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The geometry of a finite group acting on a ring

1. Introduction

In these short notes we will discuss Proposition 9.1.1 from [1] in more detail. The proposition is as follows:

Proposition 1.1. *Let $X = \mathrm{Spec}(A)$ be an affine scheme with an action by a finite group G. Then the morphism $\pi : X \to Y := \mathrm{Spec}(A^G)$ is a quotient in the category of locally ringed spaces.*

Assume in the rest of these notes that A is a ring and that a finite group G acts on this ring.

2. Integral extensions

We know that A is integral over A^G (indeed, $a \in A$ is a zero of $\prod_{g \in G}(X - g(a))$). Hence we study such an integral extensions of rings $B \subseteq A$ first. The following lemma gives us a construction of quotients in the category of locally ringed spaces.

Lemma 2.1. *Let $B \subseteq A$ be integral and assume that B is a domain. Then B is a field if and only if A is a field.*

Proof. Suppose that B is a field. Let $a \in A$ with a relation $a^n + b_{n-1}a^{n-1} + \ldots + b_0 = 0$ where $b_i \in B$ of minimal degree. As B is a domain, it follows that $b_0 \neq 0$ and $a^{-1} = -\frac{a^{n-1} + b_{n-1}a^{n-2} + b_1}{b_0}$.

Suppose that A is a field. Let $b \in B$. Then $b^{-1} \in A$ and suppose that $b^{-n} + b_{n-1}b^{1-n} + \ldots + b_0 = 0$ where $b_i \in B$. Then $b^{-1} = -b_{n-1} + \ldots + b_0 b^{n-1} \in B$. This shows that B is a field. □

Corollary 2.2. *Let $\varphi : B \to A$ be an integral morphism of rings. Then the induced map $\psi : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is closed and if φ is injective, then ψ is surjective.*

Proof. Let $Z = Z(I) \subseteq \mathrm{Spec}(A)$ be closed. Let $J = \varphi^{-1}(I)$. Notice that $Z = \mathrm{Spec}(A/I)$. Replacing B by B/J and A by A/I we may assume that $Z = \mathrm{Spec}(A)$ and φ is injective. Hence it is enough to show that $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is surjective. Let $\mathfrak{p} \in \mathrm{Spec}(B)$. Consider the inclusion $B_\mathfrak{p} \subseteq A_\mathfrak{p}$. Now take a maximal ideal of $A_\mathfrak{p}$, say $\mathfrak{m}$. Then we get an integral extension $B_\mathfrak{p}/(\mathfrak{m} \cap B_\mathfrak{p}) \to A_\mathfrak{p}/\mathfrak{m}$. As $A_\mathfrak{p}/\mathfrak{m}$ is maximal, it follow from Lemma 2.1 that $\mathfrak{m} \cap B_\mathfrak{p}$ is maximal as well, that is, $\mathfrak{m} \cap B_\mathfrak{p} = \mathfrak{p}B_\mathfrak{p}$. This finishes the proof. □

For the surjectivity, it is needed that the map φ is injective. Otherwise take a finite non-local ring and divide out by one of its maximal ideals to find a counter example.

3. Group actions on locally ringed spaces

Let G be a finite group and let A be a ring. A group action of G on A is a morphism from G to $\mathrm{Aut}(A)$. Denote the automorphism corresponding to $g \in G$ just by g. We now define $A^G = \{a \in A : \forall g \in G : g(a) = a\}$, the G-invariants of A. Let $X := \mathrm{Spec}(A)$ and let $Y := \mathrm{Spec}(A^G)$. Then G acts on the right on X, namely for $\mathfrak{p} \in X$ we set $\mathfrak{p}r(g) := g^{-1}(\mathfrak{p})$. It induces a right G-action on X. Now consider the morphism $\pi : X \to Y$ (corresponding to the inclusion $A^G \to A$, it maps $\mathfrak{p} \in X$ to $\mathfrak{p} \cap A^G \in Y$). We have $\pi \circ r(g) = \pi$, our map is invariant under G. We have the following lemma.

Lemma 3.1. *The map $\pi : X \to Y$ is the quotient for the action of G in the category of affine schemes: every G-invariant morphism $f : X \to Z$ with Z affine factors uniquely through π.*

Proof. First use the anti-equivalence of categories between the category of affine schemes and rings. The result now follows from the obvious statement: any ring morphism $\varphi : R \to A$ such that for all $g \in G$ we have $g\varphi = \varphi$ factors uniquely through A^G. □

Actually, as we will show below, $\pi : X \to Y$ is the quotient in the category of locally ringed spaces. We first have the following lemma.

Lemma 3.2. *Let X' be a locally ringed space and let a finite group G' act on it (on the right). Let $Y' = X'/G'$ as sets, let $\pi : X \to Y$ be the quotient map. For $U \subseteq Y$ define $\mathcal{O}_{Y'}(U) := \mathcal{O}_{X'}(\pi^{-1}U)^{G'}$. Then π is the quotient for the G'-action in the category of locally ringed spaces.*

Proof. First we will show that G acts naturally on $\mathcal{O}_{X'}(\pi^{-1}U)$ for $U \subseteq Y$ open. Any $g \in G'$ induces an automorphism $X' \to X'$ which for any $V \subset X'$ open induces a map $\mathcal{O}_{X'}(V) \to \mathcal{O}_{X'}(g^{-1}(V))$. Now notice that a set of the form $\pi^{-1}U$ is G'-invariant, that is, we have the induced action as claimed.

One can easily show that $(Y', \mathcal{O}_{Y'})$ is a ringed space and that it is the quotient in the category of ringed spaces (by construction we have the unique morphisms as claimed).

We now claim that $(Y', \mathcal{O}_{Y'})$ is a locally ringed space. Let $y' \in Y'$ with preimage $x' \in X$. Then we have a natural map $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$. This induces a natural map $\psi : \mathcal{O}_{Y',y'} \to k(y')$. We claim that its kernel, which is a prime ideal, is a maximal ideal. Suppose that $\psi(f) \neq 0$. This means that this element has an inverse in the stalk of x'. But this means that it has an inverse in the stalks at all points of $G'x'$. But this means that we have an inverse in $\mathcal{O}_{Y',y'}$ (here we use that inverses are unique)

Now one can check that the universal property of ringed spaces gives a morphism of locally ringed spaces, which explicitly follows from our construction. □

4. The proof

Proposition 4.1. *Let $X = \mathrm{Spec}(A)$ be an affine scheme with an action by a finite group G. Then the morphism $\pi : X \to Y := \mathrm{Spec}(A^G)$ is a quotient in the category of locally ringed spaces.*

Proof. We have already noticed that $B = A^G \subseteq A$ is integral. We will first show that $\pi : X \to Y$ is the set-theoretical quotient map, that is, the map induces a bijection between the G-orbits of X and Y. By Corollary 2.2 we know that π is surjective. The fibers of π are G-stable, so it remains to show that each fiber consists of exactly one G-orbit. Let $y = \mathfrak{p} \in Y$ be a prime of B. We have an inclusion $B_\mathfrak{p} \to A_\mathfrak{p}$ and G acts here naturally. We want to look only at the primes lying above $\mathfrak{p}$, hence we consider $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ which is integral over $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = k(y)$ (we have already shown that this is a nonzero ring). It also follows from Lemma 2.1 that any prime ideal lying above $\mathfrak{p}B_\mathfrak{p}$ in $A_\mathfrak{p}$ is maximal.

We will now show that $(A_\mathfrak{p})^G = B_\mathfrak{p}$. First let $b \in B$ and consider the natural inclusion map $B_b \to A_b$ (by exactness of localization), which factors through a map $B_b \to (A_b)^G$. We need to show that the map is surjective. Let $\frac{a}{b^m} \in (A_b)^G$. Then for $g \in G$ we have $\frac{ga}{b^m} = \frac{a}{b^m}$ which means that there exists an n such that $b^n(ga - a) = 0$, and as G is finite, we may assume this holds for all g. But then $b^n a \in A^G$ (notice that b is fixed under G) and hence $\frac{b^n a}{b^{n+m}} \mapsto \frac{a}{b^m}$. We then have

$$B_\mathfrak{p} = \varinjlim_{b \in B\setminus\mathfrak{p}} B_b = \varinjlim_{b \in B\setminus\mathfrak{p}} (A_b)^G = \left(\varinjlim_{b \in B\setminus\mathfrak{p}} A_b \right)^G = (A_\mathfrak{p})^G$$

Suppose that we have two distinct orbits x_1G and x_2G of primes lying over $\mathfrak{p}$. By the Chinese remainder theorem the map $A_\mathfrak{p} \to \prod_\sigma k(x_1\sigma) \times \prod_\sigma k(x_2\sigma)$ is surjective. Pick an element $f \in A_\mathfrak{p}$ which has image 1 in $k(x_1\sigma)$ and 0 in $k(x_2\sigma)$ (for all $\sigma \in G$). Then $f' = \prod_\sigma \sigma(f)$ has the same property and lies in $B_\mathfrak{p}$. But this is impossible: 0 and 1 are different in $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$. Hence our map is the set theoretical quotient map.

We will now show that Y has the quotient topology. First of all, the map π is continuous. By Corollary 2.2 it follows that π is closed. This shows that Y has the quotient topology.

Now we have to show that the morphism $\mathcal{O}_Y \to (\pi_*\mathcal{O}_X)^G$ is an isomorphism (it exists by the universal property of the quotient). Since both are sheaves, it suffices to verify that all $D(b)$ with $b \in B$ have the same section. On the left we have B_b. On the right we have $(A_b)^G$. We have already seen that both are equal by the natural map and hence we are done. □

References

[1] B. Edixhoven, `http://www.math.leidenuniv.nl/~edix/public_html_rennes/cours/dea9596.ps`