Maarten Derickx. 2011/02/07. 16:50.

Quotients by finite groups and symmetric products of varieties.

Example 1. X = an elliptic curve E over a number field K,
$G = \langle T \rangle$, the group generated by $T \in E(K)$ of finite order.
$G \times X \to X$, $(T, P) \mapsto T + P$. a K-rational point

Example 2. $X = \mathbb{C}^2$, $G = \mathbb{Z}^\times = \{1, -1\}$.
$G \times X \to X$, $(-1, (a,b)) \mapsto (-a, -b)$.

(without the '-1' it is a right-action)

Example 3. $X = \mathbb{C}^d$, $G = S_d$
$G \times X \to X$, $(\sigma, (a_1, \dots, a_d)) \mapsto (a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(d)})$.

If A is an affine variety and G a finite group acting on A then G acts on the coordinate ring $\mathcal{O}(A)$.
(well, from the right): $G \times \mathcal{O}(A) \to \mathcal{O}(A)$
$$(g, f) \mapsto f \circ g\cdot \qquad \left(\begin{array}{l} g\cdot : A \to A \\ \quad a \mapsto g \cdot a \end{array} \right)$$

Def. In this case, we define the quotient $G\backslash A$ of A by the G-action as the variety with coordinate ring $\mathcal{O}(A)^G = \{ f \in \mathcal{O}(A) : \forall g \in G,\ f \circ g\cdot = f \}$.

(Def. If X is a G-set (set with G-action), then $X^G = \{ x \in X : \forall g \in G,\ g \cdot x = x \}$, the set of fixed points.)

Remark: in example 1, we could consider the field of $K(E)$ rational functions on E; that has a G-action, and the quotient $E/\langle T \rangle$ is the nonsingular projective curve with function field $(K(E))^G$.

Example 2. $\mathcal{O}(A) = \mathbb{C}[x,y]$

$$(-1)\,.\,x^i y^j = (-1)^{i+j} x^i y^j$$

So $\mathbb{C}[x,y]^G = \bigoplus_{d \geq 0} \mathbb{C}[x,y]_{2d}$, only monomials of even degree.

Claim: $\mathcal{O}(A)^G$ is generated as $\mathbb{C}$-algebra by x^2, xy, y^2.

(for example: $x^4 = (x^2)^2$, $x^3y = x^2 \cdot xy$, $x^2y^2 = x^2y^2$, ---)

So: $\mathbb{C}[u,v,w] \xrightarrow{\pi} \mathcal{O}(A)^G$, $u \mapsto x^2$, $v \mapsto y^2$, $w \mapsto xy$

Then $\mathcal{O}(A)^G = \mathbb{C}[u,v,w]/\ker(\pi)$

Claim: $\ker\pi = (uv - w^2)$.

Proof. Let $f \in \ker\pi$, do div. with remainder by $w^2 - uv$ in $\mathbb{C}[u,v][w]$: $f = g + wh + i(w^2-uv)$, $g,h \in \mathbb{C}[u,v]$

then look where $g+wh$ goes under π --- $\boxtimes$

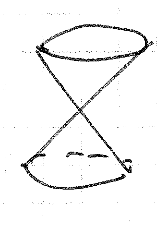
$(\pi(g+wh) = g(x^2,y^2) + xy \cdot h(x^2,y^2)$, so $g=0,\ h=0)$.

Conclusion: $G\backslash A = Z(uv - w^2) \subset \mathbb{C}^3$

$A \to G\backslash A$, $(a,b) \longmapsto (a^2, b^2, ab)$ is the quotient map.

It is a quotient map in the category of sets, but now we have it as a variety over $\mathbb{C}$.

Observe $G\backslash A$ is singular at 0:
This is the image of $0 \in A$, and that is the only point in A whose stabiliser in G is not $\{1\}$, that is, the point fixed by -1.



$z^2 = x^2 + y^2$
$\cong$
$w^2 = uv$
by linear coord. changes:
$$\begin{cases} u = x + iy \\ v = x - iy. \end{cases}$$

Example 3. $\mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_d]$ ← from algebra course: symmetric polynomials.

$\mathcal{O}(X)^G = \mathbb{C}[s_0, \dots, s_{d-1}]$

where $\prod_i (T - x_i) = T^d + \sum_{i=0}^{d-1} s_i \cdot T^i$ in $\mathbb{C}[x_1, \dots, x_d][T]$.

So we have the quotient map:

$$\varphi : \mathbb{C}^d \longrightarrow \mathbb{C}^d$$
$$a = (a_1, \dots, a_d) \longmapsto (s_0(a), \dots, s_{d-1}(a))$$

This is an example where the group action has non-trivial stabilisers but still the quotient is non-singular.

We will use this later, as follows.

Let $\mathbb{Q} \to K$ be a field extension of degree d, $g_1, \dots, g_d$ the embeddings $K \to \mathbb{C}$, and $\alpha \in K$,

then $\varphi(g_1(\alpha), \dots, g_d(\alpha))$ = the point in $\mathbb{Q}^d$ whose ~~coeff.~~ coordinates are the coefficients of the char. polynomial of $\alpha \cdot : K \to K$ as vect. sp. of dim d / $\mathbb{Q}$.

Def. For X an affine variety, we let $X^{(d)}$ be the quotient of X^d by S_d; it is called the "symmetric ~~pow~~ dth power" of X.

More on this next time.