

Bas Edixhoven, Oberwolfach, 2018/07/27, 45 min. 1

Quadratic Chabauty and the Poincaré torsor.

joint project with Guido Lido (PhD student of René Schoof, and me).
 Apologies for the "preliminary stage" of this project. ~~Started~~
 I only started thinking about it in December 2017, and only since June there was real progress as a consequence of Guido reading the work of Balakrishnan et al. (not me).

The aim: to replace everything in quadr. Chab. by ^{old-fashioned} algebraic geometry / \mathbb{Z} and over $\mathbb{Z}/p^2\mathbb{Z}$, not to ^{re-}prove general finiteness results, but to find $C(\mathbb{Q})$ for specific C .

Assumptions. C/\mathbb{Z} curve, proper, regular, flat, geom. connected fibers, genus $g \geq 2$, smooth over $\mathbb{Z}[1/n]$, $n \geq 1$.

$J :=$ Nérm model / \mathbb{Z} of $\text{Pic}_{C/\mathbb{Q}}^0$,

$P_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}} \times_{\mathbb{Q}} J_{\mathbb{Q}}^{\vee} :=$ the Poincaré Grm-torsor:

if $B_{\mathbb{Q}}$ is the Poincaré line bundle on $J_{\mathbb{Q}} \times_{\mathbb{Q}} J_{\mathbb{Q}}^{\vee}$, trivialised at $(0_{\mathbb{Q}} \times J_{\mathbb{Q}}^{\vee}) \cup (J_{\mathbb{Q}} \times 0_{\mathbb{Q}})$, then $P_{\mathbb{Q}} = \underline{\text{Isom}}_{J_{\mathbb{Q}} \times_{\mathbb{Q}} J_{\mathbb{Q}}^{\vee}}(\mathcal{O}, B_{\mathbb{Q}})$.

$P_{\mathbb{Q}}$ is a biextension of $J_{\mathbb{Q}} \times_{\mathbb{Q}} J_{\mathbb{Q}}^{\vee}$ by Grm:

for S a \mathbb{Q} -scheme, $x_1, x_2 \in J_{\mathbb{Q}}(S)$, $y \in J_{\mathbb{Q}}^{\vee}(S)$,

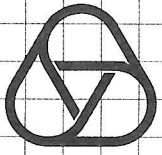
$$(x_1+x_2, y)^* B_{\mathbb{Q}} = (x_1, y)^* B_{\mathbb{Q}} \otimes_{\mathcal{O}_S} (x_2, y)^* B_{\mathbb{Q}} \quad (\text{canonical isom.})$$

(thm. of the square).

This gives $(x_1+x_2, y)^* P_{\mathbb{Q}} \leftarrow (x_1, y)^* P_{\mathbb{Q}} \times_S (x_2, y)^* P_{\mathbb{Q}}$

$$\begin{array}{cccc} P_{\mathbb{Q}} & & & \\ \downarrow & z_1 & z_2 & z_1 +_1 z_2 \\ J_{\mathbb{Q}} \times_{\mathbb{Q}} J_{\mathbb{Q}}^{\vee} & (x_1, y) & (x_2, y) & (x_1+x_2, y) =: (x_1, y) +_1 (x_2, y) \end{array}$$

And same in other coordinate. These $+_1$ and $+_2$ commute, when it makes sense.



Extension $(\mathbb{Z}, J^0 \hookrightarrow J \rightarrow \Phi, \Phi \overset{\text{finite}}{\vee} \text{skyscraper}$ 2.

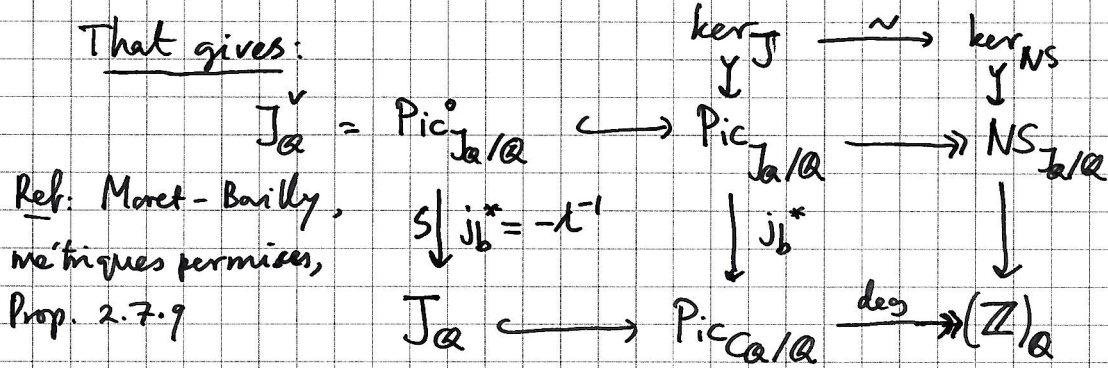
group scheme supported on $\mathbb{Z}/n\mathbb{Z}$, all fibres of J^0 connected.

Then $P_{\mathbb{Q}}$ extends uniquely to $P \rightarrow J \times_{\mathbb{Z}} J^{v_0}$ as biextension by \mathbb{G}_m .

Assumption We have a $b \in C_{\mathbb{Q}}(\mathbb{Q}) = C(\mathbb{Z}) = C^{sm}(\mathbb{Z})$.

Then $C^{sm} \xrightarrow{j_b} J, P \mapsto [P-b]$.

That gives:



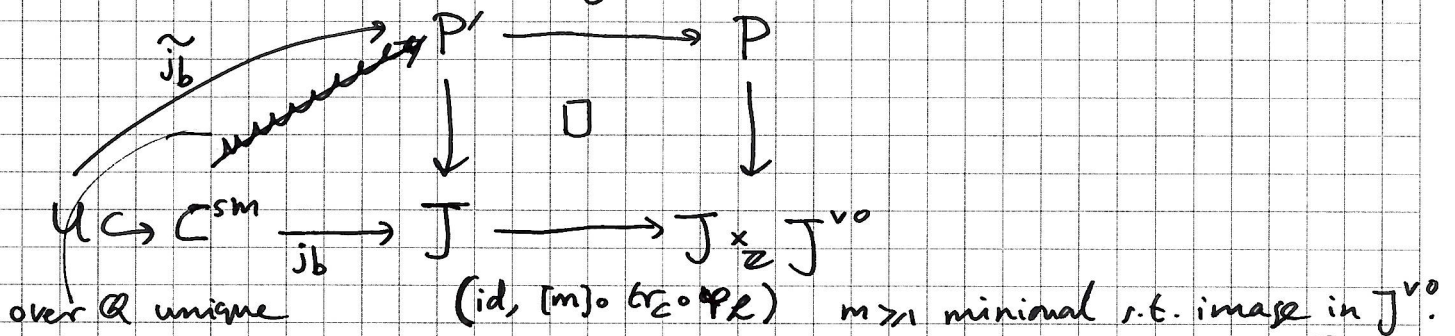
So $\ker_J(\mathbb{Q}) = \ker_{NS}(\mathbb{Q}) \subseteq NS(J_{\mathbb{Q}})^{\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} = \text{End}(J_{\mathbb{Q}})^+$

(is a free \mathbb{Z} -module of rank $p-1$.)

For any L on $J_{\mathbb{Q}}$: ~~$L^{\otimes 2} = L \otimes (-\text{id}^* L^{-1}) \otimes L \otimes (\text{id}^* L)$~~

$$L^{\otimes 2} = \underbrace{L \otimes (-\text{id}^* L^{-1})}_{c \in J_{\mathbb{Q}}^v(\mathbb{Q})} \otimes \underbrace{L \otimes (\text{id}^* L)}_{\parallel \substack{(\text{id}, \varphi_L)^* B_{\mathbb{Q}} \\ \varphi_L: J_{\mathbb{Q}} \rightarrow J_{\mathbb{Q}} \\ x \mapsto (\varphi_x^* L) \otimes L^{-1}}}$$

Assumption $p \geq 2$. Let $L \in \ker_J(\mathbb{Q}) \neq 0$.



up to \mathbb{Q}^*

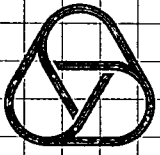
$U \subset C^{sm}$: take 1 irred. comp.

in each fibre at $q|n$.

Now try old fashioned Chabauty in P' .

$U(\mathbb{Z}) \in \tilde{j}_b U(\mathbb{Z}_p) \cap \overline{P'(\mathbb{Z})}$ in $P'(\mathbb{Z}_p)$.

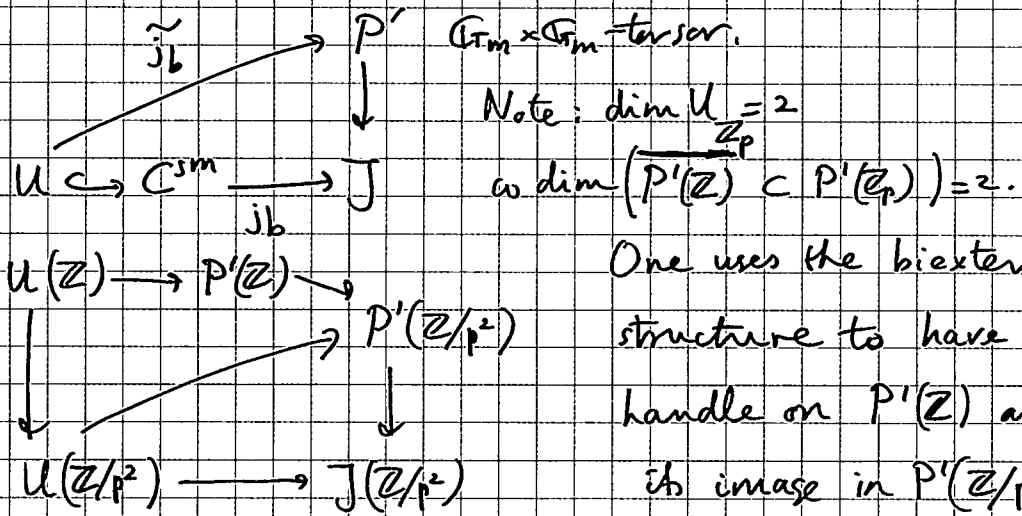
Assumption: $r \neq g$



Now assume $p \gg 3$.

Take L_1 and L_2 lin. indep. in $\ker_j(\mathcal{Q})$.

This gives:

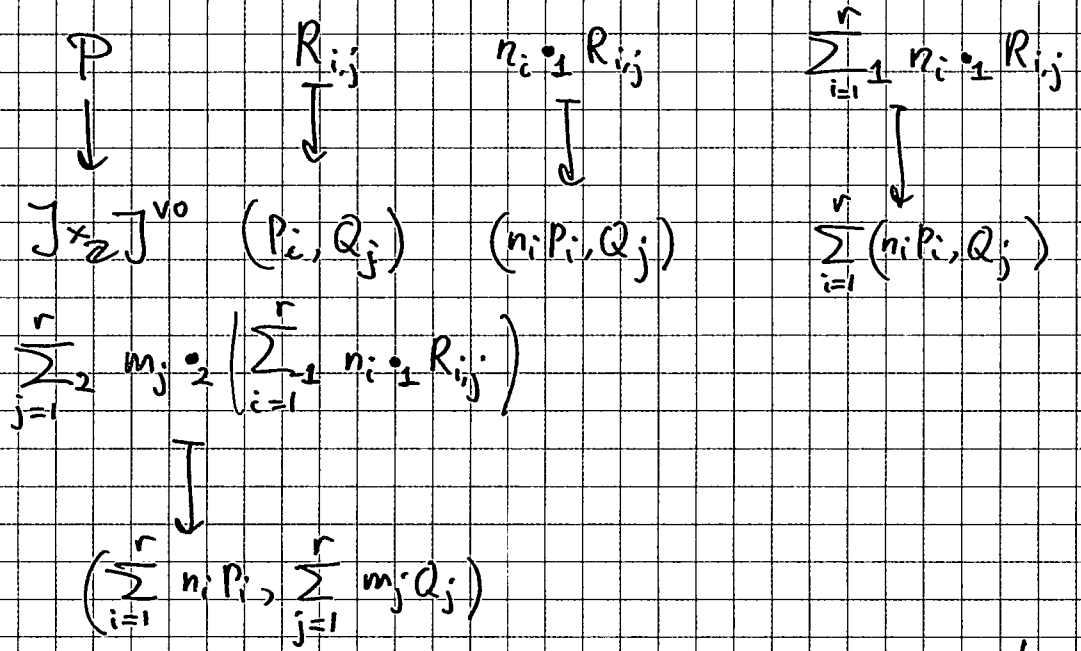
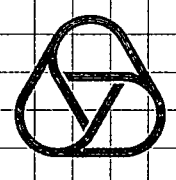


Hope: ~~the~~

$\forall u \in U(\mathbb{Z}/p^2\mathbb{Z})$ whose image in $P'(\mathbb{Z}/p^2\mathbb{Z})$ is in the image of $P'(\mathbb{Z})$, we know a $\tilde{u} \in U(\mathbb{Z})$ that lifts it, and also that it is unique by transversality with ~~the~~ only L_1 or L_2 .

If time, or if questions.

① How to use the biextension structure:



$$\left[\sum_{i=1}^k (x_i - \gamma_i) \right]$$

Explicit description of $P \rightarrow J \times J^{vo}$. For S any scheme, $x \in J(S)$, $y \in J^{vo}(S)$,

$$(x, y)^* P \cong \bigotimes_{i=1}^k (y_i^* \mathcal{L}) \otimes (x_i^* \mathcal{L})^{-1}$$

can. isom.

\mathcal{L} on $J^o(S)$

See MB, techniques pervnises, (2.7.11.2) and Cor. 2.8.6.