Bas Edixhoven, Oberwolfach, 2018/07/27, 45 min.

Quadratic Chabauty and the Poincaré torsor.

joint project with Guido Lido (PhD student of René Schoof, and me).
Apologies for the "preliminary stage" of this project. ~~I started~~
I only started thinking about it in December 2017, and only since June there was real progress as a consequence of Guido reading the work of Balakrishnan et al. (not me).

<u>The aim</u>: to replace everything in quadr. Chab. by old-fashioned algebraic geometry $/\mathbb{Z}$ and over $\mathbb{Z}/p^2\mathbb{Z}$, not to re-prove general finiteness results, but to find $C(\mathbb{Q})$ for specific C.

<u>Assumptions</u>. $C/\mathbb{Z}$ curve, proper, regular, flat, geom. connected fibers, genus $g \geqslant 2$, smooth over $\mathbb{Z}[1/n]$, $n \geqslant 1$.

$J :=$ Néron model $/\mathbb{Z}$ of $\mathrm{Pic}^0_{C_\mathbb{Q}/\mathbb{Q}}$,

$P_\mathbb{Q} \longrightarrow J_\mathbb{Q} \times_\mathbb{Q} J_\mathbb{Q}^\vee$:= the <u>Poincaré $\mathbb{G}_m$-torsor</u>:

if $B_\mathbb{Q}$ is the Poincaré line bundle on $J_\mathbb{Q} \times_\mathbb{Q} J_\mathbb{Q}^\vee$, trivialised at $(\{0\} \times J_\mathbb{Q}^\vee) \cup (J_\mathbb{Q} \times \{0\})$, then $P_\mathbb{Q} = \underline{\mathrm{Isom}}_{J_\mathbb{Q} \times_\mathbb{Q} J_\mathbb{Q}^\vee}(\mathcal{O}, B_\mathbb{Q})$.

$P_\mathbb{Q}$ is a <u>biextension</u> of $J_\mathbb{Q} \times_\mathbb{Q} J_\mathbb{Q}^\vee$ by $\mathbb{G}_m$:
for S a $\mathbb{Q}$-scheme, $x_1, x_2 \in J_\mathbb{Q}(S)$, $y \in J_\mathbb{Q}^\vee(S)$,

$(x_1+x_2, y)^* B_\mathbb{Q} = (x_1,y)^* B_\mathbb{Q} \otimes_{\mathcal{O}_S} (x_2,y)^* B_\mathbb{Q}$ (canonical isom.) (thm. of the square).

This gives $(x_1+x_2,y)^* P_\mathbb{Q} \longleftarrow (x_1,y)^* P_\mathbb{Q} \times_S (x_2,y)^* P_\mathbb{Q}$

$$\begin{array}{cccc} P_\mathbb{Q} & z_1 & z_2 & z_1 +_1 z_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ J_\mathbb{Q} \times_\mathbb{Q} J_\mathbb{Q}^\vee & (x_1,y) & (x_2,y) & (x_1+x_2,y) =: (x_1,y) +_1 (x_2,y) \end{array}$$

And same in other coordinate. These $+_1$ and $+_2$ commute, when it makes sense.

Extension / $\mathbb{Z}$. $J^\circ \hookrightarrow J \twoheadrightarrow \Phi$, Φ finite skyscraper groupscheme supported on $\mathbb{Z}/n\mathbb{Z}$, all fibres of J° connected.

Then $P_\mathbb{Q}$ extends uniquely to $P \to J \times_\mathbb{Z} J^{\vee\circ}$ as biextension by $\mathbb{G}_m$.

Assumption We have a $b \in C_\mathbb{Q}(\mathbb{Q}) = C(\mathbb{Z}) = C^{sm}(\mathbb{Z})$.

Then $C^{sm} \xrightarrow{j_b} J$, $P \mapsto [P-b]$.

That gives:

$$\begin{tikzcd}
 & \ker_J \arrow[r, "\sim"] \arrow[d, hook] & \ker_{NS} \arrow[d, hook] \\
J^\vee_\mathbb{Q} = \mathrm{Pic}^\circ_{J_\mathbb{Q}/\mathbb{Q}} \arrow[r, hook] \arrow[d, "\wr\ j_b^* = -\lambda^{-1}"] & \mathrm{Pic}_{J_\mathbb{Q}/\mathbb{Q}} \arrow[r, two heads] \arrow[d, "j_b^*"] & NS_{J_\mathbb{Q}/\mathbb{Q}} \arrow[d] \\
J_\mathbb{Q} \arrow[r, hook] & \mathrm{Pic}_{C_\mathbb{Q}/\mathbb{Q}} \arrow[r, two heads, "\deg"] & (\mathbb{Z})_\mathbb{Q}
\end{tikzcd}$$

Ref: Moret-Bailly, Métriques permises, Prop. 2.7.9

So $\ker_J(\mathbb{Q}) = \ker_{NS}(\mathbb{Q}) \subset NS(J_{\bar{\mathbb{Q}}})^{Gal(\bar{\mathbb{Q}}/\mathbb{Q})} = \mathrm{End}(J_\mathbb{Q})^+$
(is a free $\mathbb{Z}$-module of rank $\rho - 1$.

For any $\mathcal{L}$ on $J_\mathbb{Q}$: ~~[illegible]~~

$$\mathcal{L}^{\otimes 2} = \underbrace{\mathcal{L} \otimes (-id^* \mathcal{L}^{-1})}_{c \,\in\, J^\vee_\mathbb{Q}(\mathbb{Q})} \otimes \underbrace{\mathcal{L} \otimes (-id^* \mathcal{L})}_{= (id, \varphi_\mathcal{L})^* B_\mathbb{Q}}$$

$\varphi_\mathcal{L} : J_\mathbb{Q} \to J_\mathbb{Q}^\vee$, $x \mapsto (tr_x^* \mathcal{L}) \otimes \mathcal{L}^{-1}$

Assumption $\rho \geq 2$. Let $\mathcal{L} \in \ker_J(\mathbb{Q})$, $\mathcal{L} \neq 0$.

$$\begin{tikzcd}
 & & P' \arrow[rr] \arrow[d] \arrow[drr, phantom, "\square"] & & P \arrow[d] \\
U \arrow[r, hook] \arrow[urr, bend left, "\tilde{j}_b"] & C^{sm} \arrow[r, "j_b"] & J \arrow[rr, "{(id, [m] \circ tr_c \circ \varphi_\mathcal{L})}"] & & J \times_\mathbb{Z} J^{\vee\circ}
\end{tikzcd}$$

over $\mathbb{Q}$ unique up to $\mathbb{Q}^\times$

$m \geq 1$ minimal s.t. image in $J^{\vee\circ}$.

$U \subset C^{sm}$: take 1 irred. comp. in each fibre at $q \mid n$.

$\boxed{\text{Now try old fashioned Chabauty in } P'.}$

$\boxed{\text{Assumption: } r = g}$

$U(\mathbb{Z}) \subset \tilde{j}_b U(\mathbb{Z}_p) \cap \overline{P'(\mathbb{Z})}$ in $P'(\mathbb{Z}_p)$.

Now assume $p \geqslant 3$.

Take $\mathcal{L}_1$ and $\mathcal{L}_2$ lin. indep. in $\ker_J(\mathbb{Q})$.

This gives:

$$\begin{tikzcd}
 & & P' \arrow[d] \\
U \arrow[r, hook] \arrow[rru, "\tilde{j}_b"] & C^{sm} \arrow[r, "j_b"'] & J
\end{tikzcd}$$

P' $\mathbb{G}_m \times \mathbb{G}_m$-torsor.

Note: $\dim U_{\mathbb{Z}_p} = 2$

$\operatorname{codim}\left(\overline{P'(\mathbb{Z})} \subset P'(\mathbb{Z}_p)\right) = 2$.

$$\begin{tikzcd}
U(\mathbb{Z}) \arrow[r] \arrow[d] & P'(\mathbb{Z}) \arrow[r] & P'(\mathbb{Z}/p^2) \arrow[d] \\
U(\mathbb{Z}/p^2) \arrow[rru] \arrow[rr] & & J(\mathbb{Z}/p^2)
\end{tikzcd}$$

One uses the biextension structure to have a handle on $P'(\mathbb{Z})$ and its image in $P'(\mathbb{Z}/p^2)$.

Hope: ~~[illegible]~~

$\forall u \in U(\mathbb{Z}/p^2\mathbb{Z})$ whose image in $P'(\mathbb{Z}/p^2\mathbb{Z})$ is in the image of $P'(\mathbb{Z})$, we know a $\tilde{u} \in U(\mathbb{Z})$ that lifts it, and also that it is unique by transversality with ~~X~~ only $\mathcal{L}_1$ or $\mathcal{L}_2$.

If time, or if questions.

① How to use the biextension structure:

$$\begin{array}{cccc}
P & R_{i,j} & n_i \bullet_1 R_{i,j} & \sum_{i=1}^{r}{}_1\, n_i \bullet_1 R_{i,j} \\
\downarrow & \rotatebox[origin=c]{-90}{$\mapsto$} & \rotatebox[origin=c]{-90}{$\mapsto$} & \rotatebox[origin=c]{-90}{$\mapsto$} \\
J \times_2 J^{vo} & (P_i, Q_j) & (n_i P_i, Q_j) & \sum_{i=1}^{r} (n_i P_i, Q_j)
\end{array}$$

$$\sum_{j=1}^{r}{}_2\; m_j \bullet_2 \left(\sum_{i=1}^{r}{}_1\; n_i \bullet_1 R_{i,j} \right)$$

$$\rotatebox[origin=c]{-90}{$\mapsto$}$$

$$\left(\sum_{i=1}^{r} n_i P_i ,\ \sum_{j=1}^{r} m_j Q_j \right)$$

$$\left[\sum_{i=1}^{k} (x_i - y_i) \right]$$

Explicit description of $\begin{array}{c} P \\ \downarrow \\ J \times J^{vo} \end{array}$. For S any scheme, $x \in J(S)$, $y \in J^{vo}(S)$,

$$\begin{array}{cc} y \in & J^{vo}(S) \\ \updownarrow & \lambda \Big\uparrow \wr \\ \mathcal{L} & J^{0}(S) \\ \text{on } C_S & \end{array}$$

$$(x,y)^* P \overset{\text{can. isom.}}{=\!=} \bigotimes_{i=1}^{k} (y_i^* \mathcal{L}) \otimes (x_i^* \mathcal{L})^{-1} \quad \text{on } C_S$$

See MB, métriques permises, (2.7.11.2) and Cor. 2.8.6.