Assignment AAG Fall 2011.

December 8, 2011

1. Let $(X, \mathcal{O})$ be in (LRS).
   (a) Give a bijection between the set of open and closed subsets U of X, and the set of idempotents in $\mathcal{O}(X)$.
   (b) Does this also work in (RS)?
   (c) Let A be a ring. Give a bijection between the set of open and closed subsets U of $\mathrm{Spec}(A)$, and the set of idempotents in A.
2. Let (X, C_X^∞) be a real smooth manifold as in Lecture 1. Let k be in $\mathbb{Z}_{\geq 0}$.
   (a) Give a definition of the sheaf C_X^k of functions on opens of X that are k times differentiable with continuous derivatives.
   (b) The inclusion $C_X^\infty \to C_X^k$ gives a morphism in $(\mathrm{LRS}_\mathbb{R})$ from (X, C_X^k) to (X, C_X^∞) that we denote by f. Let $\mathcal{E}$ be a locally free C_X^∞-module, not necessarily of finite rank. Show that the natural morphism of C_X^∞-modules $\phi\colon \mathcal{E} \to f_*f^*\mathcal{E}$ is injective.
   (c) What is the maximal open subset of X on which ϕ is an isomorphism?
   (d) Give an example of an X and an $\mathcal{E}$ such that X is connected and $\mathcal{E}$ is not free.
3. Let k be a field. We know that the category with objects pairs (V, ϕ) with V a k-vector space and ϕ in $\mathrm{End}_k(V)$ and with morphisms k-linear maps $f\colon V \to V'$ such that $\phi' \circ f = f \circ \phi$ is isomorphic to the category of $k[x]$-modules: the multiplication by x is ϕ. This is particularly useful for finite dimensional V, because we have a classification of $k[x]$-modules that are finite dimensional as k-vector spaces and we know the morphisms between such. But now we also know that $k[x]$-mod is equivalent to $\mathrm{QCoh}(\mathbb{A}_k^1)$.
   (a) Let V be a finite dimensional k-vector space and let ϕ be in $\mathrm{End}_k(V)$. Let $\mathcal{F}$ be the quasi-coherent $\mathcal{O}$-module on $\mathbb{A}_k^1$ given by (V, ϕ). Describe the stalks of $\mathcal{F}$. Give some advantages of looking at (V, ϕ) and at $k[x]$-modules in this more geometric way. For example, is it now easier to see when $\mathrm{Hom}((V, \phi), (V', \phi'))$ is zero?
   (b) Let V be a finite dimensional k-vector space and ϕ and ψ be two commuting endomorphisms. Then view V as a $k[x, y]$-module and describe the stalks of the quasi-coherent $\mathcal{O}$-module on $\mathbb{A}_k^2$ that corresponds to (V, ϕ, ψ).

4. Give an example of an $\mathcal{O}$-module $\mathcal{F}$ on a scheme X such that $\mathcal{F}$ is not quasi-coherent. Try to make the underlying set of X as small as possible.

5. Let k be field. Let X be the closed subscheme of $\mathbb{A}^3_k$ given by the equation $xy - z^2 = 0$. Let 0 denote the closed point given by the equations $x = 0$, $y = 0$ and $z = 0$. Let $U = X - \{0\}$.

   (a) Show that X is integral, and that U is smooth over k. This implies that U is regular (you do not need to prove that).

   (b) Prove that $\mathcal{O}_X(U) = \mathcal{O}_X(X)$. Hint: use the cover $U = D(x) \cup D(y)$.

   (c) Let f be in $\mathcal{O}_X(X)$. Show that $\mathcal{O}_X(D(f) - \{0\}) = \mathcal{O}_X(D(f))$. Hint: use the cover of $D(f) - \{0\}$ by $D(xf)$ and $D(yf)$, write the usual complex for $\mathcal{O}_X(D(f) - \{0\})$, and note that it is the localisation by f for the complex for $\mathcal{O}_X(U) = \mathcal{O}_X(X)$.

   (d) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, such that $\mathcal{L}|_U$ is free. Show that $\mathcal{L}$ is free. Hint: let $s \in \mathcal{L}(U)$ be a generator. Observe that $\mathcal{L}$ is free in a neigborhood of 0 and use the previous part. Conclude that $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is injective.

   (e) Example II.6.5.2 of [H] says that $\mathrm{Pic}(U) = \mathrm{Cl}(U) = \mathrm{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$, generated by the ideal generated by y and z (recall from Lecture 11 that $\mathrm{Pic}(U) = \mathrm{Cl}(U)$ because U is locally factorial). Use the same arguments to show that $\mathrm{Pic}(U) = \mathrm{Pic}(V)$, where $V = \mathrm{Spec}(O_{X,0}) - \{0\}$.

   (f) Show that $\mathrm{Pic}(X) \to \mathrm{Pic}(U) \to \mathrm{Pic}(V)$ is zero. Conclude that $\mathrm{Pic}(X)$ is zero.

6. Let n be in $\mathbb{Z}_{\geq 1}$. Let O_n be the covariant functor from (Ring) to (Grp) sending A to the subgroup of $\mathrm{GL}_n(A)$ consisting of the g with $g^t{\cdot}g = 1$.

   (a) Show that O_n (as functor from (Ring) to (Set)) is representable by a ring H. Let now O_n denote the affine scheme $\mathrm{Spec}(H)$ that represents this functor (compare the discussion of GL_n in Lecture 5).

   (b) Let M_n be the scheme such that for any ring A the set $\mathrm{M}_n(A)$ is the set of n by n matrices with coefficients in A. Let M_n^+ be the scheme such that $\mathrm{M}_n^+(A)$ is the set of g in $\mathrm{M}_n(A)$ such that $g^t = g$. Show that M_n is isomorphic to $\mathbb{A}^{n^2}$ and M_n^+ is isomorphic to $\mathbb{A}^{n(n+1)/2}$.

   (c) Let $f\colon \mathrm{GL}_n \to \mathrm{M}_n^+$ be given by $g \mapsto g^t{\cdot}g$ (for every A, for every g in $\mathrm{GL}_n(A)$). Show that $\mathrm{O}_n \to \mathrm{Spec}(\mathbb{Z})$ is the pullback of f via the point 1 in $\mathrm{M}_n^+(\mathbb{Z})$.

   (d) Show that the restriction of f over $\mathbb{Z}[1/2]$ is smooth, using the criterion of formal smoothness and local finite presentation (Lecture 12). Here are a few hints. Let $I \hookrightarrow A \twoheadrightarrow \overline{A}$ as in the definition of formally smooth, h in $\mathrm{M}_n^+(A)$ with image $\overline{h}$ in $\mathrm{M}_n^+(\overline{A})$, and $\overline{g}$ in $\mathrm{GL}_n(\overline{A})$ such that $\overline{g}^t{\cdot}\overline{g} = \overline{h}$. Let g_0 be any element in $\mathrm{GL}_n(A)$ with image $\overline{g}$ in $\mathrm{GL}_n(\overline{A})$ (show that it exists!). Show that $g_0^t{\cdot}g_0 = h + a =$ for some a in $\mathrm{M}_n(A)$ with all coefficients in I, and that $a^t = a$. Then consider all possible g in $\mathrm{GL}_n(A)$ with image $\overline{g}$: they are the $g = g_0 + g_0 b = g_0(1+b)$ with b in $\mathrm{M}_n(A)$ with all coefficients in I. Then compute, using that $(hb)^t = b^t h$, that you may divide by 2, and that you may try to find a solution with hb symmetric.

(e) Show that the restriction to $\mathrm{Spec}(\mathbb{Z}[1/2])$ of the $\mathbb{Z}$-scheme O_n is smooth over $\mathrm{Spec}(\mathbb{Z}[1/2])$. (Prove first that smoothness is preserved under base change.)

(f) Show that the fibre over $\mathbb{F}_2$ of O_n is not smooth.

(g) On the positive side, one can show (you don't have to) that the relative codimension over $\mathbb{Z}$ of O_n in GL_n is $(n^2 - n)/2$ and that therefore O_n is a complete intersection over $\mathbb{Z}$.

7. (a) Let n and r be in $\mathbb{Z}_{\geq 1}$ and let k be a field. Let $\mathcal{E} = \oplus_{i=1}^r \mathcal{O}(d_i)$ and $\mathcal{E}' = \oplus_{i=1}^r \mathcal{O}(d_i')$ be two direct sums of r invertible $\mathcal{O}$-modules on $\mathbb{P}^n_k$. Show that they are isomorphic if and only if the sequences d and d' are equal up to permutation.

(b) Recall from Lecture 15 that on any ringed space $(X, \mathcal{O})$ and any $\mathcal{O}_X$-module $\mathcal{F}$ and for any $i \in \mathbb{Z}_{\geq 0}$ we have $\mathrm{Ext}^i(\mathcal{O}, \mathcal{F}) = \mathrm{H}^i(X, \mathcal{F})$. Recall also that $\mathrm{H}^1(\mathbb{P}^1_k, \mathcal{O}(-2))$ is of dimension 1 as k-vector space. Now give explicitly a short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{O}(-2) \to \mathcal{E} \to \mathcal{O} \to 0$ on $\mathbb{P}^1_k$ that is not split.

(c) Can you relate this short exact sequence to the closed immersion $0\colon \mathrm{Spec}(k) \to \mathbb{A}^2_k$?