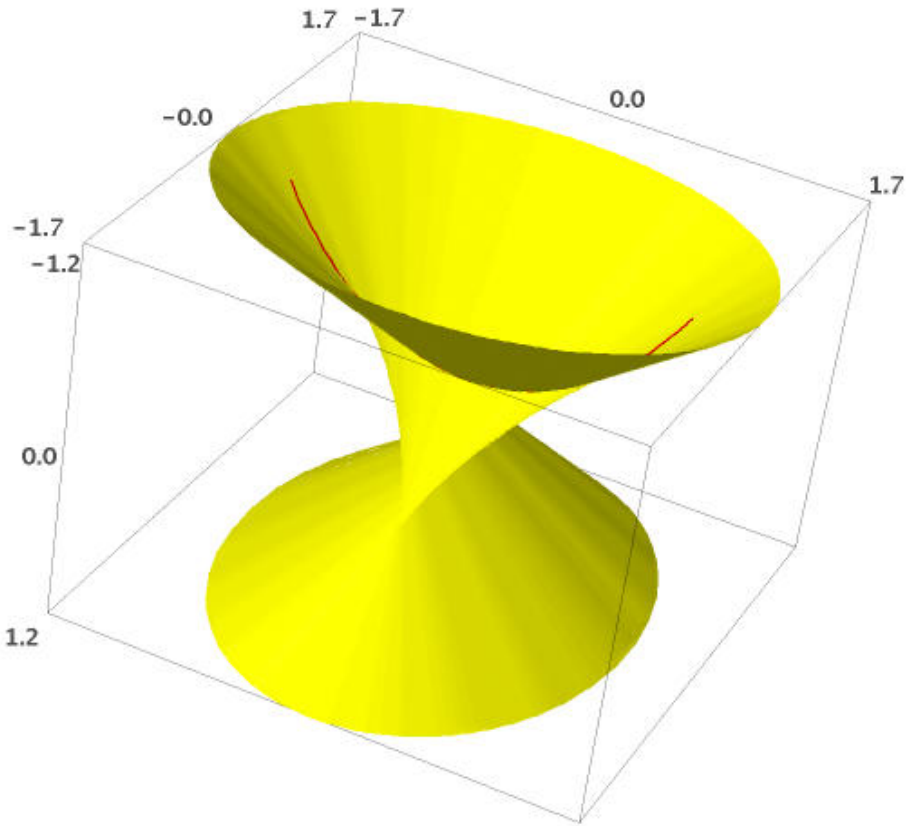
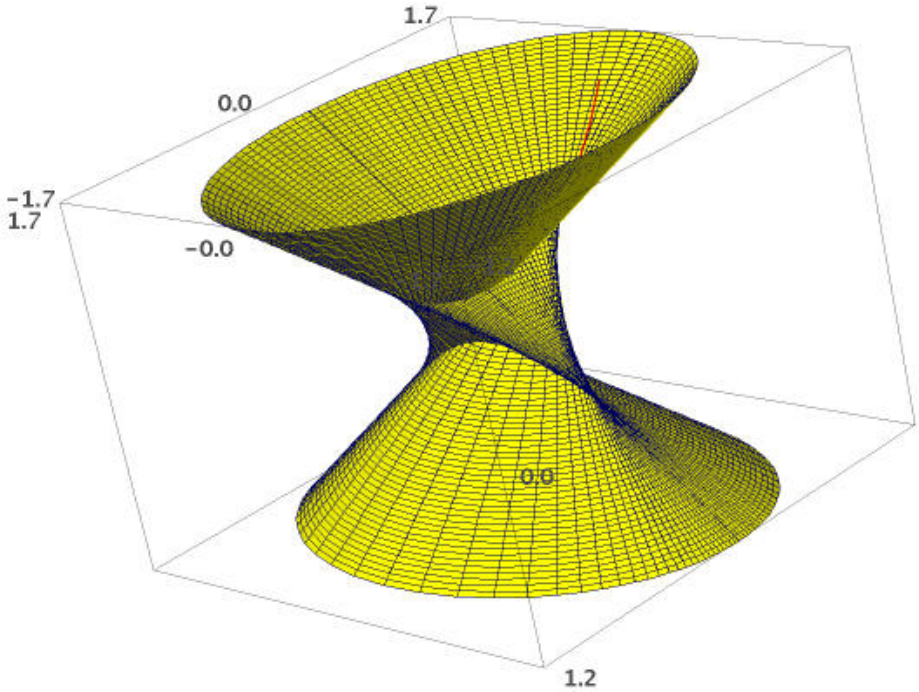
Indeed (Maarten Derickx): the reflection in $\mathbb{P}^3$ with respect to the plane $H_1$ and the center of $C_2$ does this.

# A one-dimensional part of $S(\mathbb{R})$

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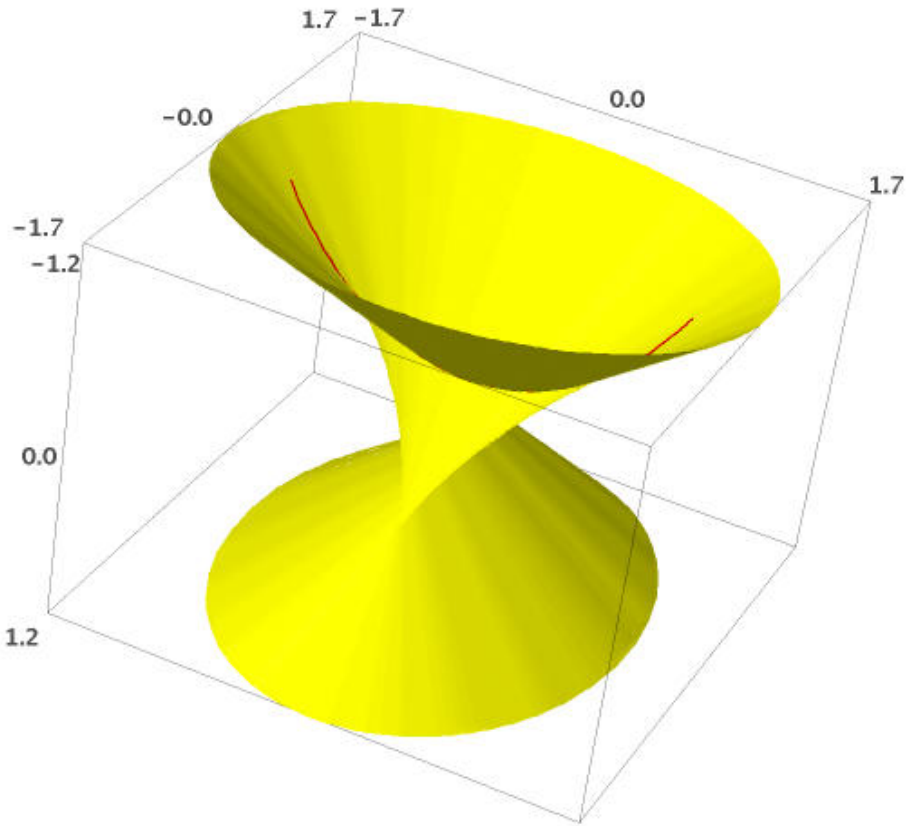


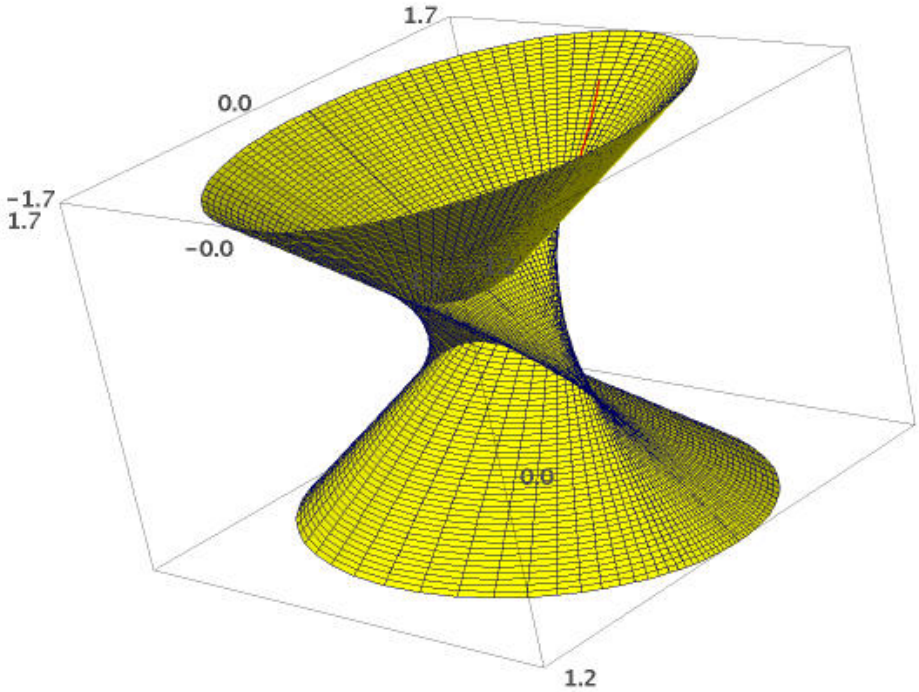


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It is explained by conjugate intersecting lines.

# Some metric properties

Away from the singularities, the yellow and blue surfaces have Gaussian curvature zero: they are locally a boundary of their convex hull, and are ruled. The Gaussian curvature is the product of the two main curvatures, one of which is zero.

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Hard to imagine, but we can imagine the case of two circles, giving a cone. The cone can roll on the plane.

# 3D-printing

Oliver Labs is a mathematician in Mainz, with an interest in computer science and design.
He converted my Sage output to input for a 3d-printer, so that I could have it printed by Shapeways.
Check it out!

`http://www.oliverlabs.net/`

`http://www.shapeways.com/art/mathematical-art?li=nav`

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- To you for your attention!