CONSTRUCTION OF CUBIC SURFACES
FROM 6 POINTS IN THE PROJECTIVE PLANE
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In this notes we will analyse a general construction of cubic surfaces. It is the blow-up of the projective plane at six points, and every cubic surface can be obtained in this way.

First we will see an example to understand better the behaviour of the blow-up.

0.1 Blow-up of $\mathbb{A}^n$ at 0

Let k an algebraically closed number field.

Definition 1. *The* blow-up *of $\mathbb{A}^n$ at 0 is the closed subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the equations*

$$X = \{x_i y_j = x_j y_i : i, j = 1, ..., n\}$$

where $x_1, ..., x_n$ are the affine coordinates over $\mathbb{A}^n$ and $y_1, ..., y_n$ are the projective coordinates of $\mathbb{P}^{n-1}$. We consider also the projection $\pi : X \to \mathbb{A}^n$, induced by the one of the cartesian product $\mathbb{A}^n \times \mathbb{P}^{n-1}$ over $\mathbb{A}^n$. We will also call $\pi^{-1}(0)$ the exceptional curve *E of the blow-up.*

We will now list some general properties of the blow-up.

- $\pi^{-1}(0) \cong \mathbb{P}^{n-1}$

  Proof. For a point $P = (a_1, ..., a_n)$, we have in general that the inverse image of it through π is

  $$\pi^{-1}(P) = \{(a_1, ..., a_n) \times [y_1, ..., y_n] : y_i \in k \text{ and } a_i y_j = a_j y_i \ \forall i, j = 1, ..., n\}$$

  Hence, for $P = 0$, we obtain

  $$\pi^{-1}(0) = \{0 \times [y_1, ..., y_n] : y_i \in k\} \cong \mathbb{P}^{n-1}$$

  □

- Let $P \in \mathbb{A}^n \backslash \{0\}$. Then $\pi^{-1}(P)$ is a single point.

  Furthermore we have

  $$X \backslash \pi^{-1}(0) \cong \mathbb{A}^n \backslash \{0\}$$

  Proof. Let $P = (a_1, ..., a_n)$ a point of $\mathbb{A}^n$ with $a_k \neq 0$ for some $1 \leq k \leq n$. Now

  $$\pi^{-1}(P) = \{(a_1, ..., a_n) \times [y_1, ..., y_n] : y_i \in k \text{ and } a_i y_j = a_j y_i \ \forall i, j = 1, ..., n\}$$

  Hence we have that $\forall j \ \ y_j = \frac{a_j}{a_k} y_k$. So $[y_1, ..., y_n] = \left[a_1 \frac{y_k}{a_k}, ..., a_n \frac{y_k}{a_k}\right] = [a_1, ..., a_n]$. It is a uniquely determined point of $\mathbb{P}^{n-1}$.

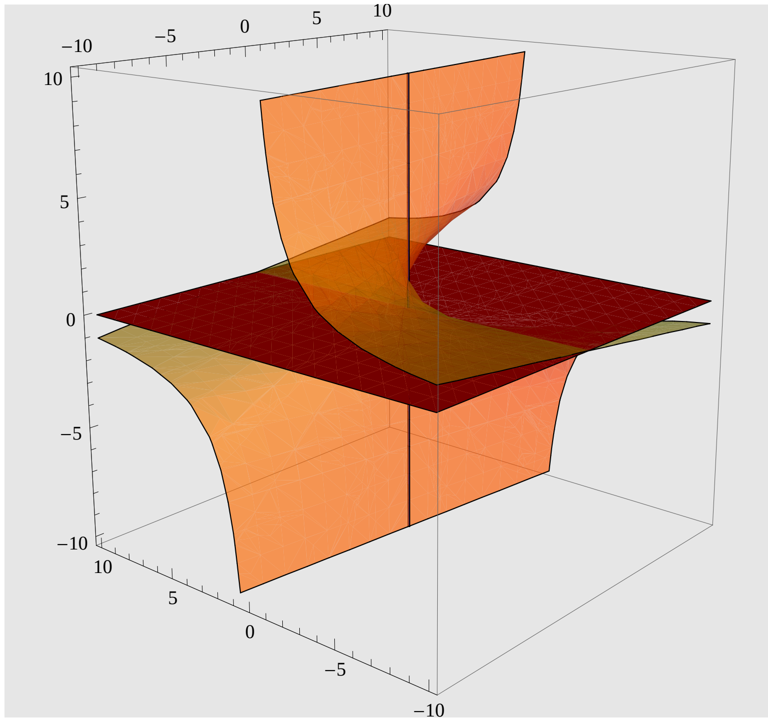

Figure 1: **Blow up of $\mathbb{A}^2$ (the red plane) at 0. Observe that $\pi^{-1}(0)$ is the blue line.**

We have shown that, for $P \in \mathbb{A}^n \backslash \{0\}$,

$$\pi^{-1}(P) = \{(a_1, ..., a_n) \times [a_1, ..., a_n]\}$$

that is a point of $\mathbb{A}^n \times \mathbb{P}^{n-1}$. $\square$

- The lines through 0 in $\mathbb{A}^n$ are in bijection with the points of $\pi^{-1}(0)$.

  Proof. Consider L a line through 0 in $\mathbb{A}^n$ with parametric equations $x_1 = l_1 t$, ..., $x_n = l_n t$, where $t \in \mathbb{A}^1$ and l_i not all zeroes.

  If $P = (a_1, ..., a_n) \in L\backslash\{0\}$, then, by definition, there is a $\tilde{t}$ such that $a_1 = l_1\tilde{t}$, ..., $a_n = l_n\tilde{t}$. Hence

$$\pi^{-1}(P) = \{(a_1, ..., a_n) \times [a_1, ..., a_n]\} = P \times \{[l_1, ..., l_n]\}$$

  and so

$$\pi^{-1}(L\backslash\{0\}) = (L\backslash\{0\}) \times \{[l_1, ..., l_n]\}$$

  It's clear that its closure in X only adds the point $0 \times [l_1, ..., l_n]$ of $\pi^{-1}(0)$. This gives our bijection. $\square$

- X is irreducible.

Proof. First we consider our blow-up as $X = (X\backslash\pi^{-1}(0)) \cup \pi^{-1}(0)$. Now $X\backslash\pi^{-1}(0) \cong \mathbb{A}^n\backslash\{0\}$ is irreducible, furthermore it is dense in X (because for the previous point its closure gives all X). Hence X is irreducible. □

0.2 Points in general position and plane cubic curves passing through them

Definition 2. *Six points of the projective plane are in* general position *if no three of them lie on a line and no six of them lie on a conic.*

We are now interested in all the cubic curves that lie in the projective plane $\mathbb{P}^2$.

We start from the space of homogeneous polynomials in three variables of fixed degree 3, that we will denote $k[x_0, x_1, x_2]_3$. It is a space of dimension 10 over k .

Indeed, giving a degree three monomial is the same as choosing three (possibly equal) elements from the three homogeneous coordinates x_0, x_1, x_2, without caring of the order. So, it is the number of combination of $n = 3$ elements in $k = 3$ positions, with repetitions permitted:

$$\binom{n+k-1}{k} = \binom{5}{3} = 10$$

We now fix six points in general position P_1, P_2, P_3, P_4, P_5, P_6.

Consider the subspace V of $k[x_0, x_1, x_2]_3$ defined by

$$V = \{f \in k[x_0, x_1, x_2]_3 : f(P_i) = 0 \quad \forall i = 1, ..., 6\}$$

that is the subspace of all cubic homogeneous equations that are satisfied by the points $P_1, P_2, P_3, P_4, P_5, P_6$. We first observe that, imposing on $k[x_0, x_1, x_2]_3$ the passage of a point make the dimension decrese of 1. The same holds imposing the passage of other points, if they are in general position with the previous ones. It follows that the dimension of V is $10 - 6 = 4$. We can then choose 4 generators of V over k, that we will call f_0, f_1, f_2, f_3. We can also observe that

$$Z(f_0, f_1, f_2, f_3) = \left\{p \in \mathbb{P}^2 : f_i(p) = 0 \quad \forall p = 0, ..., 3\right\} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

0.3 Main Construction

We will now use the previous observations to analyze the blow-up of $\mathbb{P}^2$ at these six points in general position. We start defining the following map:

$$\begin{array}{rcl} \mathbb{P}^2\backslash V(f_0, f_1, f_2, f_4) & \longrightarrow & \mathbb{P}^3 \\ P & \longmapsto & [f_0(P), f_1(P), f_2(P), f_3(P)] \end{array}$$

that takes a point $P \neq P_1, P_2, P_3, P_4, P_5, P_6$ and sends it into $[f_0(P), f_1(P), f_2(P), f_3(P)]$, that is a point in $\mathbb{P}^3$. If not, $f_i(P) = 0 \quad \forall i = 0, ...3$ and hence P belongs to $V(f_0, f_1, f_2, f_4)$. This implies that the map is well defined.

Consider the blow-up of the projective plane at $P_1, P_2, P_3, P_4, P_5, P_6$, $\pi : \widetilde{\mathbb{P}^2} \longrightarrow \mathbb{P}^2$. We recall that there is a bijection between the points of $\mathbb{P}^2 \backslash \{P_1, ..., P_6\}$ and the points of $\widetilde{\mathbb{P}^2} \backslash \{\pi^{-1}(P_1), ..., \pi^{-1}(P_6)\}$. In this way we can define a map $\widetilde{\mathbb{P}^2} \backslash \{\pi^{-1}(P_1), ..., \pi^{-1}(P_6)\} \to \mathbb{P}^3$, simply composing the two maps that we have. Furthermore $\widetilde{\mathbb{P}^2} \backslash \{\pi^{-1}(P_1), ..., \pi^{-1}(P_6)\}$ is an open dense in $\widetilde{\mathbb{P}^2}$. Hence we can take the closure of the image of that map, obtaining a closed immersion of $\widetilde{\mathbb{P}^2}$ into a surface of $\mathbb{P}^3$.

$$\begin{array}{ccc} \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^3 \\ \downarrow & \nearrow & \\ \widetilde{\mathbb{P}^2} & & \end{array}$$

This defines a closed embedding $\widetilde{\mathbb{P}^2} \longrightarrow \mathbb{P}^3$, whose image is a cubic surface.

0.4 We can obtain all the cubic surfaces in this way

Let k an algebraically closed field. The dimension of $k[x_0, x_1, x_2, x_3]_3$ (of the space of all the degree three monomials over k) is

$$\left(\begin{array}{c} 4+3-1 \\ 3 \end{array} \right) = \left(\begin{array}{c} 6 \\ 3 \end{array} \right) = 20$$

Hence the dimension of all the cubic surfaces in the projective space is $20 - 1 = 19$ (because if we multiply by a constant the equation defines the same surface, so we can fix one of the variables).

If we analyze now the family of cubic surfaces that we obtain with our construction, we see that

- choosing the six points brings 12 choices of their coordinates (two for each point)
- we have to subtract the dimension of the automorphisms of $\mathbb{P}^2$ and add the dimension of the automorphisms of $\mathbb{P}^3$, that are respectively $2^2 - 1$ and $3^2 - 1$, because we can change the coordinates of the projective space but without moving the projective plane where our points lies. This bringes $15 - 8 = 7$ other choices.

Hence we have obtained a family of cubic surfaces of dimension 19, that is equal to the dimension of all cubic surfaces in $\mathbb{P}^3$.

1 Examples

1.1 Choice of six points in general position — $k = \mathbb{C}$

We can take, for $k = \mathbb{C}$, these six points on $\mathbb{P}^2$:

$$[1,0,0],[0,1,0],[0,0,1],[1,1,1],[1,3,2],[1,4,3]$$

They are in general position Indeed clearly not three of them lies on a line. Furthermore the first five points lies and determine the plane conic $-4x_0x_1 + 3x_0x_2 + x_1x_2 = 0$, which does not contain $[1,4,3]$.

1.2 Finding the generators of space of plane cubics passing through the six points — $k = \mathbb{C}$

We will now find the pencil of cubic curve passing for our six points.

The passage of the first three points implies that we want a cubic like $ax_0^2x_1 + bx_0^2x_2 + cx_0x_1^2 + dx_0x_2^2 + ex_1^2x_2 + fx_1x_2^2 + gx_0x_1x_2$. Furthermore we have other three conditions corresponding to the passage of the remaining 3 points:

$$a + b + c + e + f + g = 0$$

$$3a + 2b + 9c + 4d + 18e + 12f + 6g = 0$$

$$4a + 3b + 16c + 9d + 48e + 36f + 12g = 0$$

We can solve the linear system, to obtain our pencil of cubic curves through the six points:

$$f[-16x_0^2x_1 + 18x_0^2x_2 + 4x_0x_1^2 - 6x_0x_2^2] + i[6x_0^2x_1 + 13x_0^2x_2 - 23x_0x_2^2 + 4x_1x_2^2] +$$

$$+h[-6x_0^2x_1 + 31x_0^2x_2 - 29x_0x_2^2 + 4x_1^2x_2] + l[-6x_0^2x_1 + 7x_0^2x_2 - 5x_0x_2^2 + 4x_0x_1x_2] = 0$$

Hence we can finally choose the generators of the four dimensional vector space generated by cubics through the six points as:

$$f_0 = x_0[-16x_0x_1 + 18x_0x_2 + 4x_1^2 - 6x_2^2]$$

$$f_1 = x_0[-6x_0x_1 + 7x_0x_2 - 5x_2^2 + 4x_1x_2]$$

$$f_2 = -6x_0^2x_1 + 31x_0^2x_2 - 29x_0x_2^2 + 4x_1^2x_2$$

$$f_3 = 6x_0^2x_1 + 13x_0^2x_2 - 23x_0x_2^2 + 4x_1x_2^2$$

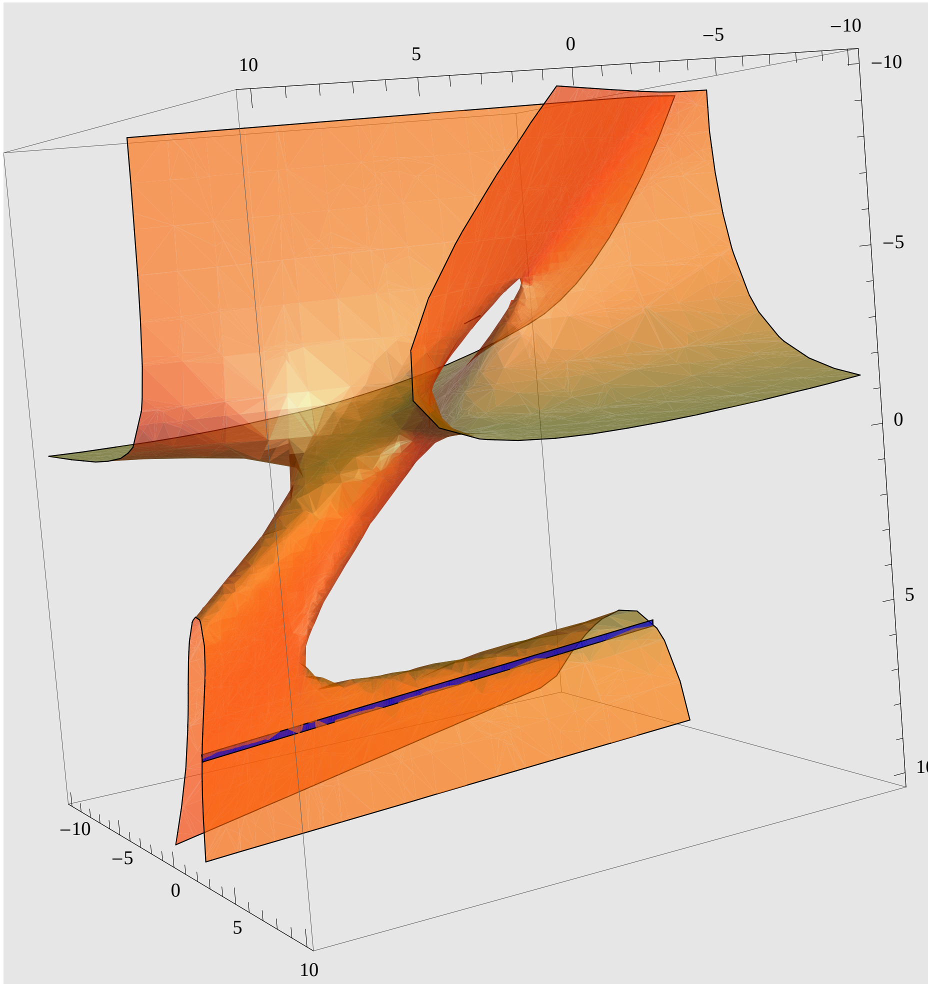

Figure 2: **The cubic surface** $70z_1^3 + 6z_0^2z_1 + 6z_0^2z_3 - 23z_0z_1^2 + 5z_0z_3^2 - 32z_1z_2 + 36z_1^2z_3 + 4z_1z_2^2 - 6z_1z_3^2 - 14z_0z_1z_3 - 4z_0z_2z_2 = 0$ **in the affine space with** $z_1 = 1$**. The blue line is the one obtained blowing up the conic that passes through** $P_1, ..., P_5$

1.3 The equation of the cubic surface — $k = \mathbb{C}$

We define the map

$$\begin{array}{rcl} \mathbb{P}^2 \backslash \{P_1, ..., P_6\} & \longrightarrow & \mathbb{P}^3 \\ P & \longmapsto & [f_0(P), f_1(P), f_2(P), f_3(P)] \end{array}$$

The image of this map stays on a cubic surface of $\mathbb{P}^3$, precisely stays on the cubic surface defined by the equation

$$70z_1^3 + 6z_0^2z_1 + 6z_0^2z_3 - 23z_0z_1^2 + 5z_0z_3^2 - 32z_1z_2 + 36z_1^2z_3 + 4z_1z_2^2 - 6z_1z_3^2 - 14z_0z_1z_3 - 4z_0z_2z_2 = 0$$

It is enought to substitute $z_0 = f_0(x_0, x_1, x_2), z_1 = f_1(x_0, x_1, x_2), z_2 = f_2(x_0, x_1, x_2), z_3 =$

$f_3(x_0, x_1, x_2)$ in the degree three equation of the surface and see that it is always verified for all x_0, x_1, x_2. It is a straightforward calculation.

Hence the blow-up of $\mathbb{P}^2$ at the six points is the closure of the image of this map that is precisely the cubic surface defined by

$$70z_1^3+6z_0^2z_1+6z_0^2z_3-23z_0z_1^2+5z_0z_3^2-32z_1z_2+36z_1^2z_3+4z_1z_2^2-6z_1z_3^2-14z_0z_1z_3-4z_0z_2z_2=0$$

1.4 Choice of six points in general position — k finite field

We will now see what happens if $k = \mathbb{F}_q$ is a finite field of order q. The projective plane over $\mathbb{F}_q$ is $\mathbb{P}^2(\mathbb{F}_q) = \{[z_0, z_1, z_2] : z_0, z_1, z_2 \in \mathbb{F}_q\}$

Observation 1. *For every point of the projective plane over $\mathbb{F}_q$ pass $q+1$ (distinct) lines. Every projective line in the plane has $q+1$ (distinct) points over $\mathbb{F}_q$.*

Proof. It's enought to see what happens in the origin in some choice of affine coordinates. The lines through 0 are the lines $x_2 = kx_1$ for $k \in \mathbb{F}_q$ plus the line $x_1 = 0$. Hence we have $q+1$ distinct projective lines. The other result follow by duality. □

With this and with the observation that every two points over $\mathbb{F}_q$ are joint by a line over $\mathbb{F}_q$, we can count the number of points of $\mathbb{P}^2(\mathbb{F}_q)$:

$$\#\mathbb{P}^2(\mathbb{F}_q) = (q+1)q + 1 = q^2 + q + 1$$

They are the number of lines through 0 (i.e. $q+1$) times the number of points of each line that are not 0 (i.e. $q = (q+1) - 1$), and 0 itself (1 point).

We now fix a certain number of points in general position and we want to count how many points of $\mathbb{P}^2(\mathbb{F}_q)$ lies on some line that joins two of the fixed points. We will count how many points are "taken"in this sense.

- if we fix **2** points, the $\mathbf{q+1}$ points on the line joining them are "taken".
- if we fix **3** points not on a line (i.e. if we add to the previous two points a new one that is not "taken") we have the previous $q+1$ points, then the new point, then the other points of the two lines that join the new point to the other two respectively and that are in number $(q+1) - 2 = q - 1$ each: we have $(q+1) + 2(q-1) + 1 = \mathbf{3q}$ points that are "taken".
- if we fix **4** points not three on a line, we have to add the points on the three lines that joins the new point to the previous ones. Each line has $q - 2 = (q+1) - 2 - 1$ new points, because we have to take out the two fixed point and one other point, that we get intersecting the considered line with the line that joins the other two fixed points: we have $3q + 3(q-2) + 1 = \mathbf{6q-5}$ points that are "taken". [obs. $q \geq 2$]

- if we fix **5** points not three on a line, we have to add the points on the three lines that joins the new point to the previous ones. Each line brings $q-4=(q+1)-2-3$ new points, because we have to take out the two fixed point and three other points, that we get intersecting the considered line with the other three lines that joins the other three fixed points: we have $(6q-5)+4(q-4)+1=\mathbf{10q}-\mathbf{20}$ points that are "taken". [obs. $q \geq 4$]

From this we can see that the only possibility to have 6 points in general position is that $q^2+q+1>10q-20$, in other words $\mathbf{q} \geq \mathbf{4}$.

We would like also that not six of them pass through a conic. This happens for $\mathbb{F}_4$ (we will see an example later), but it does not for $\mathbb{F}_5$.

For $\mathbb{P}^2(\mathbb{F}_5)$, if we allow that all the six points (not three of them on a line) belongs to a conic, we can see that the construction of the cubic surface done in the complex case also works (and the last considered point is the correct one). Doing this we obtain the cubic surface over $\mathbb{F}_5$

$$z_0^2z_1+z_0^2z_3+2z_0z_1^2+3z_1^2z_2+z_1^2z_3+4z_1z_2^2+4z_1z_3^2+z_0z_1z_3+z_0z_2z_3$$

That construction also works for finite fields with greater characteristic.

We will now consider $k=\mathbb{F}_4=\{0,1,\zeta,\bar{\zeta}\}$ where $1+\zeta+\bar{\zeta}=0$. We will take in $\mathbb{P}^2(\mathbb{F}_4)$ these points:

$$[1,0,0],[0,1,0],[0,0,1],[1,1,1],[1,\zeta,\bar{\zeta}],[1,\bar{\zeta},\zeta]$$

They are in general position. Indeed, if we consider the first five points, it's clear that no three of them lies on the same line. As seen before, there are 21 points in $\mathbb{P}^2(\mathbb{F}_4)$ and fixing 5 points in general position takes out 20. Hence there is only one possible point in general position, that is $[1,\bar{\zeta},\zeta]$. (Indeed, if we write it $[z_0,z_1,z_2]$, we have that: $z_0 \neq 0$ so we can choose $z_0=1$; $z_1 \neq 1$, $z_2 \neq 0$ and $z_1 \neq z_2$, hence it has to be $z_1=\bar{\zeta}$ and $z_2=\zeta$.) Finally the conic that passes through the first five points is $\zeta x_0x_1+\bar{\zeta}x_0x_2+x_1x_2$, and $\zeta\bar{\zeta}+\bar{\zeta}\zeta+\bar{\zeta}\zeta=1+1+1=1\neq 0$, hence this conic does not pass through the last point, so the six of them are in general position.

1.5 Finding the generators of space of plane cubics passing through the six points — $k=\mathbb{F}_4$

We will now find the pencil of cubic curve passing for our six points.

The passage of the first three points implies that we want a cubic like $ax_0^2x_1+bx_0^2x_2+cx_0x_1^2+dx_0x_2^2+ex_1^2x_2+fx_1x_2^2+gx_0x_1x_2$. Furthermore we have the other three conditions, dued imposing the passage of the other three points:

$$a+b+c+e+f+g=0$$

$$\zeta a + \bar{\zeta} b + \bar{\zeta} c + \zeta d + \zeta e + \bar{\zeta} f + g = 0$$
$$\bar{\zeta} a + \zeta b + \zeta c + \bar{\zeta} d + \bar{\zeta} e + \zeta f + g = 0$$

Solving this linear system, we can obtain the pencil of cubic curves through the six points:

$$f[x_0^2 x_2 + x_0 x_1^2] + g[x_0^2 x_1 + x_0 x_2^2] + h[x_0^2 x_1 + x_1^2 x_2] + i[x_0^2 x_2 + x_1 x_2^2] = 0$$

So we can choose the generators of the four dimensional vector space generated by plane cubics through the six points as:

$$f_0 = x_0[x_0 x_1 + x_2^2]$$
$$f_1 = x_0[x_0 x_2 + x_1^2]$$
$$f_2 = x_1[x_0^2 + x_1 x_2]$$
$$f_3 = x_2[x_0^2 + x_1 x_2]$$

1.6 The equation of the cubic surface — $k = \mathbb{F}_4$

We define the map

$$\begin{array}{rcl} \mathbb{P}^2(\mathbb{F}_4) \backslash \{P_1, ..., P_6\} & \longrightarrow & \mathbb{P}^3(\mathbb{F}_4) \\ P & \longmapsto & [f_0(P), f_1(P), f_2(P), f_3(P)] \end{array}$$

The image of this map stays on a cubic surface of $\mathbb{P}^3(\mathbb{F}_4)$, precisely stays on the cubic surface defined by the equation

$$z_0^3 + z_0^2 z_2 + z_1 z_3^2 = 0$$

It is enought to substitute $z_0 = f_0(x_0, x_1, x_2), z_1 = f_1(x_0, x_1, x_2), z_2 = f_2(x_0, x_1, x_2), z_3 = f_3(x_0, x_1, x_2)$ in the degree three equation of the surface and see that it is always verified for all x_0, x_1, x_2 in $\mathbb{F}_4$. It is a straightforward calculation.

Hence the blow-up of $\mathbb{P}^2(\mathbb{F}_4)$ at the six points is the closure of the image of this map, that is precisely the cubic surface defined by

$$z_0^3 + z_0^2 z_2 + z_1 z_3^2 = 0$$

1.7 Further observations

This construction is quite important also because it allows to prove some generical facts about cubic surfaces. For instance, in our examples, we can easily find some of the lines that are on our cubic surfaces, as the blow-up of the lines joining two of $P_1, ..., P_6$ and the blow-up of the conics that pass through five of them. [The only lines remaining are the exceptional curves (blow-up of the points $P_1, ..., P_6$).]

Here are some examples of lines that lies in our cubic surface over the complex field:
The blow-up of the line through P_2, P_3.

$$(0, a, b) \longmapsto [0, 0, 4a^2b, 4ab^2] = [0, 0, a, b]$$

The blow-up of the conic through P_1, P_2, P_3, P_4, P_5.
Suppose $-4x_0x_1 + 3x_0x_2 + x_1x_2 = 0$. Then

$$(x_0, x_1, x_2) \longmapsto [6a + b, 5a, 29a + 4b, 35a]$$

$$\text{where we have defined} \quad a = 2x_0x_1 - x_0x_2 - x_2^2, b = 4x_1^2 - 28x_0x_1 + 24x_0x_2$$