What is geometry ??

Lecture 1, 2011/09/08. 1a.

	kind	Functions/equations
∃ many kinds:	topology	continuous
examples:	differential top./geometry	differentiable
	analytic geometry $\mathbb{C}, \mathbb{R}$	analytic
	algebraic geometry	rational/polynomial

$\rightsquigarrow$ geometric objects, as manifolds, varieties and their morphisms

Common framework: ringed spaces, locally ringed spaces.

Example: Let X be a C^∞-manifold, defined ~~as~~ in a way you like.
Then $\forall\, U \subset X$ open we have $C^\infty_X(U) = \{f : U \to \mathbb{R} : f \text{ is } C^\infty\}$, an $\mathbb{R}$-sub-alg. of ~~a~~ $\{f : U \to \mathbb{R}\}$. And $\forall\, V \subset U$ incl. of opens we have $C^\infty_X(U) \to C^\infty_X(V)$, $f \mapsto f|_V$.

Finished with: RSp, LRSp were already defined. (Sch too, given local model.)

A a ring. $U \subset \operatorname{Spec} A$ open.

$$\mathcal{O}(U) = \left\{ f : U \to \coprod_{p \in A} A_p \;:\; \begin{array}{l} 1.\ \forall p \in U,\ f(p) \in A_p \\ 2.\ \forall p \in U,\ \exists\, V \subset U,\ g, h \in A \text{ s.t.} \\ \quad \text{(a) } \forall q \in V,\ h(q) \neq 0 \text{ in } \kappa(q) \\ \quad \text{(b) } \forall q \in V,\ f(q) = \psi_q(h)^{-1} \cdot \psi_q(g) \text{ in } A_q \end{array} \quad A \xrightarrow{\psi_q} A_q \right\}$$

End of week 1

Promised to do: $\mathcal{O}_p = A_p$, $\mathcal{O}(\operatorname{Spec} A) = A$

Nullstellensatz: $\forall\, a \subset A$ ideal, $I(V(a)) = \sqrt{a}$.

Proof of ↗. $I(V(a)) = \{ f \in A : \forall p \in \operatorname{Spec} A \text{ s.t. } p \supset a,\ f \in p\}$

$$= \bigcap_{p \supset a} p = q^{-1} \bigcap_{p \in \operatorname{Spec}(A/a)} p \qquad (q : A \twoheadrightarrow \bar{A} = A/a)$$

$q^{-1}b \longleftarrow b \subset \bar{A}$ ideal; ideals $\supset a$ ⟷ $q c$.

Let A be a ring. Then $\bigcap_{\substack{p \subset A \\ \text{prime}}} p = \sqrt{A}$.

Proof. $\supset$ clear. Now $\subset$. Let $f \in \bigcap_{p \text{ prime}} p$.

Claim: $A_f = 0$. Proof. Assume not, let $m \subset A_f$ max. ideal. Gives contradiction. ⊠ $A_f = 0 \Rightarrow f$ nilpotent. ⊠

$\boxed{A \xrightarrow{\Psi} A_f}$ $\boxed{\text{Study } \operatorname{Spec}(\Psi) \text{ first.}}$