Locally ringed spaces and affine schemes

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This talk was given at the seminar Topics in Arithmetic Geometry II on February 28, 2011. The seminar was aimed at master students. In this talk a ring is an associative, commutative and unitary ring. A morphism of rings respects the unit elements.

1 Motivation

Today's main theorem is

Theorem 1.1. Let $X = \operatorname{Spec} A$ be an affine scheme. Suppose that G is a finite group acting on X. There is a canonical action of G on A. Let $A^G \subset A$ be the ring of invariants of A. Define $Y = \operatorname{Spec} A^G$. The induced morphism $\pi : X \longrightarrow Y$ is a quotient in the category of locally ringed spaces.

In this talk we will define and explain the notions in the above Theorem. That is, we will define the category of locally ringed spaces and its subcategory of affine schemes. We will give examples and finish with the following useful result: the fiber of a morphism $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ in $\mathfrak{p} \in \operatorname{Spec} A$ is the affine scheme $\operatorname{Spec}(B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$.

The proof of the above Theorem will be given by Michiel Kosters in the next talk.

2 Locally ringed spaces

Let us state the definition of a "ringed space".

Definition 2.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A *morphism of ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$, where $f : X \longrightarrow Y$ is a morphism of topological spaces and $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is a morphism of sheaves of rings on Y. This defines the category of ringed spaces.

Example 2.2. Let X be a topological space and $\mathcal{O}_X$ the sheaf of continuous functions on X. That is, for every open $U \subset X$, we have that $\mathcal{O}_X(U) = C^0(U, \mathbf{R})$. The pair $(X, \mathcal{O}_X)$ is a ringed space.

Example 2.3. Let X be a C^k-manifold, where $k \in \mathbf{Z}_{\geq 0} \cup \{\infty\} \cup \{\omega\}$. The pair $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is the sheaf of C^k-functions on X, is a ringed space. (Recall that C^0-manifolds are called topological manifolds, C^∞-manifolds are called differentiable manifolds (by Lübke) and C^ω-manifolds are complex analytic manifolds.)

Example 2.4. Let k be an algebraically closed field and let X be a (quasi-projective) variety over k. Let $\mathcal{O}_X$ be the sheaf of regular functions on X. That is, for every open $U \subset X$, we have that $\mathcal{O}_X(U)$ is the ring of regular functions on U. The pair $(X, \mathcal{O}_X)$ is a ringed space.

All three examples above are "locally ringed spaces". (We will need the following notion. A morphism $A \longrightarrow B$ of local rings is *local* if the image of the maximal ideal of A is contained in that of B. This is equivalent to the inverse image of the maximal ideal of B being equal to the maximal ideal of A.)

Definition 2.5. A ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space* if for every x in X, the stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at x is a local ring. A *morphism of locally ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a morphism of ringed spaces $(f, f^\#)$ from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ such that, for every x in X, the induced homomorphism of local rings

$$f_x^\# : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is local. (Here we used that there is a canonical morphism $(f_*\mathcal{O}_X)_{f(x)} \longrightarrow \mathcal{O}_{X,x}$ constructed as follows in Remark **??** below.) This defines the category of locally ringed spaces. We denote it by $\mathfrak{Lrs}$. It is **not** a full subcategory of the category of ringed spaces. (See [Har, Example II.2.3.2].)

For X as in any of the above examples, note that $(X, \mathcal{O}_X)$ is a locally ringed space. In fact, the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$ is the set of germs[1] which vanish at x.

Remark 2.6. Let $f : X \longrightarrow Y$ a morphism of topological spaces. Let $\mathcal{F}$ be a sheaf on X and let x be in X. The stalk of $\mathcal{F}$ at $f(x)$ is not $\mathcal{F}_x$ in general. (Take a constant morphism, for example.) There is a canonical morphism from $(f_*\mathcal{F})_{f(x)}$ to $\mathcal{F}_x$ constructed as follows. Let s be a section of $\mathcal{F}$ on $f^{-1}V$, where V is an open in Y containing $f(x)$. Then, by continuity of f, there exists an open U in X containing x such that $f(U) \subset V$. In particular, we have that $U \subset f^{-1}V$. Thus, we can restrict s to U.

Here's a pathological example of a ringed space which is not a locally ringed space.

Example 2.7. Let X be a singleton. To give a sheaf of rings on X is to give a ring. Let R be any ring which is not local. Then this will not be a locally ringed space (obviously).

Here's an example of a morphism of local rings which is not local.

Example 2.8. Let A be a local integral domain with field of fractions K. Then the morphism $A \longrightarrow K$ is not local.

We finish this section with the following important remark.

Remark 2.9. Let $(X, \mathcal{O}_X)$ be a locally ringed space and $U \subset X$ an open subset. Then $(U, \mathcal{O}_U)$ is a locally ringed space, where $\mathcal{O}_U := (\mathcal{O}_X)|_U$. In fact, for every x in U, we have that $\mathcal{O}_{U,x} = \mathcal{O}_{X,x}$.

3 Affine schemes

We have seen three important examples of (full) subcategories of $\mathfrak{Lrs}$. We will now construct another subcategory of $\mathfrak{Lrs}$: the category of affine schemes.

Let A be a ring. Let $X = \operatorname{Spec} A$ be the set of prime ideals. The goal is to define a topology on X and to construct a sheaf of rings $\mathcal{O}_X$ on X such that $(X, \mathcal{O}_X)$ is a locally ringed space.

Let us recall how the topology on $\operatorname{Spec} A$ is defined. For any ideal $I \subset A$ in A, we define

$$V(I) := \{\mathfrak{p} \in \operatorname{Spec} A \mid I \subset \mathfrak{p}\}.$$

For a in A, we define $D(a) = \operatorname{Spec} A \backslash V(aA)$. If $I \subset J$, we have that $V(J) \subset V(I)$.

The closed subsets of $\operatorname{Spec} A$ are precisely the subsets $V(I)$, where $I \subset A$ is an ideal in A. This defines a topology on X. It is called the *Zariski topology.* A morphism $\varphi : A \longrightarrow B$ of rings induces a morphism $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ of topological spaces given by $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$.

Example 3.1. Consider the canonical surjection $\varphi : A \longrightarrow A/I$, where $I \subset A$ is an ideal of A. Then the induced morphism $f : \operatorname{Spec}(A/I) \longrightarrow \operatorname{Spec} A$ induces a homeomorphism of $\operatorname{Spec}(A/I)$ to the closed subset $V(I)$ of $\operatorname{Spec} A$.

[1] A germ is an equivalence class of pairs (U, f), where U is an open of X and $f \in \mathcal{O}_X(U)$...

Example 3.2. For a in A, the morphism $f : \operatorname{Spec} A_a \longrightarrow \operatorname{Spec} A$ induced by $\varphi : A \longrightarrow A_a$ induces a homeomorphism of $\operatorname{Spec} A_a$ onto the open $D(a)$ in $\operatorname{Spec} A$.

Remark 3.3. Let k be an algebraically closed field. Recall that an algebraic set in $\mathbf{A}_k^n$ is a subset of the form $\{x \in \mathbf{A}_k^n : f(x) = 0 \text{ for all } f \text{ in } S\}$ for some subset $S \subset k[x_1, \ldots, x_n]$

The Zariski topology on X defined above can also be defined in terms of "functions" having "value" zero.

Definition 3.4. Let x be in X. The *residue field* of x, denoted by $k(x)$, is defined to be

$$k(x) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}},$$

where $\mathfrak{p} \subset A$ is the prime ideal corresponding to x.

Example 3.5. The residue field of the zero ideal in $\mathbf{A}_k^1 = \operatorname{Spec}(k[t])$ is $k(t)$. The residue field of (P), where P is an irreducible polynomial, is $k[t]/(P)$. This is a finite extension of k.

Example 3.6. The residue field of (0) in $\operatorname{Spec} \mathbf{Z}$ is $\mathbf{Q}$. The residue field of (p) is $\mathbf{Z}/p\mathbf{Z}$.

Remark 3.7. Let x be in X and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to it. Each element a in A gives an element in $A_{\mathfrak{p}}$. This element should be considered as the "germ of a function". It also gives an element $a(x)$ in $k(x)$. This should be considered as the "value of the germ" at x.

Proposition 3.8. Let V be a subset of X. Then V is closed in the Zariski topology if and only if there exists a subset $S \subset A$ such that $V = \{x \in \operatorname{Spec}(A) : a(x) = 0 \text{ for all } a \text{ in } S\}$.

Proof. The easy verification is left to the reader. □

The following theorem tells us how we can make $X = \operatorname{Spec} A$ into a locally ringed space.

Theorem 3.9. There is a unique sheaf of rings $\mathcal{O} = \mathcal{O}_X$ (which we call the *structure sheaf*) on $X = \operatorname{Spec} A$ satisfying the following properties.

1. For every a in A, we have that $\mathcal{O}(D(a)) = A_a = A[t]/(at - 1)$.

2. For every a and b in A such that $D(b) \subset D(a)$, we have that the restriction morphism $\mathcal{O}(D(a)) \longrightarrow \mathcal{O}(D(b))$ is the canonical morphism $A_a \longrightarrow A_b$. (Note that $b \in \sqrt{(a)}$. Write $b^m = fa$ with $m > 0$ and $f \in A$. Then f is invertible in A_b and we obtain a canonical homomorphism $A_a \longrightarrow A_b$ (mapping ca^{-n} to $cf^n b^{-nm}$) which is an isomorphism if $D(a) = D(b)$.)

3. For every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is isomorphic to the local ring $A_{\mathfrak{p}}$, where $\mathfrak{p} \subset A$ is the prime ideal corresponding to x.

Proof. The uniqueness follows from the fact that the $(D(f))_{f \in A}$ form a basis $\mathcal{B}$ for the Zariski topology on X. In fact, to show that $\mathcal{O}$ is a sheaf it suffices to show that for an open cover $(D(a_i))$ of $U \in \mathcal{B}$ by principal opens and sections $s_i \in \mathcal{O}(D(a_i))$ such that $(s_i)|_{D(a_i) \cap D(a_j)} = (s_j)|_{D(a_i) \cap D(b_i)}$ for all i, j, there exists a unique section $s \in \mathcal{O}_X(U)$ such that $s|_{U_i} = s_i$ for all i. Property 3 follows from 1 and 2. In fact, one easily shows that, for every prime ideal $\mathfrak{p} \subset A$ in A, the canonical morphism

$$\lim_{a \notin \mathfrak{p}} A_a \longrightarrow A_{\mathfrak{p}}$$

is an isomorphism. □

Corollary 3.10. Let $X = \operatorname{Spec} A$. Then $(X, \mathcal{O}_X)$ is a locally ringed space. □

We have a (contravariant) functor Spec from the category of rings to the category of locally ringed spaces defined as follows. Let $\varphi : A \longrightarrow B$ be a morphism of rings. We already know how this induces a morphism $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ of topological spaces. Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. There is a canonical morphism $f^* : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ of sheaves constructed as follows. Firstly, note that $f^{-1}(D(a)) = D(\varphi(a))$ for all a in A. (The condition $a \notin f(\mathfrak{p})$ is equivalent to the condition $\varphi(a) \notin \mathfrak{p}$.) Therefore, we see that φ induces a morphism $\mathcal{O}_Y(D(a)) = A_a \longrightarrow B_{\varphi(a)} = (f_*\mathcal{O}_X)(D(a))$. These homomorphisms are clearly compatible with the restriction morphisms. Since the $D(a)$ form a basis of open sets of Y, this defines a morphism of sheaves $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$. Note that the morphism on stalks is a local morphism of local rings.

Theorem 3.11. The functor Spec from the category of rings to the category of locally ringed spaces is fully faithful. That is, if A and B are rings, the morphism

$$\operatorname{Hom}_{\mathfrak{Ring}}(A, B) \longrightarrow \operatorname{Hom}_{\mathfrak{Lrs}}(\operatorname{Spec} B, \operatorname{Spec} A)$$

is bijective. (Here $\mathfrak{Ring}$ is the category of rings.)

Definition 3.12. The *category of affine schemes* is the essential image of the functor Spec in the category of locally ringed spaces. (In particular, it is a full subcategory of $\mathfrak{Lrs}$.) An *affine scheme* is an object of the category of affine schemes, i.e, a locally ringed space which is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.

Remark 3.13. One should view Theorem 3.11 as a generalization of Corollary I.3.8 in Hartshorne. In fact, the category of rings is anti-equivalent to the category of affine schemes.

Example 3.14. Let $U \subset X$ be an open subset of $X = \operatorname{Spec} A$. Then U is not necessarily an affine scheme. For example, let $X = \mathbf{A}_k^2 = \operatorname{Spec} k[t_1, t_2]$ and $U = X - \{(0,0)\}$, where k is a field. Then U is not an affine scheme. To show this, let us cover U by $U_1 = D(t_1) \subset U$ and $U_2 = D(t_2) \subset U$. Note that $\mathcal{O}_X(U_1) = k[t_1, t_2]_{t_1}$ and $\mathcal{O}_X(U_2) = k[t_1, t_2]_{t_2}$. The inclusion $i : U \longrightarrow X$ is not surjective (clearly). Therefore, it is not an isomorphism of locally ringed spaces. Note that the homomorphism $\mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(U)$ is an isomorphism. (We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U_1) \oplus \mathcal{O}_X(U_2) \longrightarrow \mathcal{O}_X(U_1 \cap U_2) \oplus \mathcal{O}_X(U_2 \cap U_1)\ ,$$

where the first map is given by the restriction maps and the second by $(s_1, s_2) \mapsto (s_1 - s_2)$. The elements of $\mathcal{O}_X(U)$ are rational fractions whose denominator is a power of t_1 and t_2. Therefore, the elements of $\mathcal{O}_X(U)$ are polynomials.) Suppose that U is an affine scheme. Then, by Theorem 3.11, the inclusion $i : U \longrightarrow X$ is an isomorphism. Contradiction.

Example 3.15. Let $X = \operatorname{Spec} A$ be an affine scheme. Then there exists a unique morphism $X \longrightarrow \operatorname{Spec} \mathbf{Z}$.

Remark 3.16. Note that the structure sheaf allows us to distinguish between $\operatorname{Spec} k$, $\operatorname{Spec} L$ and $\operatorname{Spec} k[T]/(T^2)$ for a field extension $k \subset L$ even though their topological space consists of a single point.

4 Fibres of a morphism of affine schemes

Definition 4.1. Let $f : X \longrightarrow Y$ be a morphism of affine schemes, where $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Let y be a point in Y. The *fibre* of f in y is the affine scheme

$$X_y := \operatorname{Spec}(B \otimes_A k(y)).$$

Example 4.2. Let K be a number field with ring of integers O_K. Consider the morphism $\operatorname{Spec} O_K \longrightarrow \operatorname{Spec} \mathbf{Z}$ of affine schemes. The *generic fibre* $\operatorname{Spec} K$ is the fibre in the generic point of $\operatorname{Spec} \mathbf{Z}$. The fibre in the closed point (p) is $\operatorname{Spec}(O_K \otimes_{\mathbf{Z}} \mathbf{F}_p)$. Note that $\operatorname{Spec}(O_K \otimes_{\mathbf{Z}} \mathbf{F}_p)$ is the set of primes lying above p.

Proposition 4.3. Let $f : X \longrightarrow Y$ be a morphism of affine schemes, where $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Let y be a point in Y. The canonical morphism $\varphi : B \longrightarrow B \otimes_A k(y)$ induces a morphism of affine schemes $p : X_y \longrightarrow X$. This morphism induces a homeomorphism of X_y onto $f^{-1}(y) \subset X$.

Proof. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to y. Let $u : A \longrightarrow B$ be the (unique!) morphism corresponding to f. (Here we use Theorem 3.11.) The canonical morphism $\varphi : B \longrightarrow B \otimes_A k(y)$ is the composition of the localization $\varphi_1 : B \longrightarrow B \otimes_A A_\mathfrak{p}$ and the canonical surjection $\varphi_2 : B \otimes_A A_\mathfrak{p} \longrightarrow B \otimes_A k(y)$. Therefore, we have that p induces a homeomorphism of X_y onto the set of prime ideals $\mathfrak{q}$ in B which contain $u(\mathfrak{p})B$ and do not meet $u(A - \mathfrak{p})$. This means precisely that $u^{-1}(\mathfrak{q}) = \mathfrak{p}$ (or $f(\mathfrak{q}) = \mathfrak{p}$). □

5 The main theorem of the day again

We have explained what the category of locally ringed spaces is and what affine schemes are. The action of a finite group G on an affine scheme X can be transformed into an action of G on $\mathcal{O}_X(X)$. An action of G on $\mathcal{O}_X(X)$ can be transformed into an action of G on X by taking the Spec. Since Spec is an anti-equivalence this explains the following statement.

Theorem 5.1. Let $X = \operatorname{Spec} A$ be an affine scheme. Suppose that G is a finite group acting on X. There is a canonical action of G on A. Let $A^G \subset A$ be the ring of invariants of A. Define $Y = \operatorname{Spec} A^G$. The induced morphism $\pi : X \longrightarrow Y$ is a quotient in the category of locally ringed spaces.

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