Two properties of rings of invariants for finite group actions on rings

Let $X = \operatorname{Spec} A$ be an affine scheme, A an R-algebra of finite type with an action of finite group G.
(see notes of Ariyan "Locally ringed spaces and affine schemes"
Michiel " The geometry of a finite group acting on a ring")

We will discuss following proposition which based on lectures notes "Cours DEA, jacobiennes, printemps 1996." by Bas Edixhoven.

Prop 1.0 Let R be Noetherian ring. Then the sub-R-algebra A^G of G-invariants in A is of finite type, and A is finitely generated as A^G-module

First we will see some definitions and theorems that can be useful to prove our proposition.

def 1.1 a Ring R is Noetherian if all ideals of R are finitely generated.

def 1.2 a Module R is Noetherian if all its submodules are finitely generated.

def 1.3 Group action of G on A an R-algebra is defined as an action of G on A as a ring such that the following diagram:

$$\begin{tikzcd} A & A \arrow[l, "(g\cdot)^*"'] \\ R \arrow[u, "\varphi"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

is commutative.

def 1.4 an R-algebra A of finite type if there exists surjective homomorphism $R[X_1,..,X_n] \twoheadrightarrow A$.

theorem 1.5 (Hilbert basis theorem) if R is Noetherian ring, then $R[X_1,..,X_n]$ is also Netherian.

theorem 1.6 Let A be finitely generated R-module, then if R is Noetherian ring so is A. In other word A is Noetherian module.

Now we ready to prove prop 1.0.

Proof of prop 1.0 Let $x_1, x_2, .., x_n$ be R-generators of A. Let C be the sub-R-algebra of $B := A^G$ that is generated by (finitely many) coefficients of the polynomials

$$P_i(x) = \prod_{\sigma \in G} (x - \sigma(x_i)) \quad \text{for all} \quad i = 1, 2, .., n.$$

Notice that for any $h \in G$

$$h * P_i(x) = \prod_{\sigma \in G} (x - h \circ \sigma(x_i)) = \prod_{\xi \in G} (x - \xi(x_i)),$$

so the coefficients of $P_i(x)$ lying in $A^G = B$.

Then we have C is of finite type over R. By applying theorem 1.5 we deduce that, since C is quotient of polynomial ring, C is Noetherian ring itself.

Now any generators of A, i.e x_i, satisfy $P_i(x_i) = 0$ a polynomial with coefficients that generated C. Therefore A is finitely generated as C-module.

Futhermore $B \supseteq C$ as a ring then A is finitely generated as B-module.

Applying theorem 1.6 to A as C-module, we get that the sub-C-module B is finitely generated. We can write

$$B = Cb_1 + \cdots + Cb_k \quad \text{for some } b_1, \ldots, b_k \in B,$$ and

since C is R-algebra of finite type we get B is of finite type over R too.

$\square$

We will give example finite group actions on rings. This time we don't require R is Noetherian but the result still true.

<u>example 1.7</u> Let R be a ring, $n \geq 0$ and $A := R[x_1, \ldots, x_n]$. Let $G := S_n$ be the group of permutations of $\{1, 2, \ldots, n\}$. We let G act on A by R-algebra automorphism permuting the x_i: $\sigma : x_i \longmapsto x_{\sigma(i)}$. Then $B := A^G$ is finitely generated ~~(even free)~~ sub-A-algebra, the generators are elementary symmetric polynomials $P_{n,1}, \ldots, P_{nn}$. Moreover A is finitely generated (even free) B-module with basis $(x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n})_{\varepsilon_i < i}$.

<u>proof A is free B-module:</u>

We proceed by induction, for $n = 0, 1$ it's clear. Assume that it's true for $n-1$. We will prove that for any ring R, $R[x_1, \ldots, x_n] := A$ is free $R[P_{n1}, \ldots, P_{nn}] := B$-module.

Consider following expansion

$$\big((x - x_1) \ldots (x - x_{n-1})\big)(x - x_n) = (x - x_1) \cdots (x - x_n)$$, we will get identity

$$P_{n,1} = P_{n-1,1} + X_n$$
$$P_{n,2} = P_{n-1,2} + X_n \cdot P_{n-1,1}$$
$$\vdots$$
$$P_{n,n-1} = P_{n-1,n-1} + X_n \cdot P_{n-1,n-2}$$
$$P_{n,n} = P_{n-1,n-1} \cdot X_n$$

So in particular for $R = \mathbb{Z}$ we have

$$\mathbb{Z}[X_n][P_{n-1,1}, \dots, P_{n-1,n-1}] = \mathbb{Z}[P_{n,1}, \dots, P_{n,n}][X_n] \quad \dots\dots (1)$$

Also we get from expansion $X_n^n - P_{n,1} X_n^{n-1} + \dots + (-1)^n P_{n,n} = 0$,

so $\mathbb{Z}[P_{n,1}, \dots, P_{n,n}][X_n]$ generated by $1, X_n, \dots, X_n^{n-1}$ as $\mathbb{Z}[P_{n,1}, \dots, P_{n,n}]$ module.

Now applying induction hypothesis for $R = \mathbb{Z}[X_n]$ we have

$$\begin{array}{lcl}
A := \mathbb{Z}[X_1, \dots, X_n] & & \text{has basis } (X_1^{\varepsilon_1} \dots X_{n-1}^{\varepsilon_{n-1}})_{\varepsilon_i < i} \\
\cup & & \\
\mathbb{Z}[X_1, \dots, X_n]^{S_{n-1}} & = & (\mathbb{Z}[X_n])(X_1, \dots, X_{n-1})^{S_{n-1}} = \mathbb{Z}[X_n][P_{n-1,1}, \dots P_{n-1,n-1}] \\
\cup & & \qquad\qquad\qquad\qquad \| \\
B := \mathbb{Z}[X_1, \dots, X_n]^{S_n} & = & \mathbb{Z}[P_{n,1}, \dots, P_{n,n}] \subset \mathbb{Z}[P_{n,1}, \dots, P_{n,n}][X_n] \\
 & & \qquad\qquad \downarrow \\
 & & \text{with generator } 1, X_n, \dots, X_n^{n-1}
\end{array}$$

So we deduce that A generated by $(X_1^{\varepsilon_1} \dots X_{n-1}^{\varepsilon_{n-1}} X_n^{\varepsilon_n})_{\varepsilon_i < i}$ as B-module. But we can also see that

$$\begin{array}{lcl}
\mathbb{Z}[X_1, \dots, X_n] \subset & \mathbb{Q}(X_1, \dots, X_n) & \circlearrowleft S_n \text{ acts faithfully} \\
\cup & \cup \ (n-1)! & \\
\mathbb{Z}[X_1, \dots, X_n]^{S_{n-1}} \subset & \mathbb{Q}(X_1, \dots, X_n)^{S_{n-1}} & \quad n! \implies \text{hence } (X_1^{\varepsilon_1} \dots X_n^{\varepsilon_n})_{\varepsilon_i < i} \\
\cup & \cup \ n & \quad \text{must be linearly} \\
\mathbb{Z}[X_1, \dots, X_n]^{S_n} \subset & \mathbb{Q}(X_1, \dots, X_n)^{S_n} & \quad \text{independent.}
\end{array}$$

So $(x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n})_{\varepsilon_i < i}$ is the basis of A as B-module.

Now write

$$\mathbb{Z}[x_1, .., x_n] = \bigoplus_{0 \leq i_1 < i_2 < \cdots i_n < n} \mathbb{Z}[x_1, .., x_n]^{S_n} \cdot x_1^{i_1} \dots x_n^{i_n}$$

and taking the tensor product $R \otimes_{\mathbb{Z}} -$ for any ring R, we get our result (induction) for n.

$\square$

Now we will see the second proposition, namely:

Prop 2.0 Let R be a ring. Let $\pi : X \longrightarrow Y := X/G$ be the quotient and R' be an R-algebra. Then we have natural morphism

$$\mathrm{Spec}\,(R' \otimes_R A)^G \longrightarrow \mathrm{Spec}\,(R' \otimes_R A^G).$$

If R' is flat over R, this morphism is an isomorphism.

def 2.1 R' is flat over R if for every exact sequence P

$$\cdots \to N_{k+1} \longrightarrow N_k \longrightarrow N_{k-1} \to$$

of R-modules, then $R' \otimes_R P$ i.e

$$\cdots \longrightarrow R' \otimes_R N_{k+1} \to R' \otimes_R N_k \to R' \otimes_R N_{k-1} \to \cdots$$

is also exact.

And now here is the proof of prop 2.0.

Proof. First we have an exact sequence of R-modules

$$0 \longrightarrow A^G \xrightarrow{i} A \xrightarrow{f} \prod_{\sigma \in G} A,$$

where $f : A \longrightarrow \prod_{\sigma \in G} A$ sends a to $(\sigma \mapsto \sigma(a) - a)$.

(So the kernel of f is really A^G by definition)

Since G is finite, tensoring by G we will obtain

$$0 \to R' \otimes_R A^G \xrightarrow{1 \otimes_R i} R' \otimes_R A \xrightarrow{1 \otimes_R f} R' \otimes_R \prod_{\sigma \in G} A \quad \ldots (1)$$

$$R' \otimes_R \prod_{\sigma \in G} A = \prod_{\sigma \in G} (R' \otimes_R A).$$

This sequence .... (1) is not neeessary exact but

$$\mathrm{Ker}\,(1 \otimes_R f) = (R' \otimes_R A)^G \supseteq R' \otimes_R A^G$$

since $f \circ i = 0$. So there exists inclusion $R' \otimes_R A^G \hookrightarrow (R' \otimes_R A)$

This induces natural morphism

$$\mathrm{Spec}\,(R' \otimes_R A)^G \longrightarrow \mathrm{Spec}\,(R' \otimes_R A^G).$$

Futhermore if $R \to R'$ is flat, then the sequence ... (1) remains exact, proving what we need.

$\square$

We want give example what happenned when R' isn't flat over R.

<u>example 2.2</u>

Take $R = \mathbb{Z}$, $R' = \mathbb{F}_2$ then we know that $\mathbb{Z} \to \mathbb{F}^2$ isn't flat since take exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2x} \mathbb{Z} \longrightarrow \mathbb{Z}/(2) \longrightarrow 0,$$

then after tensoring with $\mathbb{F}_2$, we get

$$0 \to \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \longrightarrow \mathbb{F}_2 \longrightarrow 0$$

which is n't exact any more.

Now for $A = \mathbb{Z}[x] \circlearrowleft G := \mathbb{Z}/2\mathbb{Z}$ where $1 : x \longmapsto -x$, we have $A^G = \mathbb{Z}[x^2]$. So $\mathbb{F}_2 \otimes_{\mathbb{Z}} A^G = \mathbb{F}_2[x^2]$, while $\mathbb{F}_2 \otimes_{\mathbb{Z}} A = \mathbb{F}_2[x]$. Now G act trivially in $\mathbb{F}_2[x]$ therefore

$$(\mathbb{F}_2 \otimes_{\mathbb{Z}} A)^G = \mathbb{F}_2[x].$$

Thus implies the morphism

$$\mathrm{Spec}\,(\mathbb{F}_2[x]) \longrightarrow \mathrm{Spec}\,(\mathbb{F}_2[x^2])$$

is not isomorphism.

$\square$

This is the continuation of our last talk "two properties of rings of invariants for finite group actions on rings". We will prove this interesting theorem, still the same notation, namely

<u>thm 3.0</u> Let R be a Noetherian ring, $X = \mathrm{Spec}\, A$ an affine scheme where A is R-algebra of finite type with an action by a finite group. Let $\pi : X \to Y := X/G$ be the quotient. Let y be in Y, then the diagram

$$\begin{tikzcd} X \arrow[d] & \coprod_{x \to y} \mathrm{Spec}(\widehat{\mathcal{O}_{X,x}}) \arrow[l] \arrow[d] \\ Y & \mathrm{Spec}(\widehat{\mathcal{O}_{Y,y}}) \arrow[l] \end{tikzcd}$$

is Cartesian, and the second vertical arrow is also quotient for the action by G.

<u>def 3.1</u> Let A be a ring, $I \subseteq A$ an ideal. Then the I-adic completion $\hat{A}$ is defined by

$$\hat{A} = \varprojlim_n A/I^n$$

<u>theorem 3.2</u> Let A be a ring and $S \subset A$ a multiplicative set, then the localisation A_S of A with respect to S is flat over A.

<u>theorem 3.3</u> Let A be a Noetherian ring, I an ideal, and $\hat{A}$ the I-adic completion of A; then $\hat{A}$ is flat over A.

<u>theorem 3.4</u> Let A be ring and $m_1, .., m_r$ be maximal ideals of A. Set $I = m_1 . m_2 ... m_r$. Then the I-adic

completion $\hat{A}$ of A decomposes as a direct product.

$$\hat{A} = \hat{A}_1 \times \cdots \times \hat{A}_r$$

where $A_i = A_{m_i}$, and $\hat{A_i}$ is the completion of the local ring A_i.

Note $m_1, \ldots, m_r$ not necessary <u>all</u> maximal ideals of A.

<u>Proof thm 3.0</u> First by prop 1.0, $\pi: X \to Y$ is finite implies $\{x \in X \mid x \mapsto y\}$ is finite set. Now we want to prove that the diagram

$$\begin{tikzcd} A \arrow[r] & \prod\limits_{q : q \cap B = p} \hat{A}_q \\ A^G =: B \arrow[u, hook] \arrow[r] & \hat{B}_p \arrow[u] \end{tikzcd}$$

is cartesian. Here p is denoted prime ideal y, q denoted for prime ideals x that map to y.

In Category theoritical product of A, B_p cartesian means

$$\prod_{q : q \cap B = p} \hat{A}_q = \hat{B}_p \otimes_B A.$$

Consider following cartesian diagram:

$$\begin{tikzcd} A \arrow[r] & B_p \otimes_B A \\ B \arrow[u, hook] \arrow[r] & B_p \arrow[u] \end{tikzcd} \quad \ldots (1)$$

We have G acts on $B_p \otimes_B A$ given by $B_p \otimes_B A \xrightarrow{id \otimes_B g} B_p \otimes_B A$ for any $g \in G$.

Because the localisation B_p of B is flat over B (thm 3.2), then using prop 2.0 applying for $R' = B_p$, $R = B$ we will get

$$\left(B_p \otimes_B A\right)^G = B_p \otimes_B A^G = B_p \otimes_B B = B_p . \quad (*)$$

So the second vertical arrow of diagram ...(1) is quotient.

We use change cofficient in ring by tensoring

$$\widehat{B_p} \otimes_{B_p} (B_p \otimes_B A) = (\widehat{B_p} \otimes_{B_p} B_p) \otimes_B A = \widehat{B_p} \otimes_B A .$$

Now we have big diagram :

$$\begin{tikzcd}
A \arrow[r] & B_p \otimes_B A \arrow[r] & \widehat{B_p} \otimes_B A = \widehat{B_p} \otimes_{B_p} (B_p \otimes_B A) \\
B \arrow[u, hook] \arrow[r] \arrow[r, phantom, shift right=4ex, "\cdots(1)"] & B_p \arrow[u] \arrow[r] \arrow[r, phantom, shift right=4ex, "\cdots(2)"] & \widehat{B_p} \arrow[u]
\end{tikzcd}$$

...... (3)

Since diagram ...(1) & ...(3) cartesian then ...(2) is also cartesian.

Now we just need to concentrate in diagram ...(2).

Recall by prop 1.0, B is finite type R-algebra therefore B is Noetherian. This implies B_p is also Noetherian ring. In diagram ...(1) we can apply prop 1.0 with $R = B$ itself to deduce that $B_p \otimes_B A$ is finitely generated B_p-algebra (module).

By theorem 3.3, we have $\widehat{B_p}$ is flat over B_p. Therefore apply Prop 2.0 with $R' = \widehat{B_p}$, $R = B_p$ we get

$$\left(\widehat{B_p} \otimes_{B_p} (B_p \otimes_B A)\right)^G = \widehat{B_p} \otimes_{B_p} (B_p \otimes_B A)^G = \widehat{B_p} \otimes_{B_p} B_p = \widehat{B_p}$$

(*)

So we have proved one thing i.e the map

$$\mathrm{Spec}(\hat{B_p} \otimes_B A) \longrightarrow \mathrm{Spec}(\hat{B_p}) \text{ is a quotient.}$$

Now to finish the proof. we noticed that since B_p is Noetherian ring, ~~then~~ $B_p \otimes_B A$ is fin. generated B_p-module

$$\hat{B_p} \otimes_B A = \hat{B_p} \otimes_{B_p} (B_p \otimes_B A)$$

will be products completion of local rings. To see this first consider exact sequence $B_p^k \longrightarrow B_p^l \longrightarrow B_p \otimes_B A \longrightarrow 0$; and we get commutative diagram with exact rows

$$\begin{tikzcd}
\widehat{B_p^k} \arrow[r] & \widehat{B_p^l} \arrow[r] & \widehat{B_p \otimes_B A} \arrow[r] & 0 \\
\hat{B_p} \otimes_{B_p} B_p^k \arrow[r] \arrow[u, "\wr"] & \hat{B_p} \otimes_{B_p} B_p^l \arrow[r] \arrow[u, "\wr"] & \hat{B_p} \otimes_{B_p} (B_p \otimes_B A) \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Here the first two vertical arrows are natural map; since completion commutes with direct sum. They are isomorphism, so the third arrow is an isomorphism too.

Let $m_B = p \cdot B_p$ the maximal ideals of B_p, and $m_1, \dots, m_r$ are ~~maximal~~ prime which is max ideals of $B_p \otimes_B A \longmapsto m_B$. Notice that

$$\widehat{B_p \otimes_B A} = \varprojlim_n \left(B_p \otimes_B A / m_B^n (B_p \otimes_B A) \right) = \varprojlim_n (B_p \otimes_B A) \otimes_{B_p} B_p / m_B^n$$

We can find n such that $I := m_1 \dots m_r \supseteq m_B (B_p \otimes_B A) \supseteq (m_1 \dots m_r)^n$, thus I & $m_B(B_p \otimes A)$ gives the same completion. But in $B_p \otimes_B A / I^n$ will equal to product of localization in m_i, since completion & product commutative then (Thm 3.4) $\widehat{B_p \otimes_B A} = \prod \widehat{(B_p \otimes_B A)_{m_i}} = \prod \hat{A_{q_i}}$ where q_i is prime ideal of A s.t $q_i \cap A = p$.