Jinbi Jin

Construction of some modular curves.

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Definition The modular group $SL_2(\mathbb{Z})$ is ~~the~~ $\{\gamma \in M_2(\mathbb{Z}) : \det \gamma = 1\}$

It is a group under multiplication.

Definition. The principal congruence subgroup $\Gamma(N)$ of level N ($N \in \mathbb{Z}_{>0}$).
is $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$.

($\Gamma(N)$ is $\ker(SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z}/N\mathbb{Z}))$)

Definition. A congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is a subgroup that contains a $\Gamma(N)$ for some N.

Let $SL_2(\mathbb{Z})$ act on $\{$lattices + oriented basis in $\mathbb{C}\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \sigma \\ \tau \end{pmatrix} = \begin{pmatrix} a\sigma + b\tau \\ c\sigma + d\tau \end{pmatrix}$$

We also have the action by $\mathbb{C}^\times$:

$$\left(\lambda, \begin{pmatrix} \sigma \\ \tau \end{pmatrix}\right) \mapsto \begin{pmatrix} \lambda\sigma \\ \lambda\tau \end{pmatrix}.$$

This gives an action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$, the upper half plane ($\{$latt + pos. basis$\}/\mathbb{C}^\times$: $(\sigma, \tau) \mapsto \frac{\tau}{\sigma} \in \mathbb{H}$.

$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tau = \frac{a\tau + b}{c\tau + d} \right)$

Def. For Γ a congruence subgroup $Y(\Gamma) := \Gamma \backslash \mathbb{H}$, for the moment as a topological space.

$\Gamma_0(N)$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ mod N. $\quad Y_0(N)$

$\Gamma_1(N)$: $\equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ mod N $\quad Y_1(N)$

$\Gamma(N)$ $\quad Y(N)$.

$Y(1) \longleftrightarrow \{$elliptic curves $/\mathbb{C}\}/\cong$

the same!

$\mathbb{C}/\Lambda$ up to $\underset{\mathbb{C}^\times}{\underline{\text{scaling}}}$.

$$Y_1(N) \xleftrightarrow{\sim} \left\{(E,P) : \begin{array}{l} E \text{ ell. curve}/\mathbb{C}, \\ P \in E \text{ order } N \end{array}\right\} / \cong$$

Let us describe this map.

Let (E,P) be given, $\tau \in \mathbb{H}$, $E = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) = \mathbb{C}/\Lambda_\tau$,

$P = \frac{c\tau + d}{N}$, gcd $\mathbb{Z}c + \mathbb{Z}d + \mathbb{Z}N = \mathbb{Z}$,

then $\exists$ $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ (replace c, d in their classes mod N).

Let $\tau' = \gamma\tau$ $\quad m = c\tau + d$,

then $m\Lambda_{\tau'} = m \cdot (\mathbb{Z}\tau' + \mathbb{Z}) = \mathbb{Z}(a\tau + b) + \mathbb{Z}(c\tau + d) = \Lambda_\tau$.

and $m \cdot \left(\frac{1}{N} + \Lambda_{\tau'}\right) = \frac{c\tau + d}{N} + \Lambda_\tau = P$.

So: $(E,P) \cong \left(\mathbb{C}/\Lambda_{\tau'}, \left[\frac{1}{N}\right]\right)$. $\quad \exists$ isom. $E_\tau \to E_{\tau'}$ s.t. $P_\tau \mapsto P_{\tau'}$.

And: $(E_\tau, P_\tau) \cong (E_{\tau'}, P_{\tau'}) \iff \Gamma_1(N)\tau = \Gamma_1(N)\cdot\tau'$.

<u>Lemma</u>. Let $N \in \mathbb{Z}_{\geq 4}$. Then every pair (E,P) is uniquely ~~uniquely~~ isomorphic to some $(E_{s,t}, (0,0))$ with

$$E_{s,t}: \quad y^2 + s \cdot xy + ty = x^3 + tx^2, \qquad s \in \mathbb{C},\ t \in \mathbb{C}^\times.$$

<u>Facts</u> $\exists$ $F \in \mathbb{C}[s,t]$ s.t. ((0,0) in $E_{s,t}$ has order N $\iff$ $F(s,t) = 0$).

Next time: determination of F for $N \in \{4,5,6\}$.

A fundamental domain for $SL_2(\mathbb{Z})$-action on $\mathbb{H}$: