Jinbi Jin, Construction of some modular curves, II.

From last time:

For $N \in \mathbb{Z}_{\geq 1}$, $\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$

$$\mathbb{H} \xrightarrow{\text{quotient}} Y_1(N)$$

Lemma 1. We have a bijection: $Y_1(N) \rightleftarrows \left\{ (E,P) : \begin{array}{l} E/\mathbb{C} \text{ ell. curve} \\ P \in E \text{ order } N \end{array} \right\}/\cong$

Lemma 2. Let $N \in \mathbb{Z}_{\geq 4}$. Then every pair (E,P) is uniquely isomorphic to ~~some~~ a unique $(E_{s,t}, (0,0))$, with:

$E_{s,t}$ given by $y^2 + sxy + ty = x^3 + tx^2$. $s \in \mathbb{C}$, $t \in \mathbb{C}^\times$ and $(\Delta(s,t) \in \mathbb{C}^\times)$

Proof. Any isomorphism is of the following form:

$(x,y) \longmapsto (\alpha^2 x + a, \alpha^3 y + bx + c)$, $\alpha \in \mathbb{C}^\times$, $a,b,c \in \mathbb{C}$.

(here E is also given by a Weierstrass equation).

General Weierstrass equation by which E can be given:

$E: y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$.

By a unique translation, we can make $P = (0,0)$. Then $\boxed{a_6 = 0}$. We used that $N \neq 1$.

Now note that $N \neq 2$. So the line $x = 0$ is not the tangent of E at $P = (0,0)$. So $a_3 \neq 0$.

By applying a unique map of the form $(x,y) \mapsto (x, y + bx)$ we can map the tangent at P to the line $y = 0$. Then $\boxed{a_4 = 0}$

Now we use that $N \neq 3$: the line $y + a_1 x + a_3$ is not a flex, hence $a_2 \neq 0$. So there is a unique $\alpha \in \mathbb{C}^\times$ s.t. after $(x,y) \mapsto (\alpha^2 x, \alpha^3 y)$ we have $\boxed{a_3 = a_2}$. $\boxtimes$.

Now we must determine for which (s,t) the point $(0,0)$ on $E_{s,t}$ is of order N.

Fact. $\exists\, F \in \mathbb{C}[S,T]$ s.t. $(0,0)$ in $E_{s,t}$ is of order N if and only if $F(s,t)=0$.

We will compute this F for some values of N.

N=4. Assume $(0,0)$ on $E_{s,t}$ has order 4. Then $\exists\, f$ rational function on E with $\mathrm{div}(f) = 4\cdot[P] - 4\cdot[0]$.

Then, considering the order at 0 of ~~elements of~~ the $x^i y^j$, $i \in \mathbb{N}$, $j \in \{0,1\}$, gives that $f = x^2 + \alpha y + \beta x + \gamma$ for certain α, β, γ in $\mathbb{C}$. As ~~$\&$~~ $v_P(f) = 4$, $\gamma = 0$ and $\beta = 0$.

We get $f = x^2 + \alpha y$, and see that $\alpha \in \mathbb{C}^\times$.

We substitute $y = -\alpha^{-1}x^2$ in the equation for E; that gives: $0 = \alpha^{-2}x^4 - (1+\alpha^{-1}s)x^3 - (t+\alpha^{-1}t)x^2$,

hence: $1+\alpha^{-1}s = 0$, $t\cdot(1+\alpha^{-1}) = 0$. Hence $\alpha = -1$, and $s = 1$.

So $F_4 = s-1$.

N=5. $\exists\, f$ with $\mathrm{div}(f) = 5\cdot[P] - 5\cdot[0]$, $f = xy + \alpha x^2 + \beta y$, $\alpha\beta \in \mathbb{C}^\times$.

We work with the ideal in $\mathbb{C}[x,y]$ generated by f and the W-eq. of E, and want to see what it means that $(0,0)$ has mult. 5.

We can express y in x, using f: $y = \dfrac{-\alpha x^2}{x+\beta}$, note that as we are interested in what happens at $(0,0)$ we can divide by $x+\beta$. Substitute $y = \dfrac{-\alpha x^2}{x+\beta}$ in the eq'n for $E_{s,t}$, and asking that it has order 5 at $x=0$, gives:

N=6. $F = s^2 - 3s + 2 + t$