Symmetric products of varieties

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Outline

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Quotients of varietes by finite groups

X is a variety, with ring of regular functions R. G is a group acting on X hence also on R.

- If X is affine then X/G corresponds to R^G
- Can also be done for projective varieties.
  - take an affine cover $(A_i)_{i \in I}$ of X such that G acts on the A_i.
  - X/G is covered by A_i/G
  - if X/G smooth curve: just take the subfield of invariant rational functions.

Definition (Symmetric Product)

The d-th symmetric product of X is $X^{(d)} := X^d/S_d$.

Outline

1. About the Previous Talk
   - Recap of Previous talk
   - Is the Symmetric Product Smooth?
2. Prerequisite Knowledge
   - Abelian Varieties
   - Jacobians
3. Symmetric Products

The awnser is NO for general symmetric products

Example: $X = \mathbb{C}^2$ is smooth, $X^{(2)}$ is not smooth.
Proof:

- Write the action of S_2 on $X^2 = \mathbb{C}^4$ w.r.t. $(1,0,1,0),(0,1,0,1),(1,0,-1,0),(0,1,0,-1)$
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  - $\sigma(1,0,1,0) = (1,0,1,0)$
  - $\sigma(0,1,0,1) = (0,1,0,1)$
  - $\sigma(1,0,-1,0) = -(1,0,-1,0)$
  - $\sigma(0,1,0,-1) = -(0,1,0,1)$
- σ acts like $(Id,-1)$ where -1 is as in example 2 of last time.
- conclusion: $X^{(d)} \cong \mathbb{C}^2 \times Cone$ hence singular.

...However it is true for smooth projective irreducible curves.

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Abelian Varieties

A generalization of elliptic curves

Definition (Abelian Variety)

It is a connected and projective variety X with a group law such that:

- $+ : X \times X \to X$
- $- : X \to X$

Are given by morphims of varieties, i.e., locally by regular functions.

Properties of Abelian Varieties

Let A be an abelian variety defined over a number field K.

- Mordell-Weil holds: $A(K)$ is finitely generated.
- $A(\mathbb{C}) \cong \mathbb{C}^n/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}^n$
- Not all $\mathbb{C}^n/\Lambda$ are A.V. since not all $\mathbb{C}^n/\Lambda$ embed into projective space.

Outline

Divisors

Formal sums of points

Let X be a smooth projective curve over k, with function field K.

Definition (Weil Divisors)

They are of the form $\sum_{p \in X} a_p p$ with $a_p \in \mathbb{Z}$ and for almost all $p \in X$: $a_p = 0$.

- Set of all divisors: $\operatorname{div} X := \bigoplus_{p \in X} \mathbb{Z}$
- Degree of a divisor: $\deg \sum_{p \in X} a_p p := \sum_{p \in X} a_p \deg p$, with $\deg p$ the degree of the residue field at p over k
- $\sum_{p \in X} a_p p$ is effective if $\forall p \in X : a_p \geq 0$

Principal Divisor

Definition (Divisor of a Function)

For $f \in K^\times$: $\operatorname{div} f := \sum_{p \in X} v_p(f)p$. Where v_p is the valuation at p.

Fact: $\deg \operatorname{div} f = 0$

Jacobians of Curves

The degree zero divisors in the Picard Group.

Let X be a smooth projective curve, with function field K.

Definition (Picard Group of a Curve)

- $\mathrm{Pic}(X) := \operatorname{coker} \operatorname{div}$
- $K^\times \overset{\operatorname{div}}{\to} \operatorname{div} X \to \mathrm{Pic}(X) \to 0$

Definition (Jacobian Variety of a Curve)

- $J(X) = \mathrm{Pic}^0(X)$ i.e the classes of degree zero divisors
- $0 \to J(X) \to \mathrm{Pic}(X) \overset{\deg}{\to} \mathbb{Z}$

Divisors and Symmetric Products

Effective divisors correspond to points on symmetric products

Let $K \subseteq L$ be number fields and X/K a curve then: And let $\operatorname{div}^{d,+} X$ be set the effective divisors of degree d

$$
\begin{array}{lcl}
\operatorname{div}^{d,+} X(\overline{K}) & \stackrel{1:1}{\longleftrightarrow} & X^{(d)}(\overline{K}) \\
p_1 + \ldots + p_d & \longleftrightarrow & [(p_1, \ldots, p_d)] \\
D \text{ is defined over } L & \Longleftrightarrow & p \in X^{(d)}(L)
\end{array}
$$

Symmetric Products map to Jacobians

Every $D \in \operatorname{div} X$ with $\deg D = d$ defines a map.

$$\begin{aligned} \phi_D : \quad X^{(d)} &\to J(X) \\ [(p_1, \ldots, p_d)] &\mapsto [p_1] + \ldots + [p_d] - D \end{aligned}$$

Summary

Where is all this good for

Let $\mathbb{Q} \overset{d}{\subseteq} L$ be a field extension. And $X/\mathbb{Q}$ a curve.

- Instead of studying $X(L)$ for all L of degree d study $X^{(d)}(\mathbb{Q})$
- Study $X^{(d)}(\mathbb{Q})$, in a point p by studying $\phi_p : X^{(d)} \to J(X)$

- Outlook
  - Application to torsion points on elliptic curves over L
  - Find better bounds for $S(5)$.