Locally ringed spaces and manifolds

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In this lecture we shall define the category of Manifolds $\mathscr{M}_{at}$ giving a k-differential atlas on a topological space X and the category of Manifolds $\mathscr{M}_{Sh}$ as a subcategory of the category of locally ringed spaces and then we shall prove that $\mathscr{M}_{at}$ and $\mathscr{M}_{Sh}$ are actually isomorphic.
We shall reserve the symbol k to denote an integer $k \geq 0$ or ∞.

Definition 0.1. Let X be a topological space. A k-differentiable atlas A_X^k on X is a family $\{(U_i, h_i, X_i, n_i)\}_{i \in I}$ of quadruples, where I is a set and for each $i \in I$ X_i is an open subset of X, $n_i \geq 0$ is an integer, U_i is an open subset of $\mathbb{R}^{n_i}$ and $h_i : U_i \to X_i$ is a homeomorphism such that $X = \cup_i X_i$, and for i and j in I, if we use use the notations: $X_{ij} := X_i \cap X_j =: X_{ji}$, the restrictions $h_i : U_{ij} \to X_{ij}$ and $h_j : U_{ji} \to X_{ji}$ are compatible in the sense that the homeomorphism $h_j^{-1} \circ h_i : U_{ij} \to U_{ji}$ is k-differentiable.

Whenever required for the sake of clarity, we shall denote an atlas whose data is given using the symbols above by the symbol $A_X^k(I, U, h, n)$, but when there is no risk of confusion we shall continue using A_X^k.

A pair (X, A_X^k) of a set X and a k-differentiable atlas A_X^k on X we call a topological space with a k-differentiable atlas. A morphism, or k-differentiable map $\phi : (X, A_X^k(I, U, h, n)) \to (Y, A_Y^k(J, V, g, m))$ of such pairs is a continuous map $\phi : X \to Y$ such that for each $(i, j) \in I \times J$,

$$g_j^{-1} \circ \phi \circ h_i : h_i^{-1}(\phi^{-1}(Y_j) \cap X_i) \to V_j$$

is a k-differentiable map.

Lemma 0.2. *If $\phi : (X, A_X^k) \to (Y, A_Y^k)$ and $\psi : (Y, A_Y^k) \to (Z, A_Z^k)$ are morphisms of topological spaces with k-differentiable atlases on them, $\psi \circ \phi : (X, A_X^k) \to (Z, A_Z^k)$ is also a morphism of the same. Also, for every (X, A_X^k), $1_X : (X, A_X^k) \to (X, A_X^k)$ is a morphism.*

The Lemma above proves that the topological pairs, defined above, together with their morphisms form a category. Lets name it *category of topological spaces with a k-differentiable atlas*.

Lemma 0.3. *Let $A^k_X(I, U, h, n)$ and $A^k_X(J, U, h, n)$ are two of the k differentiable atlases on X, the relation $\sim$ given by*

$$A^k_X(I, U, h, n) \sim A^k_X(J, U, h, n) \Leftrightarrow A^k_X(I \sqcup J, U, h, n) \text{ is also a k-differentiable atlas}$$

is an equivalence relation. More over, the equivalence class of A^k_X is partially ordered under the relation $\subset$ and has a unique maximal element still denoted by A^k_X.

We are ready to define what is called a k-differentiable manifold.

Definition 0.4.

- Let (X, A^k_X) be in the category of topological spaces with k-differentiable atlases and let A^k_X is a maximal atlas on X. A maximal atlas on X we call a *k-differentiable structure on X* and we define *k-differentiable manifold* to be a pair (X, A^k_X) of a topological space with a k-differentiable structure on it.

- A morphism $\phi : (X, A^k_X) \to (Y, A^k_Y)$ of k-differentiable manifolds is a morphism in the the category of topological spaces with k-differentiable atlas.

Definition 0.5. Let $A^k_X(I, U, h, n)$ be a k-differentiable manifold and let $X' \subset X$ open in X. A map $f : X' \to \mathbb{R}$ is said to be a k-differentiable map if for every $i \in I$ the map $f \circ h_i : h_i^{-1}(X_i \cap X') \to \mathbb{R}$ is a k differentiable.

Definition 0.6. Let $U \subset \mathbb{R}^n$ open in $\mathbb{R}^n$. The sheaf of k-differentiable functions C^k_U on U is the sheaf defined

$$C^k_U(V) = \{ f : V \to \mathbb{R} \mid f \text{ is a } k\text{-differentiable map}\}$$

for every $V \subset U$ open in U. For $W \subset V \subset U$ open in U the restriction maps are given by the usual restrictions functions; that is $\rho^V_W(f) = f\big|_W$ for every $f \in C^k_U(V)$.

Definition 0.7. We define a category $\mathscr{P}$ whose objects are the pairs $(X, \mathcal{F}_X)$ where X is a topological space and $\mathcal{F}_X$ is a sheaf of $\mathbb{R}$-algebras on X which is a subsheaf of sheaf of $\mathbb{R}$-valued functions.

A morphism $(f, f^*) : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ in $\mathscr{P}$ consists of a continuous map f and the morphism $f^* : \mathcal{F}_Y \to f_*\mathcal{F}_X$ of sheaves of $\mathbb{R}$-algebras on Y induced by f and is defined by

$$f^*_V : \mathcal{F}_Y(V) \to f_*\mathcal{F}_X(V), \quad s \mapsto s \circ f.$$

Identity of an object $(X, \mathcal{F}_X)$ is $(1_X, 1^*_X)$ is given by continuous map $1_X : X \to X$ and the composition $(f, f^*) \circ (g, g^*) = (f \circ g, (f \circ g)^*)$.

Definition 0.8. A k- differentiable manifold is an object $(M, C^k_M) \in \mathscr{P}$ such that for every $p \in M$ there exists an open subset M_p of M, an integer n_p, an open subset U_p of $\mathbb{R}^{n_p}$ and an isomorphism $(h_p, h^*_p) : (U_p, C^k_{U_p}) \to (M_p, C^k_M\big|_{M_p})$ in $\mathscr{P}$.
A morphism of k- differentiable manifolds $(f, f^*) : (M, C^k_M) \to (M', C^k_{M'})$ is a morphism in $\mathscr{P}$.

So we have another, I must say much better, way of defining a manifold. To distinguish is from $\mathscr{M}_{at}$ we denote it by $\mathscr{M}_{Sh}$. Now we shall show that two categories are isomorphic.

Theorem 0.9. *The categories $\mathscr{M}_{at}$ and $\mathscr{M}_{Sh}$ are isomorphic.*

Proof. Let $(X, A^k_X(I, U, h, n))$ in $\mathscr{M}_{at}$. Consider the sheaf of functions on $\mathcal{A}^k_X$ on X defined as follows:

$$\mathcal{A}^k(X') = \{ f : X' \to \mathbb{R} \mid f \text{ is a } k\text{-differentiable map}\}$$

for every $X' \subset X$ open in X and restrictions being usual restrictions of functions. $\mathcal{A}^k_X$ is clearly a sheaf and we shall show that $(X, \mathcal{A}^k_X)$ is an object of $\mathscr{M}_{Sh}$. To show that, the required data is available from the data of $A^k_X(I, U, h, n)$. All we need to show that for an arbitrary $i \in I$

$$(h_i, h^*_i) : (U_i, C^k_{U_i}) \to (X_i, \mathcal{A}^k_X\big|_{X_i}) \text{ is an isomorphism in } \mathscr{P};$$

that is, we need to prove that for arbitrary $X' \subset X_i$ open,

$$h^*_{iX'} : \mathcal{A}^k_X\big|_{X_i}(X') \to h_{i*}C^k_{U_i}(X'), \quad f \mapsto f \circ h_i$$

is a well defined isomorphism. X' open in X_i and X_i open in X gives X' open in X which further gives $\mathcal{A}^k_X\big|_{X_i}(X') = \mathcal{A}^k_X(X')$. By definition $f \in \mathcal{A}^k_X(X')$ means that $f \circ h_i \in C^k_{U_i}(h^{-1}_i(X')) = h_{i*}C^k_{U_i}(X')$; that is, $h^*_{i\,X'}$ well defined. The morphism of k algebras $h^*_{iX'}$ clearly an injection since h_i is a homeomorphism and it is a surjection as $g \in h_{i*}C^k_{U_i}(X')$ is the image of $g \circ h^{-1}_i$ which clearly is in $\mathcal{A}^k_X(X')$. So we have a map

$$E : \mathrm{Ob}(\mathscr{M}_{at}) \to \mathrm{Ob}(\mathscr{M}_{Sh}), \quad (X, A^k_X) \mapsto (X, \mathcal{A}^k_X).$$

Now let $\phi : (X, A^k_X(I, U, h, n)) \to (Y, A^k_Y(J, V, g, m))$ is a morphism in $\mathscr{M}_{at}$. We shall prove that $(\phi, \phi^*) : (X, \mathcal{A}^k_X) \to (Y, \mathcal{A}^k_Y)$ is well defined morphism in $\mathscr{P}$. ϕ is surely continuous so we just need to show that $\phi^* : \mathcal{A}^k_Y \to \phi_*\mathcal{A}^k_X$ is well defined. that is, we need to show that for arbitrary open subset $Y' \subset Y$ the morphism $\phi^*_{Y'} : \mathcal{A}^k_Y(Y') \to \phi_*\mathcal{A}^k_X(Y')$ which maps $f \mapsto f \circ \phi$ is well defined. For the coming statements please keep in mind that, for $A \subset B$ and a map $l : B \to \mathbb{R}$ when we, abusing the notation, write $l : A \to \mathbb{R}$ we actually mean $l|_A : A \to B$. Now observe the following:

$$\begin{aligned} f \in \mathcal{A}^k_Y(Y') &\Leftrightarrow \forall j \in J \quad f \circ g_j : g_j^{-1}(Y_j \cap Y') \to \mathbb{R} \text{ is differentiable} \\ &\Rightarrow \forall (i,j) \in I \times J \quad f \circ g_j \circ g_j^{-1} \circ \phi \circ h_i : h_i^{-1}(X_i \cap \phi^{-1}(Y_j \cap Y')) \to \mathbb{R} \\ &\qquad \text{is differentiable since } g_j^{-1} \circ \phi \circ h_i \text{ is differentiable, as is } \phi \text{ given} \\ &\Leftrightarrow \forall (i,j) \in I \times J \quad f \circ \phi \circ h_i : h_i^{-1}(X_i \cap \phi^{-1}(Y_j \cap Y')) \to \mathbb{R} \\ &\qquad \text{is differentiable} \\ &\Leftrightarrow \forall i \in I \quad f \circ \phi \circ h_i : h_i^{-1}(X_i \cap \phi^{-1}(Y')) \to \mathbb{R} \text{ is differentiable} \\ &\Leftrightarrow f \circ \phi \in \mathcal{A}^k_Y(\phi^{-1}(Y')) = \phi_* \mathcal{A}^k_X(Y'). \end{aligned}$$

In the equivalences above we proved that if a continuous map $\phi : X \to Y$ of topological space induces a morphism $\phi : (X, A^k_X) \to (Y, A^k_Y)$ in $\mathscr{M}_{at}$ then it also induces a morphism $(\phi, \phi^*) : (X, \mathcal{A}^k_X) \to (Y, \mathcal{A}^k_Y)$ in $\mathscr{M}_{Sh}$. Thus we have a map

$$E : \mathrm{Hom}_{\mathscr{M}_{at}}((X, A^k_X), (Y, A^k_Y) \to \mathrm{Hom}_{\mathscr{M}_{Sh}}((X, \mathcal{A}^k_X), (Y, \mathcal{A}^k_Y)), \qquad \phi \mapsto (\phi, \phi^*)$$

$E(\phi) \circ E(\psi) = (\phi, \phi^*) \circ (\psi, \psi^*) = (\phi \circ \psi, (\phi \circ \psi)^*) = E(\phi \circ \psi)$ and $E(1_X) = (1_X, 1^*_X)$. So we have a functor $E : \mathscr{M}_{at} \to \mathscr{M}_{Sh}$. It remains to prove that it is a it is an isomorphism. Lets first prove that the functor is full. Choose an arbitrary $i \in I$ and $j \in J$ we shall prove that $g_j^{-1} \circ \phi \circ h_i : h_i^{-1}(X_i \cap \phi^{-1}(Y_j)) \to V_j$ is differentiable as required for ϕ to be a morphism of morphism in $\mathscr{M}_{at}$. Let, for integers $1 \leq r \leq m_j$, $\pi_r : V_j \to \mathbb{R}$ denote the projection on r-th coordinate. $\pi_r \circ g_j^{-1} \in \mathcal{A}^k_Y(Y_j)$ which implies $\phi^*_{Y_j}(\pi_r \circ g_j^{-1}) = \pi_r \circ g_j^{-1} \circ \phi \quad \in \quad \phi_* \mathcal{A}_X(Y_j) = \mathcal{A}_X(\phi^{-1}(Y_j))$. By definition this means that $\pi_r \circ g_j^{-1} \circ \phi \circ h_i : h_i^{-1}(X_i \cap \phi^{-1}(Y_j)) \to \mathbb{R}$ is differentiable. Since that holds for every r, we have $g_j^{-1} \circ \phi \circ h_i : h_i^{-1}(X_i \cap \phi^{-1}(Y_j)) \to \mathbb{R}^{m_j}$ is differentiable, and clearly we can restrict the codomain toV_j to get the required. It is clear that E is also faithful; so we have that E is fully faithful. To prove that E is an Isomorphism of categories we also need to prove that

$$E : \mathrm{Ob}(\mathscr{M}_{at}) \to \mathrm{Ob}(\mathscr{M}_{Sh}), \quad (X, A^k_X) \mapsto (X, \mathcal{A}^k_X).$$

is a bijection. To prove that it is injective lets assume that $E(A^k_X(I, U, h, n)) = E(A^k_X(J, U, h, n))$. Which means that $1_X \in \mathrm{Hom}_{\mathscr{M}_{at}}(A^k_X(I, U, h, n), A^k_X(J, U, h, n))$ But we have already discusses that it means $A^k_X(I \sqcup J, U, h, n)$ is also an atlas and that contradicts the maximality of $A^k_X(I, U, h, n)$ and $A^k_X(I, U, h, n)$ unless they are same.

□