Bas Edixhoven, Lausanne, 2017/12/12, Conf. Recent devel. on arithm. of special values ...., 11:00 – 12:00.

Pink's conjecture on unlikely intersections and fam's of semi-ab.-var's.

Joint work with Daniel Bertrand, arxiv 2011 (app. by me). I just got back working on this again, I hope it will be finished soon.

1. **Context**. Pink 2005 preprint "A common generalisation of André-Oort, Manin-Mumford and Mordell-Lang", also called "Pink's unlikely intersection conjecture". (unlikely: expected $\emptyset$ bec. of dim.)

Thm 6.3. conj. $\Rightarrow$ relative Manin-Mumford for fam's of semi-ab. varieties. (SARMM)

Problem: Bertrand gave a counterexample to SARMM: "Ribet sections".

Solution: Thm 6.3 ($\Rightarrow$) is wrong (o.k. for ARMM) ~~...~~ and Ribet sections are in fact evidence for Pink's conjecture.

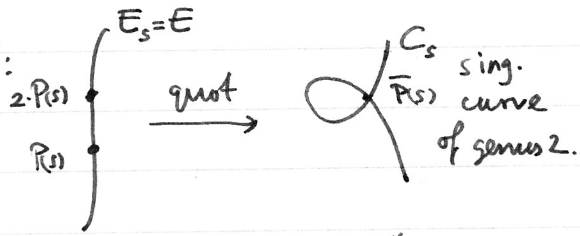
Conclusion: MHS are the right context for SARMM, not only subgr. are special subv's.

Remark: very recent preprint by Bruno Klingler on Hodge loci & atypical intersections. atypical: of higher dim. than expected. Generalises Pink's conj. ~~to~~ (mixed Shim. var's) to arbitrary VMHS.

2. **Simplest example**. $E/\mathbb{C}$ ell. curve with CM, $a \in \mathrm{End}(E)$ with $a \notin \mathbb{Z}$, $S := E - $ suitable fin. set.

$$E_S = E \times S \downarrow S, \quad P := \Delta \text{ (section)} \qquad S \overset{2P}{\underset{P}{\rightrightarrows}} E_S \xrightarrow{\text{quot}} C, \quad E_S \downarrow S, \ C \to S$$

$\forall s \in S$:



$J := \mathrm{Pic}^0_{C/S}$: line bundles on E, ~~...~~ glued at $2P$ and P: $\mathbb{G}_{m,S} \hookrightarrow J \overset{\text{quot}^*}{\twoheadrightarrow} E_S$

$J_s = \{\text{degr. 0 div. on } E - \{2P(s), P(s)\}\} / \{\mathrm{div}(f) : f \in \mathbb{C}(E)^\times \text{ reg. at } P(s), 2\cdot P(s) + \text{same value}\}$.

Let $r_a \in J(S)$ given by $D_a := a_*((2P) - (P)) - a^*((2P) - (P))$ (shrink S ...).

$$J \twoheadrightarrow E_S, \quad r_a \text{ section of } J \to S, \quad \overline{r_a} = (a - \overline{a})P \text{ not constant.}$$

Then: 1. $\overline{\bigcup_{n \in \mathbb{Z}} (n \cdot r_a)(S)}^{\mathrm{Zar}} = J$

2. $\forall n \geq 1$ $\forall s \in S$ with order$(s) = n$: $n^2 r_a(s) = 0$ in J_s, and if $\gcd(n, \deg(a - \overline{a})) = 1$, then $\exists$ such s with order$(r_a(s)) = n^2$.

Proof of 1: $[J] \in \mathrm{Ext}(E_S, \mathbb{G}_{m,S})$ is not torsion.

Proof of 2. Let $n \geq 1$, $s \in S \subset E$ order n, write P for $P(s)$.

Let $f \in \mathbb{C}(E)^\times$ s.t. $\mathrm{div}(f) = n\cdot[2P] - n\cdot[P]$ on E.

Then $\mathrm{div}(f\circ a) = \mathrm{div}(a^*(f)) = a^*(\mathrm{div}(f)) = a^*(n\cdot[2P] - n\cdot[P])$.

$\mathrm{div}(\mathrm{Norm}_a f) = a_*(\mathrm{div} f) = a_*(n\cdot[2P] - n\cdot[P])$.

Let $g_a := \mathrm{Norm}_a(f)/a^*f \in \mathbb{C}(E)^\times$. Then $\mathrm{div}(g_a) = n\cdot D_a$ on E.

Hence $n\cdot r_a(s) = g_a(2P)/g_a(P) \in \mathbb{C}^\times$. (by def. of S: $\mathrm{div}(g_a)$ and $\mathrm{div}(f)$ disjoint).

Weil reciprocity gives:

$$\left(\frac{g_a(2P)}{g_a(P)}\right)^n = g_a(\mathrm{div} f) = f(\mathrm{div}\, g_a) = f\left(\mathrm{div}(\mathrm{Norm}_a f) - \mathrm{div}(a^*f)\right) =$$

$$= \frac{f(a_*\mathrm{div}(f))}{(a^*f)(\mathrm{div} f)} = \frac{f(a_*(\mathrm{div} f))}{f(a_*(\mathrm{div} f))} = 1.$$

For order $= n^2$ part: use Weil pairing.

3. Generalisation to Poincaré tensors. Situation: S a scheme, A/S ab. scheme, A'/S its dual ($\mathrm{Pic}^0_{A/S}$), P the Poincaré $\mathbb{G}_m$-torsor on $A\times_S A'$ (univ. line bdl. − 0-section, trivialised on $A\times\{0\} \cup \{0\}\times A'$, thm. of the cube gives biextension structure), and $f: A'\to A$, $f': A'\to A'' = A$ its dual. Let $\alpha := f - f' : A' \to A$.

Ribet observed: $P|_{\text{graph of } \alpha}$ is trivial. $\mathbb{G}_{m,A'} \hookrightarrow P \twoheadrightarrow A_{A'} = A\times_S A'$, $P \to A'$ with section r_f (Ribet section attached to f), $(\alpha, \mathrm{id}): A' \to A_{A'}$.

Same properties: "r_f(torsion) = torsion of quadratic order".

(Proof: $\forall T\to S$, $\forall x,y \in A'(T)$: $P(f'y, x) = P(fx, y)$ (can. isom.! of $\mathbb{G}_m$-torsors on T)

Take $y = x$: $P(f'x, x) = P(fx, x)$, hence $P((f-f')x, x) = \mathbb{G}_{m,T}$, giving $r_f(x) \in P(\alpha x, x)(T)$.)

Now we want to explain that this is all explained by special subvarieties in mixed Shimura varieties, so we have to start talking about Mixed Hodge Structures. This must be known to experts of Mixed Shimura Varieties, but Daniel and I want the people working on unlikely and atypical intersections to understand this. (Indeed there is Milan Lopuhaä's MSc thesis, directed by Lenny Taelman, 2014, that gives this, but I want it in terms of moduli of MHS.)

3. Mixed Hodge structures.

For M a free $\mathbb{Z}$-module of finite rank, a MHS on M is:

(1) an increasing filtration $W_\bullet$ on M, $n \ll 0 \Rightarrow W_n = 0$, $n \gg 0 \Rightarrow W_n = M$, $\forall n$ $\mathrm{Gr}_n W$ free.

(2) a decreasing filtr. $F^\bullet$ on $M_{\mathbb{C}} := \mathbb{C}\otimes M$, $n \gg 0 \Rightarrow F^n M_{\mathbb{C}} = \{0\}$, $n \ll 0 \Rightarrow F^n M_{\mathbb{C}} = M_{\mathbb{C}}$, s.t. $\forall n \in \mathbb{Z}$ F induces $(\mathrm{Gr}_n W)_{\mathbb{C}} = \bigoplus_{p+q=n} (\mathrm{Gr}_n W)^{p,q}$ (pure H.S. of weight n).
$= F^p(\mathrm{Gr}_n W)_{\mathbb{C}} \cap \overline{F^q(\mathrm{Gr}_n W)_{\mathbb{C}}}$.

We are only interested in such of type $\{(-1,-1), (-1,0), (0,-1), (0,0)\}$, even more precisely: $W_{-3} = 0$, $W_{-2} \cong \mathbb{Z}(1)$, $\mathrm{Gr}_{-1} W \sim$ princ. polarisable ab.v., dim. d. $\mathrm{Gr}_0(W) \cong \mathbb{Z}(0)$. $\mathrm{rank}(M) = 2d+2$.

Note: $F^{-1} = M_{\mathbb{C}}$, $\dim F^0 = d+1$, $F^1 = \{0\}$.

4. The universal Poincaré torsor as mixed Shimura variety.

Let $d \geq 0$, $M = \mathbb{Z}^{2d+2}$ + st. basis $e_0, e_1, \ldots, e_{2d+1}$, with $W_{-3} = 0$, $W_{-2} = \mathbb{Z}\cdot e_0$, $W_{-1} = \mathbb{Z}\cdot e_0 \oplus \cdots \oplus \mathbb{Z}\cdot e_{2d}$, $W_0 = M$.

$$D := \left\{ F^0 \subset M_{\mathbb{C}} \text{ s.t. } (M, W_\bullet, F^\bullet) \text{ is a MHS of type } \{(-1,-1),(-1,0),(0,-1),(0,0)\} \text{ s.t. } \Psi := 2\pi i \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{J} : \mathrm{Gr}_{-1}(M) \longrightarrow \mathrm{Gr}_{-1}(M^\vee \otimes \mathbb{Z}(1)) \text{ is a polarisation} \right\}$$

Let $P \subset \underline{\mathrm{Aut}}(M) = GL_{2d+2,\mathbb{Z}}$ be the stabiliser of $W_\bullet$, $\mathbb{Z}(1) \xrightarrow{\sim} W_{-2}$, $\mathbb{Z}(0) \xrightarrow{\sim} \mathrm{Gr}_0 W$, and Ψ.

Then, $\forall$ $\mathbb{Z}$-alg. R:
$$P(A) = \left\{ \begin{pmatrix} \mu(g) & x & z \\ 0 & g & y \\ 0 & 0 & 1 \end{pmatrix} : \begin{array}{l} g^t J g = \mu(g)\cdot J \ (g \in GSp_J(R)) \\ x \in M_{1,g}(R),\ y \in M_{g,1}(R) \\ z \in R \end{array} \right\}$$

Example: $d = 0$, $M = \mathbb{Z}\cdot e_0 \oplus \mathbb{Z}\cdot e_1$, $D \rightleftarrows \mathbb{C}$, $\mathbb{C}\cdot(\lambda e_0 + e_1) \longleftarrow \lambda$, $P(\mathbb{Z}) = \begin{pmatrix} \pm 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \cap P(\mathbb{R}) = \begin{pmatrix} \mathbb{R}^\times & \mathbb{R} \\ 0 & 1 \end{pmatrix}$.

with $\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} t\lambda + z \\ 1 \end{pmatrix}$, $\lambda \mapsto t\lambda + z$.

The quotient $P(\mathbb{Z})\backslash D$: $\mathbb{C} \to \mathbb{C}^\times \to \mathbb{C}$ ($\lambda \mapsto e^{2\pi i \lambda}$, $u \mapsto u + u^{-1}$) is the simplest modular curve I've ever seen! Note: $P(\mathbb{R}) \circlearrowleft D$, 2 orbits $\mathbb{R}$, $\mathbb{C}-\mathbb{R}$. But $\begin{pmatrix} \mathbb{R}^\times & \mathbb{C} \\ 0 & 1 \end{pmatrix} \circlearrowleft D$ transitively.

Back to general d. $U(\mathbb{C}) = \begin{pmatrix} 1 & 0 & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ acts freely on D, $D' :=$ quotient. Then $(W/U)(\mathbb{R}) = \begin{pmatrix} 1 & \mathbb{R}^{2d} & \\ & 1 & \mathbb{R}^{2d} \\ & & 1 \end{pmatrix}$ acts freely on D', with quotient D''

$\mathbb{H}_d^+ \xrightarrow{\sim} D''$

$\tau \mapsto \mathbb{C}^d \cdot (\tau\ 1) \subset \mathbb{C}^{2d}$.

Conclusion (well known example):

$P(\mathbb{R})^+ \cdot U(\mathbb{C})$ acts transitively on D, (P, D) is a mixed Shim. datum, and $P(\mathbb{Z})^+ \backslash D \longrightarrow Sp_J(\mathbb{Z}) \backslash \mathbb{H}_d^+$ is the universal Poincaré torsor. $\big[\,(Gr_{-1}W)_{\mathbb{C}}$

(Indeed: $D \cong \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C} \times \mathbb{H}_d^+$: given τ; $\tau\ I_d$ $u \in \mathbb{C}^d$, and 1 more basis vector $z e_0 + v_1 e_1 + \dots + v_d e_d + e_{2d+1}$ (unique, modulo F^0).)

5. Ribet sections (Poincaré torsors) as special subvarieties.

Recall in §2 we had $f : A' \to A$, $\alpha := f - f'$, $\alpha' = -\alpha$. That we now describe here.
As before: $M = \mathbb{Z}^{2d+2}$, D etc. A principally polarised (A', λ, α) as in §2, with α an isogeny, gives us a $\tau \in \mathbb{H}_d$ and $\alpha : Gr_{-1}(M) \to Gr_{-1}(M^\vee(1))$ s.t. $\alpha^\vee = \alpha$ (the matrix of α w.r.t. $e_1, \dots, e_{2d}, e_1^\vee, \dots, e_{2d}^\vee$ is symmetric).
Then $\beta := J^{-1}\alpha$ satisfies $J^{-1}\beta^\vee J = -\beta$ ($\beta \in \mathrm{End}(A)$ anti-symm. for Rosatti).

Let $\tilde{\alpha} := \begin{pmatrix} & & -1 \\ & \alpha & \\ -1 & & \end{pmatrix} : M \to M^\vee(1)$. Claim: this tensor defines the Ribet section of §2.

Let $P_{\tilde{\alpha}} \subset P$ be the stab. of $\tilde{\alpha}$, then $P_{\tilde{\alpha}} = \left\{ \begin{pmatrix} \mu(g) & +y^t \alpha g & +\frac{1}{2} y^t \alpha y \\ 0 & g & y \\ 0 & 0 & 1 \end{pmatrix} : g^t \cdot \alpha \cdot g = \mu(g) \cdot \alpha \right\}$. (equation is: $\tilde{\alpha} \cdot p = \mu(p) \cdot p^{-1,t} \cdot \tilde{\alpha}$) (equiv: $p^t \cdot \tilde{\alpha} \cdot p = \mu(p) \cdot \tilde{\alpha}$)

Let $\tilde{\tau} \in D$ corr. to $\mathbb{Z}(1) \oplus H_1(A, \mathbb{Z}) \oplus \mathbb{Z}(0)$, $\tilde{F}^0 = F^0 + \mathbb{C} \cdot e_{2d+1} \subset M_{\mathbb{C}}$, where $F^0 \subset \mathbb{C} \cdot e_1 \oplus \dots \oplus \mathbb{C} \cdot e_{2d}$.

Then
$$\begin{array}{ccc} D_{\tilde{\alpha}} := P_{\tilde{\alpha}}(\mathbb{R}) \cdot \tilde{\tau} & \subset & D \\ \downarrow & & \downarrow \\ P_{\tilde{\alpha}}(\mathbb{Z}) \backslash D_{\tilde{\alpha}} & \subset & P(\mathbb{Z})^+ \backslash D \\ \shortparallel & & \shortparallel \\ \text{Ribet section} & & \text{univ. Poincaré torsor.} \end{array}$$

Proof. it is a section of the univ. Poincaré torsor, and it agrees with the Ribet section at $(0,0) \in$ graph of α, hence they are equal. ($\mathbb{G}_m(A') = \mathbb{C}$) ☒

Conclusion: now we see precisely what kind of selfduality of MHS (1-motives) is behind Ribet sections!

Remark The algebr. descr. of §2 (generalised jacobians) makes me think of quadratic Chabauty: extending J (jac of sm. curve) by $\mathbb{G}_m$ over $\mathbb{Z}$ makes the group higher dimensional without increasing the rank!