

§2 of Gröchenig, Wyss, Ziegler.

1.

Thm. 2.11. Let  $R \subset \mathbb{C}$  subalgebra of finite type. For  $i \in \{1, 2\}$ , let  $X_i$  be a separated  $R$ -scheme, of finite type, with generically free action by a finite commutative group  $\Gamma_i$ . Let  $\mathcal{X}_i := [X_i / \Gamma_i]$ , and let  $\alpha_i$  be a  $\mu_r$ -gerbe on  $\mathcal{X}_i$ . Assume that  $\forall R \twoheadrightarrow k$  ( $k$  finite field): with  $\#\Gamma_i$  in  $R^\times$ .

$$\#_{st}^{\alpha_1} \mathcal{X}_1(k) = \#_{st}^{\alpha_2} \mathcal{X}_2(k).$$

Then  $E_{st}(\mathcal{X}_1, \mathbb{C}, \alpha_1; x, \gamma) = E_{st}(\mathcal{X}_2, \mathbb{C}, \alpha_2; x, \gamma)$ .

In my 1 hour, I just will try to explain what all these things are.

$\mathcal{X} := [X / \Gamma]$ : stacky quotient. Let us look at a more general case. Let  $S_0$  be a scheme,  $X$  an  $S_0$ -scheme,  $G$  an affine  $S_0$ -groupscheme acting on  $X$  on the right:  $\forall S \rightarrow S_0, X(S) \supseteq G(S)$ , functorial in  $S$ .

Then  $\forall S \rightarrow S_0: [X(S) / G(S)] :=$  the groupoid objects  $X(S)$ , arrows:

$(x, y, g) \in X(S) \times X(S) \times G(S)$  s.t.  $x \cdot g = y$ .  $x \xrightarrow{g} y$ , composition: obvious.

Then for  $S' \xrightarrow{f} S$ , we have  $[X(S) / G(S)] \rightarrow [X(S') / G(S')]$ .

Philosophy: do not bluntly take a quotient, but do a better administration of the action. The data we have now: a presheaf of groupoids on  $Sch/S_0$ .

$[X/G]$  is then the sheafification of this, for a topology that should be specified, say étale, or fppf or Zar. Let us take it here.

For  $S \rightarrow S_0, x, y \in X(S), \underline{Isom}(x, y): (Sch/S)^{op} \rightarrow Set, (T \rightarrow S) \mapsto Isom(x_T, y_T) = \{g \in G(T) : x_T \cdot g = y_T\}$  is a sheaf:  $G$  is a sheaf, and it is a subsheaf.

However, descent data for objects are not nec. effective.

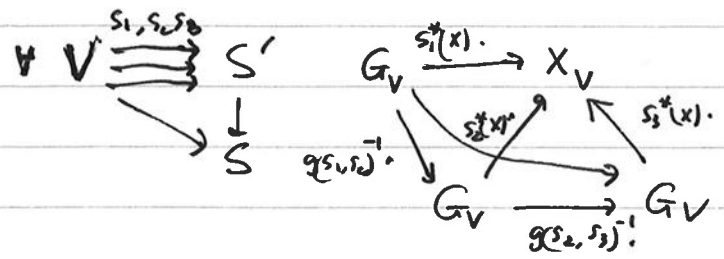
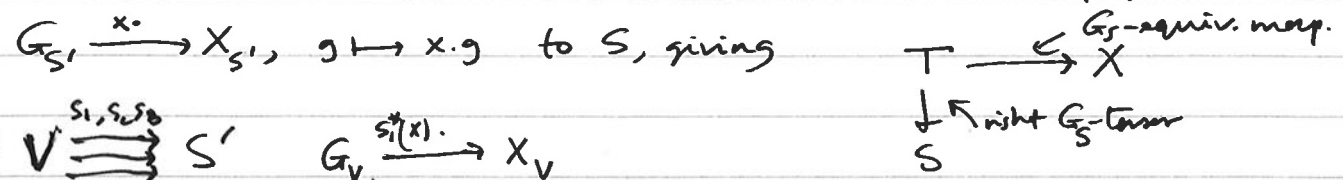
Let  $S' \twoheadrightarrow S$  be étale surjective, and  $x \in X(S')$ , object in  $[X(S') / G(S')]$ .

Then a descent datum for  $x$  rel. to  $S' \twoheadrightarrow S$  is:

$\forall V \begin{matrix} \xrightarrow{s_1^*} \\ \xrightarrow{s_2^*} \end{matrix} S' \xrightarrow{f} S$  a given  $g(s_1^*, s_2^*)$  in  $G(V)$  s.t.  $s_1^{**}(x) \cdot g(s_1^*, s_2^*) = s_2^{**}(x)$  in  $X(V)$

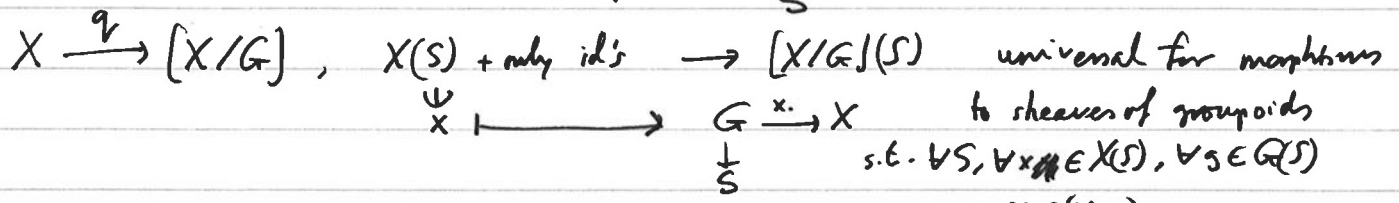
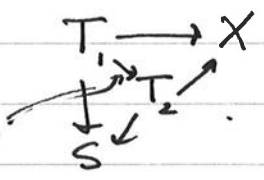
such that  $\forall V \begin{matrix} \xrightarrow{s_1^*} \\ \xrightarrow{s_2^*} \\ \xrightarrow{s_3^*} \end{matrix} S' \xrightarrow{f} S$  :  $g(s_1^*, s_3^*) = g(s_1^*, s_2^*) \cdot g(s_2^*, s_3^*)$  in  $G(V)$ .

With this descent datum for  $x$  we can descend the orbit map of  $x$ :



Rem: we asked  $G/S_0$  affine in order to descend  $G$ -torsors. In general: can work with algebraic spaces, or even with adic sheaves.

So, one defines  $[X/G](S) :=$  objects morphisms



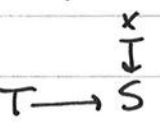
This is our first example of a stack (over  $(\text{Sch}/S_0)_{\text{ét}}$ ).  $q(x) \cong q(x \cdot g)$ .

General case:  $\mathcal{X}$  cat,  $\forall S \text{ in } C: \mathcal{X}(S) :=$  objects  $x \in \mathcal{X}$  with  $p(x) = S$   
 morphism:  $x \xrightarrow{f} y$  with  $K(f) = \text{id}_S$ .

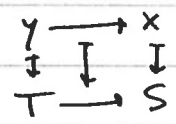
$p \downarrow$  functor

$C$  site

Cartesianness:  $\forall$



Univ.



consequences:  $\forall S \text{ in } C, \mathcal{X}(S)$  is groupoid,

$\forall T \xrightarrow{f} S$  in  $C$  have  $f^*: \mathcal{X}(S) \rightarrow \mathcal{X}(T)$ . Up to here called a pre-stack.

For a stack, TFMH: (1)  $\forall S \text{ in } C, \forall x, y \text{ in } \mathcal{X}(S)$ ,  $\text{Isom}(x, y)$  is a sheaf on  $C/S$ .

(2) all descent data for objects are effective:  $\forall S' \xrightarrow{f} S$  in  $C, \forall x \in \mathcal{X}(S') +$  desc. dat. to  $S, \exists x_S \text{ in } \mathcal{X}(S)$  s.t.  $f^* x_S + \text{nat. desc. dat.} \cong x + \text{desc. datum}$ .

The stacks over  $C$  form a 2-category:  $\mathcal{X} \xrightarrow{f} \mathcal{Y}$  comm. diagram  
 $\mathcal{X} \xrightarrow{f_1} \mathcal{Y} \xrightarrow{f_2} \mathcal{Z}$  2-morphisms, always isomorphisms.  
 (1-morphisms).

Exercise: Yoneda embedding  $C \rightarrow \text{Stack}/C, X \mapsto (X(S) + \text{id. only})_{S \in C}$ .  
 and  $\mathcal{X}(S) = (\text{Stack}/C)(S, X)$ . (assuming every  $S \text{ in } C$  is a sheaf).

If nec. I skip this.

Fibered products:  $P \xrightarrow{p_1} X_1$   $\forall$   $S$  in  $C$  objects  $P(S) : (x_1, x_2, \eta), x_i \in X_i(S), \underline{3}$   
 $p_2 \downarrow \searrow \eta \downarrow f_1$   $\eta: f_1(x_1) \xrightarrow{\sim} f_2(x_2)$   
 $X_2 \xrightarrow{f_2} Y$  morphisms:  $(x_1, x_2, \eta)$  s.t. ....  
 not nec. comm.  $\begin{matrix} \eta \downarrow & p_2 \downarrow \\ x_1' & x_2' \end{matrix}$   $(x_1', x_2', \eta')$

Representable morphisms:  $X_1 \rightarrow X$   $X_1$  is in  $C$  (and automatically  $f_1$  too).  
 $f_1 \downarrow \searrow \eta \downarrow f$   $\forall S \xrightarrow{f} Y$

Then  $f$  is smooth... iff  $\forall S \xrightarrow{f} Y$   $f_1$  is smooth.  
 $X \rightarrow (Sch/S_0)_{\text{et}}$  is a Deligne-Mumford stack if:  $\left\{ \begin{array}{l} X \xrightarrow{\Delta} X \times X \text{ is repr. by ds. spaces} \\ \exists X \rightarrow X \text{ etale surjective.} \end{array} \right.$

Example:  $[X/\Gamma]$  is a DM stack,  $X \xrightarrow{\pi} [X/\Gamma]$  is  $\overset{\text{repr.}}{\text{etale surjective}}$ , a  $\Gamma$ -torsor.

Sheaves on  $[X/\Gamma]$ :  $\Gamma$ -equivariant sheaves  $\overset{F}{\text{on } X}$ , cohom: derived functor of  $(X_{\text{et}})$   
 $F \mapsto F(X)^\Gamma$ .

Example: let  $n \geq 4$ ,  $E \xrightarrow{\alpha} E(n)$  univ. ell. curve with  $\alpha: @/n @ \xrightarrow{\sim} E(n)$   
 $F(n) \subset GL_2(@/n @)$ . Then  $[Y(n)/GL_2(@/n @)] \xrightarrow{\sim}$  stack of all-curve with level  $n$  str.

The gerbe  $\alpha$

Let  $\alpha \in H^2(X, \mu_r)$ . That gives a  $\mu_r$ -gerbe  $G \xrightarrow{f} X$ :  
 $\forall S \rightarrow S_0, \forall x$  in  $X(S)$ ,  $f_x$  locally on  $S$   $\begin{matrix} G & \xrightarrow{f} & X \\ \uparrow & & \uparrow x \\ G_S & \xrightarrow{f_x} & S \end{matrix}$  and  $\forall \gamma \in G(S): \mu_r, S \subset \text{Aut}(\gamma) \rightarrow \text{Aut}(x)$ .  
 $G_S$  has a unique object  $y$ .

Say:  $H^1 \Leftrightarrow$  torsors,  $H^2 \Leftrightarrow$  gerbes; example: fibers of  $X \rightarrow$  coarse mod. space central extension.

Note: then  $\text{Aut}(\gamma)$  is a  $\mu_r$ -torsor over  $\text{Aut}(x)$ , denoted  $P_x(x)$  (indep. of  $\gamma$ ) Example:  $(X/S, \mathcal{L})$

Fermionic shift

~~stably  $\mathcal{L}$ -trivial  $E$ -equivariant~~  $X$  over  $\mathbb{C}$ ,  $X := [X/\Gamma]$  smooth

~~$\forall x \in X$~~   $\forall x \in X$ : loc. etale at  $x$ , the  $\Gamma_x$ -action can be linearized:  $(X, x, \Gamma_x) \sim (\mathbb{C}^d, 0, \Gamma_x \text{ lin. act.})$ .  
 Also true for  $X/S_0$ , if  $\# \Gamma$  in  $\mathcal{O}(S_0)^\times$ .

Consequence:  $\forall \gamma \in \Gamma, X^\gamma$  is smooth,  $T_X(x) = T_{X^\gamma}(x) \oplus$  the isotypical components of non-triv. repr of  $\langle \gamma \rangle$ .  
 For  $\gamma \in \Gamma, x \in X^\gamma$ , let  $\zeta(\gamma, x)_1, \dots, \zeta(\gamma, x)_d$  be the eigenvalues of  $\gamma$  on  $T_X(x)$ . For  $\xi \in \mathbb{C}^\times$  of order  $\text{order}(\gamma)$ , write  $\zeta(\gamma, x)_i = \xi^{w_i}$  and  $F_\xi(\gamma, x) := \frac{1}{\text{order}(\gamma)} \cdot \sum_i w_i \in \mathbb{Q}$ .  $0 \leq w_i < \text{order}(\gamma)$

Then  $F_\xi(\gamma, \cdot): X^\gamma \rightarrow \mathbb{Q}$  loc. const. on  $X^\gamma$ , is called "Fermionic shift".

Such expressions occur when one studies  $H^*(X, \mathbb{C})$  as  $\Gamma$ -repr, or  $H^*(X, F)$   $F$  coherent  $\Gamma$ -equiv. (holom. Lefschetz, Woods Hole formula...)

Stringy  $\alpha$ -twisted  $E$ -polynomial

$$E_{st}(\mathcal{X}, \alpha; x, y) := \sum_{\gamma \in \Gamma / \text{conj}} \sum_{Y \in \pi_0(X^\sigma) / C_\gamma} E(Y, L_\gamma)^{C_\gamma} \cdot (xy)^{F_\xi(\gamma, \chi)}$$

where:  $L_\gamma$  on  $X_{\text{ét}}^\Gamma$ : loc. const. sheaf of 1-dim  $\mathbb{C}$ -ved. sp.  $L_\gamma = P_\alpha \otimes_{\mu_r} \mathbb{C}_{X^\gamma}$

$$E(Y, L_\gamma)^{C_\gamma} = \sum_{n=0}^{2 \dim Y} (-1)^n \cdot \sum_{p, q \geq 0} \dim \left( \underbrace{H_c^n(Y, L_\gamma)^{p, q}}_{\text{MHS}} \right)^{C_\gamma} \cdot x^p y^q \quad \parallel \quad (\pi_* \mathbb{C}_{P_\alpha})^\lambda$$

Also:  $H_c^n(Y, L_\gamma) = H_c^{n+2}(L_\gamma, \mathbb{C})^\lambda(-1)$   $(-1: H_c^2(A^1_{\mathbb{C}}, \mathbb{C}) = \mathbb{C}(1))$   $\lambda: \mu_r \hookrightarrow \mathbb{C}^\times$  emb.  
 $\pi: P_\alpha \rightarrow X^\sigma$

$$E_{st}(\mathcal{X}, \alpha; x, y) = E\left( \coprod_{\gamma \in \Gamma} L_\gamma \times_{X^\sigma} A^{\frac{F_\xi(\gamma, \chi)}{r}} \right)^{\Gamma=1, \mu_r=\chi} \cdot (xy)^{-1}$$

formal as far as fractional part is concerned.

Point counting:  $\#_{st}^\alpha \mathcal{X}(\mathbb{F}_q) = \text{Tr}(\text{Frob}_q, H_c^0(\coprod_{\gamma \in \Gamma} L_\gamma, \mathbb{F}_q, \alpha)) \cdot q^{\frac{F_\xi(\gamma, \chi) - 1}{r}}$

Now for the proof of Thm. 2.11: the principle is: point count equivalent at all  $R \rightarrow k_p \Rightarrow$  same element in Kgrp of  $\mathbb{C}$ -adic Galois repr.  $\Rightarrow$  via Fontaine's  $V_{DR}$ -functor: same element in Kgrp of filtered v-sp.

Remark:  $\left[ \left( \coprod_{\gamma \in \Gamma} X^\sigma \right) / \Gamma \right]$  is the inertia stack of  $[X/\Gamma]$ .  
 With this, can generalise the statement and proof to  $\mu_r$  DM stacks  $\mathcal{X}$  over  $\mathbb{C}$ : smooth

speculation!

$$E(A^{\frac{F_\xi}{r}}, L) \downarrow I_{\mathcal{X}}$$