ICGS May 4, 2018, Amsterdam, Bas Edixhoven, 1 hour.

§2 of Gröchenig, Wyss, Ziegler.

Thm. 2.11. Let $R \subset \mathbb{C}$ subalgebra of finite type $/\mathbb{Z}$. For $i \in \{1,2\}$, let X_i be a separated R-scheme, of finite type, with generically free action by a finite commutative group Γ_i with $\#\Gamma_i$ in $R^\times$. Let $\mathcal{X}_i := [X_i/\Gamma_i]$, and let α_i be a μ_r-gerbe ($r \geq 1$, $r \in R^\times$) on $\mathcal{X}_i$. Assume that $\forall R \twoheadrightarrow k$ (k finite field):

$$\#_{st}^{\alpha_1} \mathcal{X}_1(k) = \#_{st}^{\alpha_2} \mathcal{X}_2(k).$$

Then

$$E_{st}(\mathcal{X}_{1,\mathbb{C}}, \alpha_1; x, y) = E_{st}(\mathcal{X}_{2,\mathbb{C}}, \alpha_2; x, y).$$

In my 1 hour, I just will try to explain what all these things are.

$\mathcal{X} := [X/\Gamma]$: stacky quotient. Let us look at a more general case. Let S_0 be a scheme, X an S_0-scheme, G an affine S_0-groupscheme acting on X on the right: $\forall S \to S_0$, $X(S) \circlearrowleft G(S)$, functorial in S.

Then $\forall S \to S_0$: $[X(S)/G(S)] :=$ the groupoid objects $X(S)$, arrows: $(x, y, g) \in X(S) \times X(S) \times G(S)$ s.t. $x \cdot g = y$. $x \xrightarrow{\cdot g} y$, composition: obvious.

Then for $S' \xrightarrow{f} S$, we have $[X(S)/G(S)] \xrightarrow{f^*} [X(S')/G(S')]$.

<u>Philosophy</u>: do not bluntly take a quotient, but do a better administration of the action. <u>The data we have now</u>: a presheaf of groupoids on Sch/S_0. $[X/G]$ is then the sheafification of this, for a topology that should be specified, say étale, or fppf or Zar. Let us take ét here.

For $S \to S_0$, $x, y \in X(S)$, $\underline{\mathrm{Isom}}(x,y) : (Sch/S)^{op} \to Set$, $(T \to S) \mapsto \mathrm{Isom}(x_T, y_T) = \{g \in G(T) : x_T \cdot g = y_T\}$ is a sheaf: G is a sheaf, and it is a subsheaf.

However, descent data for objects are not nec. effective.

Let $S' \twoheadrightarrow S$ be étale surjective, and $x \in X(S')$, object in $[X(S')/G(S')]$.

Then a descent datum for x rel. to $S' \twoheadrightarrow S$ is:

$\forall\ V \overset{s_1^*}{\underset{s_2^*}{\rightrightarrows}} S' \to S$ a <u>given</u> $g(s_1^*, s_2^*)$ in $G(V)$ s.t. $s_1^{**}(x) \cdot g(s_1^*, s_2^*) = s_2^{**}(x)$ in $X(V)$,

such that $\forall\ V \overset{s_1^*, s_2^*, s_3^*}{\longrightarrow} S' \to S$: $g(s_1^*, s_3^*) = g(s_1^*, s_2^*) \cdot g(s_2^*, s_3^*)$ in $G(V)$.

With this descent datum for x we can descend the orbit map of x:

$G_{S'} \xrightarrow{x\cdot} X_{S'}$, $g \mapsto x\cdot g$ to S, giving

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \\ S & \end{tikzcd}$$

G_S-equiv. map. right G_S-torsor

$$\begin{tikzcd} V \arrow[r, shift left=3, "s_1{,}s_2{,}s_3"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[dr] & S' \arrow[d] \\ & S \end{tikzcd} \qquad \begin{tikzcd} G_V \arrow[rr, "s_1^*(x)\cdot"] \arrow[d, "g(s_1{,}s_2)^{-1}\cdot"'] \arrow[drr, bend left=15] & & X_V \\ G_V \arrow[urr, "s_2^*(x)\cdot"'] \arrow[rr, "g(s_2{,}s_3)^{-1}\cdot"'] & & G_V \arrow[u, "s_3^*(x)\cdot"'] \end{tikzcd}$$

$\forall V$

Rem: we asked G/S_0 affine in order to descend G-torsors. In general: can work with algebr. spaces, or even with ~~[illegible]~~ sheaves.

So, one defines $[X/G](S)$:= objects

morphisms $\begin{tikzcd} T_1 \arrow[r] \arrow[d] \arrow[dr] & X \\ S & T_2 \arrow[u] \arrow[l] \end{tikzcd}$.

$X \xrightarrow{q} [X/G]$, $X(S)$ + only id's $\longrightarrow [X/G](S)$ universal for morphisms to sheaves of groupoids

$x \longmapsto \begin{tikzcd} G \arrow[r, "x\cdot"] \arrow[d] & X \\ S & \end{tikzcd}$ s.t. $\forall S, \forall x \in X(S), \forall g \in G(S)$ $q(x) \cong q(x\cdot g)$.

This is our first example of a <u>stack</u> (over $(Sch/S_0)_{et}$).

<u>General case</u>: $\begin{tikzcd} \mathcal{X} \text{ cat} \arrow[d, "p"', "\text{functor}"] \\ C \text{ site} \end{tikzcd}$, $\forall S$ in C: $\mathcal{X}(S)$: objects $x \in \mathcal{X}$ with $p(x) = S$

morphisms: $x \xrightarrow{f} y$ with $p(f) = id_S$.

Cartesianness: $\forall$ $\begin{tikzcd} & x \arrow[d, maps to] \\ T \arrow[r] & S \end{tikzcd}$ $\exists$ univ. $\begin{tikzcd} y \arrow[r] \arrow[d, maps to] \arrow[dr, phantom, "\downarrow"] & x \arrow[d, maps to] \\ T \arrow[r] & S \end{tikzcd}$

consequences: $\forall S$ in C, $\mathcal{X}(S)$ is groupoid,

$\forall T \xrightarrow{f} S$ in C have $f^*: \mathcal{X}(S) \to \mathcal{X}(T)$. Up to ~~here~~ called a pre-stack.

For a stack, TFMH: (1) $\forall S$ in C, $\forall x, y$ in $\mathcal{X}(S)$, <u>Isom</u>(x,y) is a sheaf on C/S.

(2) all descent data for objects are effective: $\forall S' \xrightarrow{f} S$ in C, $\forall x \in \mathcal{X}(S')$ + desc. dat. to S, $\exists x_S$ in $\mathcal{X}(S)$ s.t. $f^* x_S$ + nat. desc. dat. $\cong$ x + desc. datum.

The stacks over C form a 2-category: $\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{Y} \arrow[dl, "q"] \\ & C & \end{tikzcd}$ comm. diagram (1-morphisms).

$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathcal{Y} \end{tikzcd}$ 2-morphisms, always isomorphisms.

<u>Exercise</u>: Yoneda embedding $C \longrightarrow \mathrm{Stack}/C$, $X \mapsto (X(S) + \text{id. only})_{S \in C}$.

and $X(S) = (\mathrm{Stack}/C)(S, X)$. (assuming every ~~[illegible]~~ S in C is a sheaf).

If nec. I skip this.

Fibered products:

$$\begin{tikzcd} \mathcal{P} \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathcal{X}_1 \arrow[d, "f_1"] \arrow[dl, phantom, "\Swarrow \eta"] \\ \mathcal{X}_2 \arrow[r, "f_2"'] & \mathcal{Y} \end{tikzcd}$$

not nec. comm.

$\forall S$ in C objects $\mathcal{P}(S)$: (x_1, x_2, η), $x_i \in \mathcal{X}_i(S)$, $\eta : f_1(x_1) \xrightarrow{\sim} f_2(x_2)$.

morphisms: $(x_1, x_2, \eta) \xrightarrow{g_1, g_2} (x_1', x_2', \eta')$ s.t. ....

Representable morphisms:

$$\begin{tikzcd} \mathcal{X}_y \arrow[r] \arrow[d, "f_y"'] & \mathcal{X} \arrow[d, "f"] \arrow[dl, phantom, "\Swarrow \eta"] \\ \forall\ S \arrow[r, "y"'] & \mathcal{Y} \end{tikzcd}$$

$\mathcal{X}_y$ is in C (and automatically f_y too).

Then f is smooth ... iff $\forall S \xrightarrow{y} \mathcal{Y}$ f_y is smooth.

$\mathcal{X} \to (Sch/S_0)_{et}$ is a Deligne-Mumford stack if: $\begin{cases} \mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X} \text{ is repr. by alg. spaces} \\ \exists X \to \mathcal{X} \text{ etale surjective.} \end{cases}$

Example: $[X/\Gamma]$ is a DM stack, $X \xrightarrow{q} [X/\Gamma]$ is (repr.) etale, surjective, a Γ-torsor.

Sheaves on $[X/\Gamma]$: Γ-equivariant sheaves $\mathcal{F}$ on X (X_{et}), cohom: derived functor of $\mathcal{F} \mapsto \mathcal{F}(X)^\Gamma$.

Example: let $n \geq 4$, $\begin{smallmatrix} E \\ \downarrow \uparrow 0 \\ Y(n) \end{smallmatrix}$ univ. ell. curve with $\alpha : (\mathbb{Z}/n\mathbb{Z})^2_S \xrightarrow{\sim} E[n]$, $Y(n) \circlearrowleft GL_2(\mathbb{Z}/n\mathbb{Z})$. Then $[Y(n)/GL_2(\mathbb{Z}/n\mathbb{Z})] \xrightarrow{\sim}$ stack of ell. curves with level n str.

$\boxed{\text{The gerbe } \alpha.}$

Let $\alpha \in H^2(\mathcal{X}, \mu_r)$. That gives a μ_r-gerbe $\mathcal{G} \xrightarrow{f} \mathcal{X}$:
$\forall S \to S_0$, $\forall x$ in $\mathcal{X}(S)$, locally on S

$$\begin{tikzcd} \mathcal{G} \arrow[r, "f"] & \mathcal{X} \\ \mathcal{G}_S \arrow[u] \arrow[r, "f_x"] & S \arrow[u, "x"'] \end{tikzcd}$$

and $\mathcal{G}_S$ has a unique object y, $\forall y \in \mathcal{G}(S)$: $\mu_{r,S} \hookrightarrow \underline{Aut}(y) \twoheadrightarrow \underline{Aut}(x)$. central extension.

Say: $H^1 \leftrightarrow$ torsors, $H^2 \leftrightarrow$ gerbes; example: fibers of $\mathcal{X} \to$ coarse mod. space.

Note: then $\underline{Aut}(y)$ is a μ_r-torsor over $\underline{Aut}(x)$, denoted $P_\alpha(x)$ (indep. of y)

Example: $\begin{matrix} (X/S, \mathcal{L}) \\ \downarrow \\ X/S \\ \downarrow \\ S \end{matrix}$ X/S genus 0, $\mathcal{L}$ inv. $\mathcal{O}$-mod. degree 1.

Fermionic shift

~~Stacky the α-twisted E-polynomial~~ X smooth over $\mathbb{C}$, $\mathcal{X} := [X/\Gamma]$

$\forall x \in X$: loc. etale at x, the Γ_x-action can be linearised: $(X, x, \Gamma_x) \sim (\mathbb{C}^d, 0, \Gamma_x + \text{lin. act.})$. Also true for X/S_0, if $\#\Gamma$ in $\mathcal{O}(S_0)^\times$.

Consequence: $\forall \gamma \in \Gamma$, X^γ is smooth, $T_X(x) = T_{X^\gamma}(x) \oplus$ the isotypical components of non-triv. repr of $\langle \gamma \rangle$.

For $\gamma \in \Gamma$, $x \in X^\gamma$, let $\zeta(\gamma, x)_1, \ldots, \zeta(\gamma, x)_d$ be the eigenvalues of γ on $T_X(x)$. For $\xi \in \mathbb{C}^\times$ of order $\text{order}(\gamma)$, write $\zeta(\gamma, x)_i = \xi^{w_i}$, $0 \leq w_i < \text{order}(\gamma)$, and $F_\xi(\gamma, x) := \frac{1}{\text{order}(\gamma)} \cdot \sum_i w_i \in \mathbb{Q}$.

Then $F_\xi(\gamma, \cdot) : X^\gamma \to \mathbb{Q}$ loc. const. on X^γ, is called "Fermionic shift".

Such expressions occur when one studies $H^\cdot(X, \mathbb{C})$ as Γ-repr, or $H^\cdot(X, \mathcal{F})$ $\mathcal{F}$ coherent Γ-equiv. (holom. Lefschetz, Woods Hole formula...)

Stringy α-twisted E-polynomial

$$E_{st}(\mathcal{X},\alpha;x,y) := \sum_{\gamma\in\Gamma/\mathrm{conj}} \ \sum_{Y\in\pi_0(X^\gamma)/C(\gamma)} E(Y, L_\gamma)^{C(\gamma)_Y}\cdot(xy)^{F_\xi(\gamma,Y)}$$

where: L_γ on X^γ_{et}: loc. const. sheaf of 1-dim $\mathbb{C}$-vect. sp. $L_\gamma = P_\alpha \otimes_{\mu_r} \mathbb{C}_{X^\gamma} = (\pi_* \mathbb{C}_{P_\alpha})^\chi$

$$E(Y,L_\gamma)^{C(\gamma)_Y} = \sum_{n=0}^{2\dim Y} (-1)^n \sum_{p,q\geq 0} \dim\Big(\underbrace{H^n_c(Y,L_\gamma)^{p,q}}_{MHS}\Big)^{C(\gamma)_Y} x^p y^q$$

$\chi: \mu_r \hookrightarrow \mathbb{C}^\times$ emb.

$\pi: P_\alpha \to X^\gamma$.

Also: $H^n_c(Y,L_\gamma) = H^{n+2}_c(L_\gamma,\mathbb{C})^\chi(-1)$ $\quad$ $(-1: H^2_c(\mathbb{A}^1_{\mathbb{C}},\mathbb{C}) = \mathbb{C}(1).)$

$$E_{st}(\mathcal{X},\alpha;x,y) = E\Big(\coprod_{\gamma\in\Gamma} L_\gamma \times_{X^\gamma} \mathbb{A}^{F_\xi(\gamma,\cdot)}\Big)^{\Gamma=1,\,\mu_r=\chi}\cdot(xy)^{-1}$$

formal as far as fractional part is concerned

Point counting: $\#^\alpha_{st}\,\mathcal{X}(\mathbb{F}_q) = \mathrm{Tr}\Big(\mathrm{Frob}_q, H^\bullet_c\Big(\coprod_{\gamma\in\Gamma} L_{\gamma,\overline{\mathbb{F}}_q,et}\Big)^{\Gamma=1,\,\mu_r=\chi}\Big)\cdot q^{F_\xi(\gamma,\cdot)-1}$

Now for the proof of Thm. 2.11: the principle is: point count equivalent at all R-pts$_\alpha$ $\Rightarrow$ same element in K grp of ℓ-adic Galois repr. $\Rightarrow$ via Fontaines V_{DR}-functor: same element in Kgrp of filtered v.sp.

<u>Remark</u>: $\Big[\Big(\coprod_{\gamma\in\Gamma} X^\gamma\Big)/\Gamma\Big]$ is the inertia stack of $[X/\Gamma]$. With this, <u>can generalise</u> the statement and proof to smooth DM stacks $\mathcal{X}$ over $\mathbb{C}$:

<u>speculation!</u>

$$\begin{array}{l} E(\mathbb{A}^{F_\xi}, L). \\ \quad\downarrow \\ \quad I_{\mathcal{X}} \end{array}$$