Bas Edixhoven. Topics in Arithm. Geom. 2011/04/07.

Part 1. Daniel Bertrand's counterexample for relative Manin-Mumford for semi-abelian varieties.

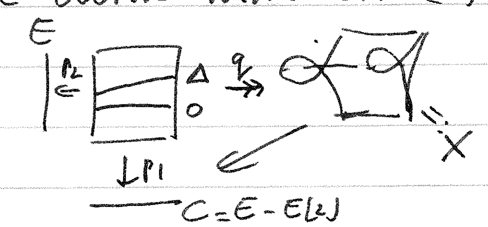
References: Richard Pink, A common generalization of ....
David Masser - Umberto Zannier, ① Torsion anomalous points and ...
② Torsion points on families of squares of elliptic curves.

Thm (Masser-Zannier) Let C be a nonsingular affine curve over $\mathbb{C}$, $E \to C$ an elliptic curve, and $P, Q \in E(C)$ $\mathbb{Z}$-linearly independent. Then there are only finitely many $c \in C(\mathbb{C})$ such that P_c and Q_c both have finite order in E_c.

Such a result was also conjectured for semi-abelian families. Here we give an example of Daniel Bertrand that shows that conjecture wrong; but he also knows how to "repair" the conjecture.

Let $E/\mathbb{C}$ be the elliptic curve with $\mathrm{End}(E) = \mathbb{Z}[i]$; $\left\{\begin{array}{l} Y^2 = X^3 - X. \\ E = \mathbb{C}/\mathbb{Z}[i] \end{array}\right.$

Let $C = E - E[2]$.
Then we have $\begin{array}{c} E_C \\ \downarrow \\ C \end{array}$:



q: glue, in E_P, P and 0.

$X \to C$ is a family of singular curves of genus 2 ($\dim H^1(X_P, \mathcal{O}) = 2$).
$J \to C$:= family of jacobians of $X \to C$, the fibres are of dim. 2.
Note: $J \to C$ is the analog of $E \times_C E \to C$ in the thm. above.
$\forall P \in C$: $\mathbb{G}_m \rightarrowtail J_P \twoheadrightarrow E_P$.
We have a section of $J \to C$ given by $P \mapsto [iP - i'P]$, divisor class.
Note that for all $P \in C = E - E[2]$, $\{iP, i'P\} \cap \{0, P\} = \emptyset$: iP and $i'P$ are in the smooth locus of X_P.

Claim. Let $n \in \mathbb{Z}_{\geq 2}$ and $P \in E$ of order n. Then $[iP - i'P]$ is torsion in J_P.

<u>Proof.</u> Choose $f \in \mathbb{C}(E)^\times$ s.t. $\mathrm{div}(f) = n\cdot P - n\cdot O$.
Then $\mathrm{div}(f\circ i) = n\cdot i^{-1}P - n\cdot O$, $\mathrm{div}(f\circ i^{-1}) = n\cdot iP - n\cdot O$.
Put $g = f\circ i^{-1}/f\circ i$. Then $\mathrm{div}(g) = n\cdot iP - n\cdot i^{-1}P$. Hence in E we have, of course, $niP - ni^{-1}P = 0$.
But: $\big([iP - i^{-1}P]$ is torsion in $J_P\big) \iff \big(g(P)/g(O) \in \mathbb{C}^\times_{\text{tors}}\big)$.

Compute: $g(P) = f(i^{-1}P)/f(iP)$, $g(O) = (-1)^n$.

($f = t^{-n}$ + h.o.t near 0, $f\circ i = (i^{-1}t)^{n}$ + h.o.t, $f\circ i^{-1} = (it)^{-n}$ + h.o.t ...)

So: $\cancel{\dfrac{g(P)}{g(O)}}$ $\dfrac{f(i^{-1}P)}{f(iP)} = (-1)^n \cdot \dfrac{g(P)}{g(O)}$.

<u>Weil reciprocity</u>: $\mathrm{div}(f)$ and $\mathrm{div}(g)$ are disjoint, so:

$$1 = \frac{f(\mathrm{div}\, g)}{g(\mathrm{div}\, f)} = \frac{f(iP)^n \cdot f(i^{-1}P)^{-n}}{g(P)^n \cdot g(O)^{-n}} = \cancel{\frac{f(iP)^n \cdot f(i^{-1}P)^{-n}}{f(i^{-1}P)^n \cdot f(iP)^{-n}} \cdot (-1)^{n^2} = (-1)^n \cdot f(iP)}$$

$$= \frac{g(O)^n \cdot g(P)^{-n} \cdot (-1)^{n^2}}{g(P)^n \cdot g(O)^{-n}} = (-1)^n \cdot \left(\frac{g(O)}{g(P)}\right)^{2n}.$$

<u>Conclusion</u>: in J_P: $4n^2 [iP - i^{-1}P] = 0$. $\boxtimes$

On the other hand, if $P \in E$ is <u>not</u> torsion, then the set $\{m\cdot[iP - i^{-1}P] : m \in \mathbb{Z}\} \subset J_P$ is <u>Zariski dense</u>.

Argument: let Z be the Zar. closure. It is a subgr. of J_P.
If $Z \to E$ is finite, then $\mathbb{G}_m \hookrightarrow J_P \twoheadrightarrow E$ is torsion, contradicting that P is <u>not</u> torsion ($\mathrm{Ext}^1(E, \mathbb{G}_m) = E$).
$$[J_P] \longleftrightarrow \overset{\in}{P}$$

<u>Conclusion</u>: subgroupschemes do not explain this example. One needs something <u>quadratic</u>: biextensions. Indeed, D. Bertrand explains it in the context of Mixed Shimura Varieties, beautifully!!

Part 2. Manin-Mumford for tori.

Let $n \in \mathbb{Z}_{\geq 1}$, $T := \mathbb{G}_{m,\mathbb{Q}}^n$, $x = (x_i)_{i \in \mathbb{N}}$ a seq. in $T(\overline{\mathbb{Q}})_{tors}$

Special subvarieties of $T_{\overline{\mathbb{Q}}}$: $\zeta \cdot T'$, T' a subtorus and $\zeta \in T(\overline{\mathbb{Q}})_{tors}$.

Assume that x is strict: $\forall\, Z \subset T_{\overline{\mathbb{Q}}}$ special, $\{i \in \mathbb{N} : x_i \in Z(\overline{\mathbb{Q}})\}$ is finite.

Thm. The sequence of probability measures on $(S^1)^n$ given by the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x_i$ converges weakly to the Haar measure.

This means: $\forall\, f : (S^1)^n \to \mathbb{C}$ continuous,

$$i \mapsto \frac{1}{\#(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x_i)} \cdot \sum_{y \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x_i} f(y) \quad \text{conv. to} \int_{(S^1)^n} f \cdot d\mu.$$

Cor. The set $\{x_i : i \in \mathbb{N}\}$ is Zariski dense in $T_{\overline{\mathbb{Q}}}$.

Cor. $\forall\, S \subset T(\overline{\mathbb{Q}})_{tors}$, each irred. component of $\overline{S}^{Zar}$ is special.

Proof of thm. Stone-Weierstrass: the image of $\mathbb{C}[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}]$ in $\mathrm{Cont}((S^1)^n, \mathbb{C})$ is dense for $\|\cdot\|_{sup}$.

So, it suffices to consider f of the form $z_1^{m_1} \cdots z_n^{m_n}$, $m_i \in \mathbb{Z}$.

Then: $(\ker f)^\circ \hookrightarrow \mathbb{G}_m^n \xrightarrow{f} \mathbb{G}_m$. The problem is reduced to the case $n=1$.

Then easy: for $\zeta \in \overline{\mathbb{Q}}^\times_{tors}$ of order n: write $n = \prod_{i=1}^r p_i^{e_i}$, then $\sum_{y \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \zeta} y = \begin{cases} 0 & \text{if } e_i > 1 \text{ for some } i \\ (-1)^r & \text{otherwise.} \end{cases}$

For abelian varieties: Ullmo-Szpiro, Zhang.

For Shimura varieties: only some GL_2-cases known (Duke, Zhang).
(the analogous conjecture is called "André-Oort".)