

Register with mastermath, also PhD-students, also if you do not want/need credit points (why? well, we want to prove that such courses are useful).

Proof of "Nullstellensatz for rings": \forall ring A , \forall ideal $a \subset A$, $I(V(a)) = \sqrt{a}$.

Recall: $V(a) = \{p \in \text{Spec } A : \forall f \in a, f(p) = 0 \text{ in } k(p) = \text{Frac}(A/p)\}$
 $= \{p \in \text{Spec } A : a \subset p\}$
 $I(V(a)) = \{f \in A : \forall p \in V(a), \overbrace{f(p) = 0 \text{ in } k(p)}^{f \in p}\} = \bigcap_{p \supset a} p$

Consider $q: A \rightarrow A/a =: \bar{A}$, $\{ \text{ideals } b \supset a \} \xrightarrow[q^{-1}]{q} \{ \text{ideals } \bar{b} \supset a \}$
 So: $\bigcap_{p \supset a} p = q^{-1}(\bigcap_{\bar{p} \in \text{Spec}(\bar{A}/a)} \bar{p})$
 inverses, prime ideals \downarrow prime ideals

This reduces the problem to the case $a=0$, which we now assume.

Let $f \in \bigcap_{p \in \text{Spec } A} p$. Consider $\psi: A \rightarrow A_f = \{1, f, f^2, \dots\}^{-1}A$

Claim: $A_f = 0$. Proof Assume not. Let $m \subset A_f$ be a maximal ideal, then $\psi^{-1}m \subset A$ is a prime ideal and $f \notin \psi^{-1}m$ bec. $\psi f \notin m$. That contradicts $f \in p$ for all p . Hence $A_f = 0$, f is nilpotent \square .

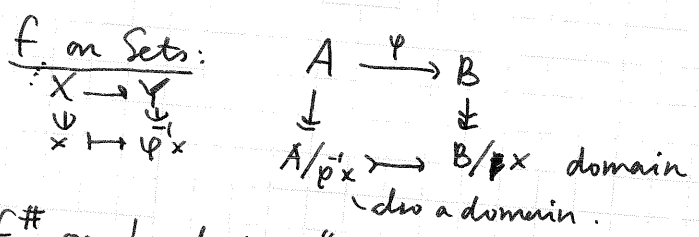
Lemma (p. 70-71 of Hartshorne). For $f \in A$, $D(f) := \{x \in \text{Spec } A : f(x) \neq 0 \text{ in } k(x)\}$

Then the $D(f)$ form a basis for the topology of $\text{Spec } A$. Proof They are open. Let $U \subset \text{Spec } A$ be open, $x \in U$. Put $a := I(\text{Spec } A - U)$.

Then $x \notin V(a)$, so $\exists f \in a$ with $f(x) \neq 0$. Take such an f . Then $x \in D(f) \subset U$. \square

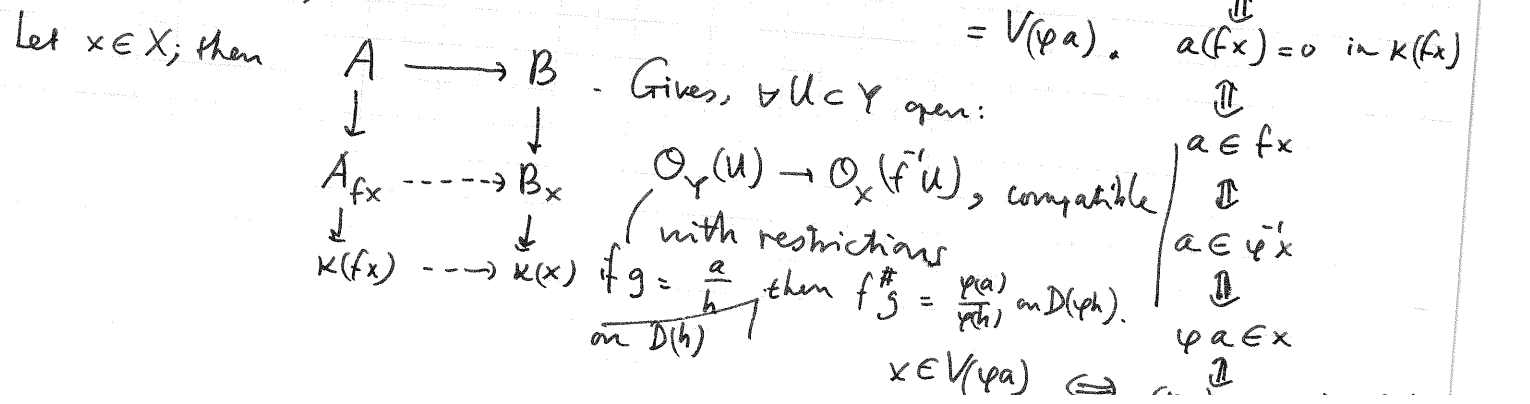
Remark: $D(f) \cap D(g) = D(fg)$; $\text{Spec } A$ is quasi-compact

Def. (functoriality of Spec). $A \xrightarrow{\psi} B \xrightarrow{\text{Spec}} (\text{Spec } A, \mathcal{O}_Y) \xleftarrow{(f, f^\#)} (\text{Spec } B, \mathcal{O}_X)$



f continuous: for $a \in A$, $f^{-1}V(a) = \{x \in X : f(x) \in V(a)\} = V(\psi a)$. $a(fx) = 0$ in $k(fx)$

" $f^\#$ on local rings"

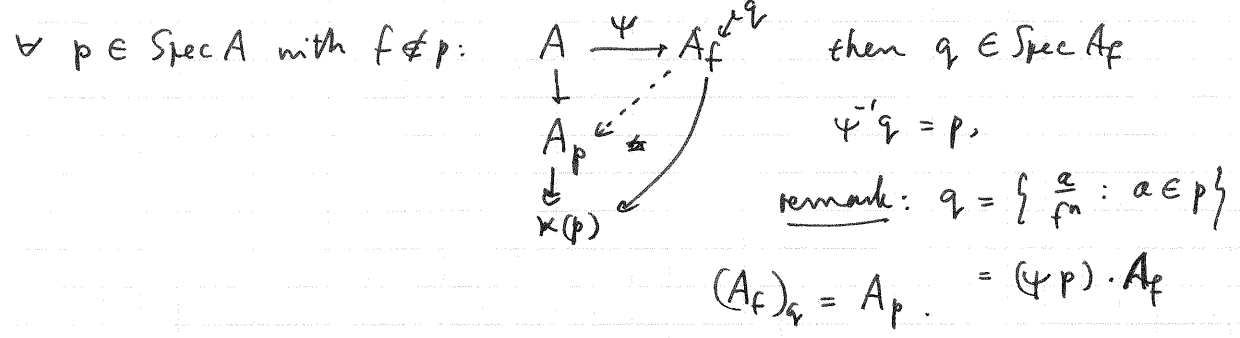


Example 1. $\varphi: A \twoheadrightarrow B$ (surjective). Then $\text{Spec } A \xleftarrow{\text{Spec } \varphi} \text{Spec } B$ is a closed immersion: ① $f: X \rightarrow Y$ is closed in Y , f injective, induced top. from Y is that of X
 ② $\forall x \in X: \mathcal{O}_{Y,fx} \rightarrow \mathcal{O}_{X,x}$ is surjective

Example 2. A a ring, $f \in A$, $\varphi: A \rightarrow A_f$.

Then $\text{Spec } \varphi$ is an isomorphism $(\text{Spec } A_f, \mathcal{O}) \rightarrow (D(f), \mathcal{O}|_{D(f)})$.

Here's the reason: $\forall p \in \text{Spec } A_f: \varphi^{-1}p \in \text{Spec } A$ and $f \notin p$ hence $f \notin \varphi^{-1}p$.



Prop. (2.2. of Hartshorne). $\mathcal{O}(\text{Spec } A) = A$.

We have $A \rightarrow \mathcal{O}(\text{Spec } A) \rightarrow \prod_{p \in \text{Spec } A} A_p$ hence $A \rightarrow \mathcal{O}(\text{Spec } A)$ is injective.

~~$(A \subset \mathcal{O}(\text{Spec } A)) = \{ f \in A : \exists g \in A, \exists p \text{ such that } f = \frac{g}{p} \}$~~
 (let $f \in A, f \neq 0$, then let $a := \ker(f: A \rightarrow A)$, then $a \subsetneq A$ ideal. take $a \subset m \subset A$ maximal. then $f/1 \in A_m$ is not zero. ...) So it remains to show surjectiveness.

Let $f \in \mathcal{O}(\text{Spec } A)$. For $p \in \text{Spec } A \exists a, h$ in $A, f = \frac{a}{h}$ on $D(h)$. ($\frac{a}{h} = \frac{ah'}{hh'}$)

Get: I finite set, a_i, h_i s.t. $\text{Spec } A = \bigcup_{i \in I} D(h_i), f = \frac{a_i}{h_i}$ on $D(h_i)$.

Then $\forall i, j \in I: \frac{a_i}{h_i} = \frac{a_j}{h_j}$ in $\mathcal{O}(D(h_i h_j))$ hence in $A_{h_i h_j}$ (injectivity!), hence $\exists n$ s.t. $(h_i h_j)^n (h_j a_i - h_i a_j) = 0$ in A . Take one n for all i, j .

Note: $\frac{a_i}{h_i} = \frac{a_i h_i^{n+1}}{h_i^{n+1}}$, so we may (and do!) assume that $\forall i, j: h_j a_i - h_i a_j = 0$.

Consider the complex $A \xrightarrow{s'} \prod_{i \in I} A \xrightarrow{s''} \prod_{i, j \in I} A$ We want an $a \in A$ s.t. $\forall i: a_i = h_i a$ in A .

"homotopy" \swarrow $A \xrightarrow{s'} \prod_{i \in I} A \xrightarrow{s''} \prod_{i, j \in I} A$

Note: $\sum_i A h_i = A$. Take $b_i \in A$ s.t. $\sum_i b_i h_i = 1$ in A .

Put $s'((a_i)_{i \in I}) := \sum_i b_i a_i$. Then $h_i a = \sum_j h_i b_j a_j = \sum_j b_j h_j a_i = a_i \quad \square$.