Mathematics and Richard Serra's torqued ellipse in the Guggenheim in Bilbao

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Abstract

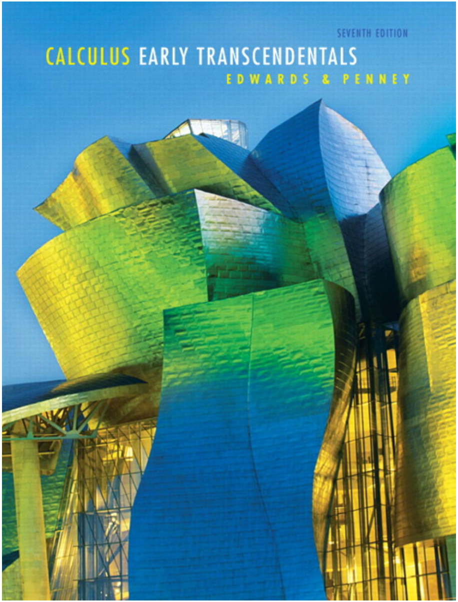
We are going to take a look in the Guggenheim museum in Bilbao, and the mathematics behind a “minimal art” sculpture by Richard Serra.

It will appear that this object is but half of a siamese twin, of which the new half looks much more interesting.

These slides and the sage-cocalc worksheet are on my homepage `http://pub.math.leidenuniv.nl/~edixhovensj/`, under ‘talks…’.

The Guggenheim museum in Bilbao

A calculus book



Richard Serra's "torqued ellipse"

How was this surface made?

Let us watch Serra's explanation in
http://www.youtube.com/watch?v=iRMvqOwtFno&feature=youtube_gdata_player:
(minutes 16–18).

The surface is obtained from 2 identical ellipses in horizontal planes, 1 on the ground and 1 at the top, with their long axes in different directions.

But Serra did not say *how* the surface connects the two ellipses. The contours being straight lines reveals the process.

For every line in a contour determines a plane through our eye containing it. Such a plane is tangent to both ellipses. The intersection of such a plane with the horizontal planes containing the ellipses are tangent lines to the ellipses.

How was this surface made?

The surface is the union of the line segments that connect the points of the 2 ellipses where the tangent lines are parallel.

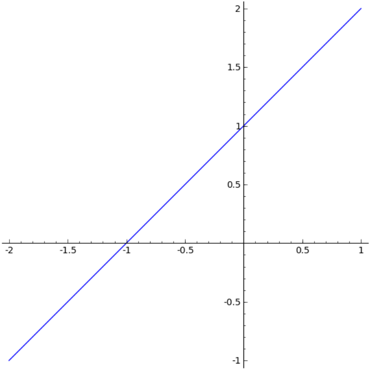
The surface is a part of the boundary of the convex hull of the union of the 2 ellipses.

Serra describes this in mechanical terms:
`http://www.youtube.com/watch?v=G-mBR26bAzA`
Start at 1:35.

He rolls a plane around the 2 ellipses, or he rolls his wheel over a slate of lead. Or, think of his wheel with glue on it that rolls over a sheet of paper.

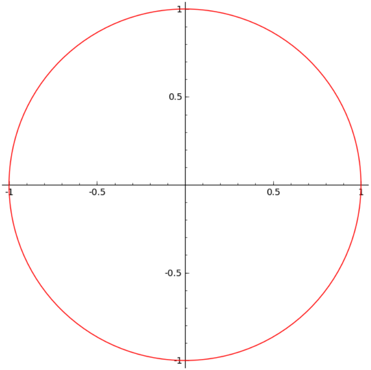
An equation for the surface?

First an exercise in dimension 2.
Back to Nicolas Oresme (14th century). A plane curve of degree 1.



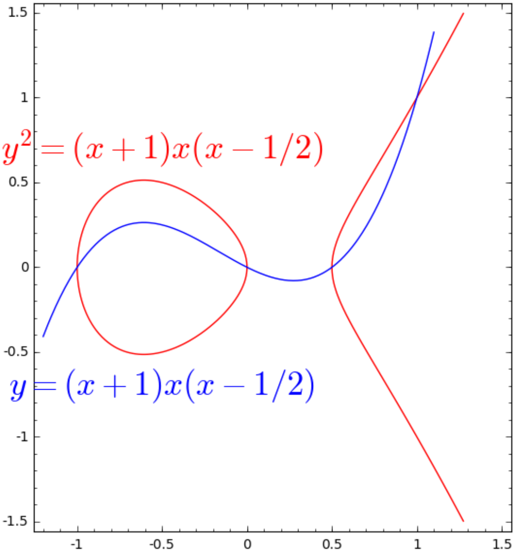
The line given by the equation $y = x + 1$.

A plane curve of degree 2



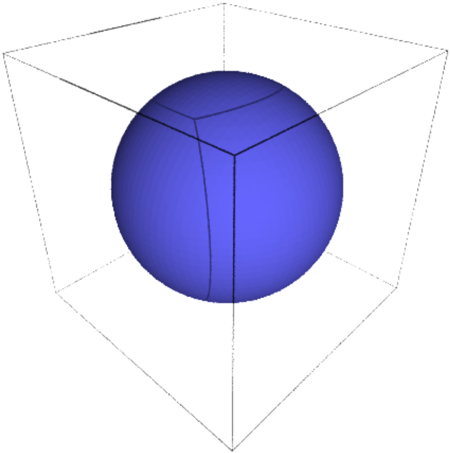
The circle given by the equation $x^2 + y^2 = 1$.

A plane curve of degree 3



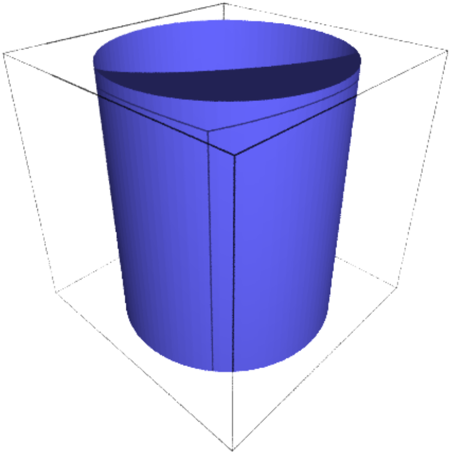
The sphere

The equation of the sphere is $x^2 + y^2 + z^2 = 1$.

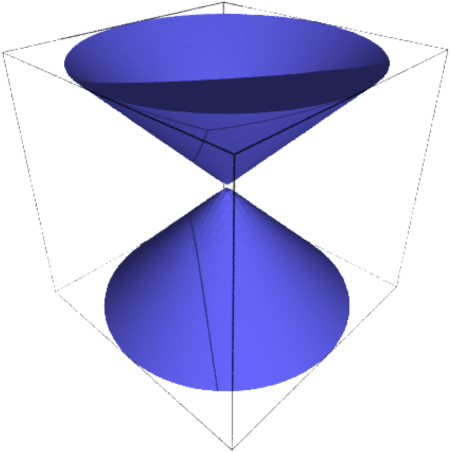


The cylinder

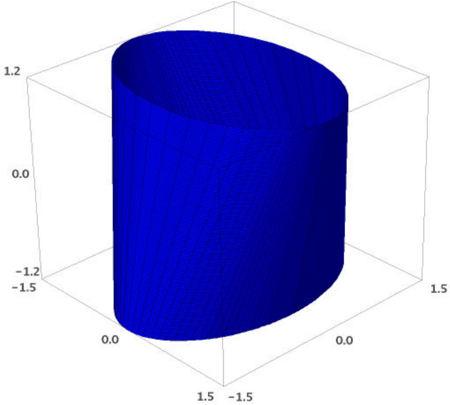
The equation of the cylinder is $x^2 + y^2 = 1$.



The cone



A parametrisation of Serra's surface



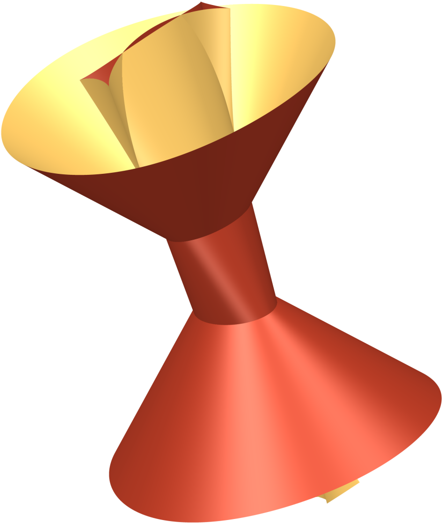
Pictures and computations in sage/cocalc (`https://cocalc.com`).

The equation of Serra's surface

$$
\begin{aligned}
&16384x^8 + 81920x^6y^2 + 135168x^4y^4 + 81920x^2y^6 + 16384y^8\\
&- 6144x^6z^2 - 76800x^4y^2z^2 - 76800x^2y^4z^2 - 6144y^6z^2\\
&- 3776x^4z^4 + 24832x^2y^2z^4 - 3776y^4z^4 - 336x^2z^6 - 336y^2z^6 + 9z^8\\
&+ 45056x^6z + 30720x^4y^2z - 30720x^2y^4z - 45056y^6z - 20736x^4z^3\\
&+ 20736y^4z^3 - 5472x^2z^5 + 5472y^2z^5 - 6144x^6 - 76800x^4y^2\\
&- 76800x^2y^4 - 6144y^6 + 40832x^4z^2 + 77312x^2y^2z^2 + 40832y^4z^2\\
&- 16560x^2z^4 - 16560y^2z^4 - 612z^6 - 20736x^4z + 20736y^4z\\
&+ 20160x^2z^3 - 20160y^2z^3 - 3776x^4 + 24832x^2y^2\\
&- 3776y^4 - 16560x^2z^2 - 16560y^2z^2 + 10422z^4 - 5472x^2z\\
&+ 5472y^2z - 336x^2 - 336y^2 - 612z^2 + 9 = 0
\end{aligned}
$$

Computed by Joan-Carles Lario (UPC Barcelona).

Surfer plot of Lario's equation

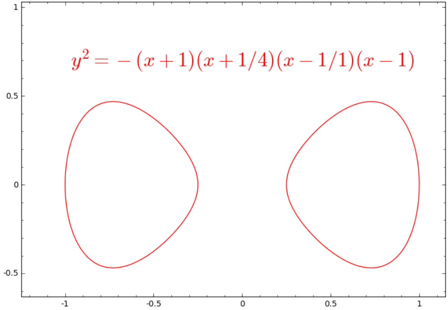


Origin of the 'other part'

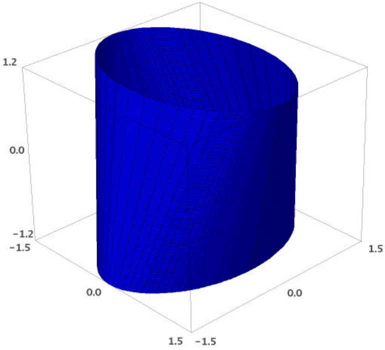
Where does the new part come from?

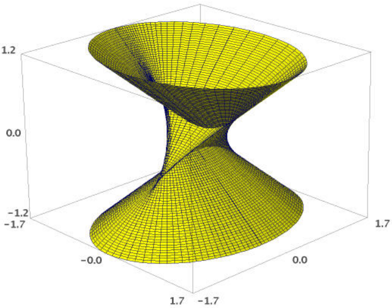
For every point on the bottom ellipse there are *two* points on the top ellipse where the tangent lines are parallel to the tangent at the bottom ellipse.

The union of all lines on the surface is parametrised by a curve of the form:



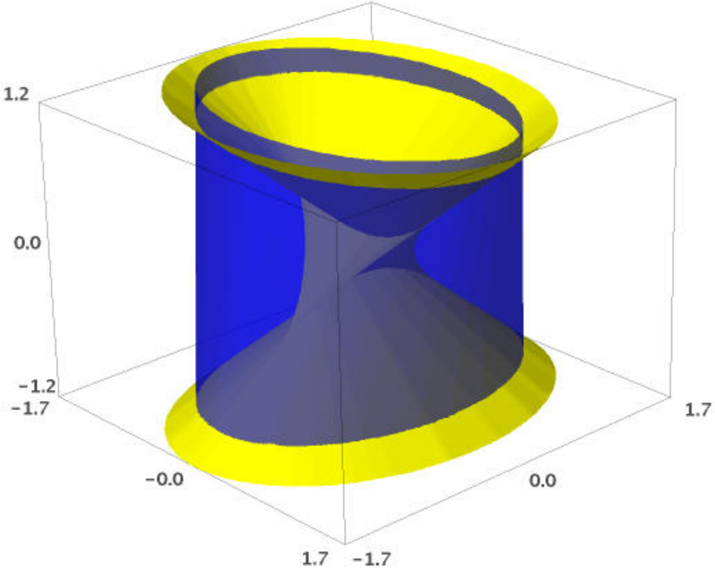
The siamese twins, apart





Process: let Serra's wheel roll, 1 wheel above the paper, the other under the paper!

The siamese twins



3D-printing, Imaginary

Oliver Labs is a mathematician in Mainz, with interests in computer science and in design.

He has coverted my sage output into input for a 3d-printer, so that I could have the siamese twins printed by Shapeways.

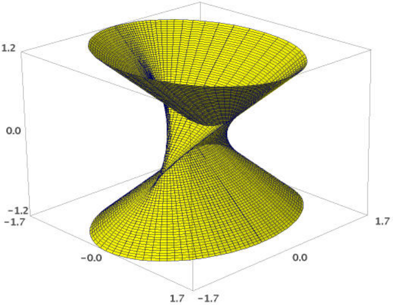
A visit here is recommended:
`http://www.oliverlabs.net/`
`http://www.shapeways.com/art/mathematical-art?li=nav`

Another nice place: `https://imaginary.org`.

The surfer plot program:
`https://imaginary.org/program/surfer`

A project?

Is anyone interested in realising the new part, at a large scale?



Dank u voor uw aandacht! (This was the end for my presentation for Ars et mathesis.)

And now more mathematics,for algant alumni!

Of course we now use complex projective geometry.

We have 2 planes H_1 and H_2 in $\mathbb{P}^3(\mathbb{C})$, with $H_1 \neq H_2$.

Let $L := H_1 \cap H_2$, this is a line.

Let C_1 and C_2 be irreducible conics in H_1 and H_2, respectively, such that $\#((C_1 \cap L) \cup (C_2 \cap L)) = 4$.

Then $\exists!\, \phi_1 \colon C_1 \to L$, such that for all $P_1 \in C_1$, $\{\phi_1(P_1)\} = L \cap T_{C_1}(P_1)$.

And $\exists!\, \phi_2 \colon C_2 \to L$, such that for all $P_2 \in C_2$, $\{\phi_2(P_2)\} = L \cap T_{C_1}(P_2)$.

More complex projective geometry

Let $E := C_1 \times_L C_2 = \{(P_1, P_2) \in C_1 \times C_2 : \phi_1(P_1) = \phi_2(P_2)\}$.

Note that both ϕ_i are of degree 2, ramified precisely over $C_i \cap L$. Hence $p_i \colon E \to C_i$ is of degree 2 and ramified over 4 points, hence of genus 1.

For P_1 and P_2 in $\mathbb{P}^3(\mathbb{C})$ distinct let $L(P_1, P_2)$ be the line in $\mathbb{P}^3(\mathbb{C})$ that contains P_1 and P_2.

Let $S \subset \mathbb{P}^3$ be Serra's surface. Then $S = \cup_{(P_1,P_2)\in E} L(P_1, P_2)$.

Let $\widetilde{S} = \{(P_1, P_2, Q) \in E \times \mathbb{P}^3(\mathbb{C}) : Q \in L(P_1, P_2)\}$.

Then $\phi \colon \widetilde{S} \to S$, $(P_1, P_2, Q) \mapsto Q$ is surjective, birational, and finite.

And $\pi \colon \widetilde{S} \to E$, $(P_1, P_2, Q) \mapsto (P_1, P_2)$ is a $\mathbb{P}^1(\mathbb{C})$-bundle (trivial locally for the Zariski topology).

The bundle is trivial

Theorem: the $\mathbb{P}^1(\mathbb{C})$-bundle $\pi\colon \widetilde{S} \to E$ is trivial.

Proof. The morphism π has 2 disjoint sections:
$\infty\colon (P_1, P_2) \mapsto (P_1, P_2, P_1)$ and $0\colon (P_1, P_2) \mapsto (P_1, P_2, P_2)$.

This makes $\widetilde{S} - (\infty(E) \cup 0(E))$ into a principal $\mathbb{C}^\times$-bundle.

A generic plane $H \subset \mathbb{P}^3(\mathbb{C})$ meets C_1 in 2 points, say Q_1 and R_1, and meets C_2 in 2 points, say Q_2 and R_2, with distinct images in L.

Then H contains no $L(P_1, P_2)$ (P_i in C_i), hence gives a 3rd section s such that $L(P_1, P_2) \cap H = \{s(P_1, P_2)\}$.

Then s meets ∞ in the 2 points (Q_1, P_2, Q_1) with $P_2 \in \phi_2^{-1}\{\phi_1(Q_1)\}$ and in the 2 points (R_1, P_2, R_1) with $P_2 \in \phi_2^{-1}\{\phi_1(R_1)\}$. Hence (as divisors on E) $s^*\infty(E) = p_1^*(Q_1 + R_1)$.

Similarly, $s^*0(E)$ is the pullback under p_2 of the sum of 2 points on C_2.

The divisor $s^*0(E) - s^*\infty(E)$ on E is the divisor of an $f \in \mathbb{C}(E)^\times$ (use that $E = C_1 \times_L C_2$).

Then $s' := f{\cdot}s$ is a section of π that is disjoint from ∞ and 0. $\square$

The ultimate picture/sculpture

Now we know more about $\widetilde{S}$ and about S, so we know how we should make a sculpture that illustrates that: the group law of E (after choice of a 0), and the triviality of the $\mathbb{P}^1(\mathbb{C})$-bundle.
To illustrate the group law of E we take a base point 0 in $E(\mathbb{R})$ (say on Serra's component), a point P of order 2 on the non-Serra component of $E(\mathbb{R})$, and a point Q of suitable finite order (60, say) in $E(\mathbb{R})^0$. Then we draw/construct/build the lines $L(nQ)$ for Serra's component, and $L(P+nQ)$ for the non-Serra component. One could use iron rods for this. These are fixed to the two conics C_1 and C_2.
As transversal coordinate axes we use a real section s of π, and a bunch of multiples of it by suitable elements of $\mathbb{R}^\times$. The BSc student Mats Beentjes has produced formulas/algorithms to compute a section, and made pictures. Think of steel cables or chains passing through holes in the rods, so that the rods and cables form a grid.
Unfortunately, the ultimate sculpture has not yet been build. Anyone interested?
And now really thank you for your attention!