__Week 2__ Register with mastermath, also PhD-students, also if you do not want/need credit points (why? well, we want to prove that such courses are useful).

__Proof of "Nullstellensatz for rings"__: $\forall$ ring A, $\forall$ ideal $a \subset A$, $I(V(a)) = \sqrt{a}$

Recall: $V(a) = \{p \in \operatorname{Spec} A : \forall f \in a,\ f(p) = 0 \text{ in } \kappa(p) = \operatorname{Frac}(A/p)\}$

$= \{p \in \operatorname{Spec} A : a \subset p\}$

$I(V(a)) = \{f \in A : \forall p \in V(a),\ \overbrace{f(p) = 0 \text{ in } \kappa(p)}^{f \in p}\} = \bigcap p.$

Consider $q : A \twoheadrightarrow A/a =: \bar{A}$, $\{\text{ideals } b \supset a\} \underset{q^{-1}}{\overset{q}{\rightleftarrows}} \{\text{ideals } \bar{b} \text{ in } \bar{A}\}$ inverses, prime ideals $\Updownarrow$ prime ideals

So: $\bigcap_{p \supset a} p = q^{-1}\Big(\bigcap_{\bar{p} \in \operatorname{Spec}(\bar{A})} \bar{p}\Big).$

This reduces the problem to the case $a = 0$, which we now assume.

Let $f \in \bigcap_{p \in \operatorname{Spec} A} p$. Consider $\varphi : A \to A_f = \{1, f, f^2, \ldots\}^{-1} A$

__Claim__: $A_f = 0$. __Proof__ Assume not. Let $m \subset A_f$ be a maximal ideal, then $\varphi^{-1} m \subset A$ is a prime ideal and $f \notin \varphi^{-1} m$ bec. $\varphi f \notin m$.

That contradicts $f \in p$ for __all__ p. Hence $A_f = 0$, f is nilpotent $\boxtimes$.

__Lemma__ (p. 70-71 of Hartshorne). For $f \in A$, $D(f) := \{x \in \operatorname{Spec} A : f(x) \neq 0 \text{ in } \kappa(x)\}$ = complem. of $V(f)$.

Then the $D(f)$ form a basis for the topology of $\operatorname{Spec} A$.

__Proof__ They are open. Let $U \subset \operatorname{Spec} A$ be open, $x \in U$. Put $a := I(\operatorname{Spec} A - U)$.

Then $x \notin V(a)$, so $\exists f \in a$ with $f(x) \neq 0$. Take such an f.

Then $x \in D(f) \subset U$. $\boxtimes$ __Remark__: $D(f) \cap D(g) = D(fg)$; $\boxed{\operatorname{Spec} A \text{ is quasi-compact}}$

__Def. (functoriality of Spec)__. $A \xrightarrow{\varphi} B \overset{\operatorname{Spec}}{\rightsquigarrow} (\underset{Y}{\underset{\shortparallel}{\operatorname{Spec} A}}, \mathcal{O}_Y) \xleftarrow{(f, f^\#)} (\underset{X}{\underset{\shortparallel}{\operatorname{Spec} B}}, \mathcal{O}_X)$

__f on Sets__:

$X \to Y$

$x \mapsto \varphi^{-1} x$

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A/\varphi^{-1}x \arrow[r, hook] & B/x \end{tikzcd}$$

B/x domain, $A/\varphi^{-1}x$ also a domain.

__f continuous__: for $a \in A$,

$f^{-1} V(a) = \{x \in X : f x \in V(a)\}$

$= V(\varphi a)$.

$x \in V(\varphi a) \iff \varphi a \in x \Updownarrow a \in \varphi^{-1} x \Updownarrow a \in fx \Updownarrow a(fx) = 0 \text{ in } \kappa(fx) \Updownarrow fx \in V(a)$

"$f^\#$ on local rings"

Let $x \in X$; then

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A_{fx} \arrow[r, dashed] \arrow[d] & B_x \arrow[d] \\ \kappa(fx) \arrow[r, dashed] & \kappa(x) \end{tikzcd}$$

Gives, $\forall U \subset Y$ open: $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$, compatible with restrictions if $g = \frac{a}{h}$ on $D(h)$ then $f^\# g = \frac{\varphi(a)}{\varphi(h)}$ on $D(\varphi h)$.

Example 1. $\varphi : A \twoheadrightarrow B$ (surjective). Then $\operatorname{Spec} A \xleftarrow{\operatorname{Spec}\varphi} \operatorname{Spec} B$ is

a closed immersion: ① fX is closed in Y, f injective, induced top. from Y is that of X

② $\forall x \in X$: $\mathcal{O}_{Y,fx} \to \mathcal{O}_{X,x}$ is surjective

Example 2. A a ring, $f \in A$, $\psi : A \to A_f$.

Then $\operatorname{Spec}\psi$ is an isomorphism $(\operatorname{Spec} A_f, \mathcal{O}) \to (D(f), \mathcal{O}|_{D(f)})$.

Here's the reason: $\forall\, p \in \operatorname{Spec} A_f$: $\psi^{-1}p \in \operatorname{Spec} A$ and $\psi f \notin p$ hence $f \notin \psi^{-1}p$.

$\forall\, p \in \operatorname{Spec} A$ with $f \notin p$:

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d] & A_f \supseteq q \arrow[dl, dashed, "\ast"] \arrow[ddl, bend left] \\ A_p \arrow[d] & \\ \kappa(p) & \end{tikzcd}$$

then $q \in \operatorname{Spec} A_f$

$\psi^{-1}q = p$,

remark: $q = \{ \frac{a}{f^n} : a \in p \}$

$= (\psi p) \cdot A_f$

$(A_f)_q = A_p$.

Prop. (2.2. of Hartshorne). $\mathcal{O}(\operatorname{Spec} A) = A$.

We have $A \to \mathcal{O}(\operatorname{Spec} A) \to \prod_{p \in \operatorname{Spec} A} A_p$ hence $A \to \mathcal{O}(\operatorname{Spec} A)$ is injective.

~~$\ker(A \to \prod A_p) = \{ f \in A : \exists g \in A, g \notin p \text{ and } gf = 0 \}$~~

(let $f \in A$, $f \neq 0$, then let $\mathfrak{a} := \ker(f\cdot : A \to A)$, then $\mathfrak{a} \subsetneq A$ ideal. take $\mathfrak{a} \subset \mathfrak{m} \subset A$ $\mathfrak{m}$ maximal. then $f/1 \in A_\mathfrak{m}$ is not zero...) So it remains to show surjectiveness.

Let $f \in \mathcal{O}(\operatorname{Spec} A)$. For $p \in \operatorname{Spec} A$ $\exists\, a, h$ in A, $f = \frac{a}{h}$ on $D(h)$. $\left(\frac{a}{h} = \frac{ah'}{hh'} \right)$

Get: I finite set, a_i, h_i s.t. $\operatorname{Spec} A = \bigcup_{i \in I} D(h_i)$, $f = \frac{a_i}{h_i}$ on $D(h_i)$.

Then $\forall\, i,j \in I$: $\frac{a_i}{h_i} = \frac{a_j}{h_j}$ in $\mathcal{O}(D(h_ih_j))$ hence in $A_{h_ih_j}$ (injectivity!),

hence $\exists\, n$ s.t. $(h_ih_j)^n(h_ja_i - h_ia_j) = 0$ in A. Take one n for all i,j.

Note: $\frac{a_i}{h_i} = \frac{a_ih_i^n}{h_i^{n+1}}$, so we may (and do!) assume that $\forall i,j$: $h_ja_i - h_ia_j = 0$.

Consider the complex

"homotopy"

$$\begin{tikzcd} A \arrow[r, "a \mapsto (h_ia)_i"] & \prod_{i \in I} A \arrow[r, "(a_i)_{i\in I} \mapsto (h_ja_i - h_ia_j)_{i,j}"] \arrow[dl, "s^1"'] & \prod_{i,j \in I} A \arrow[dl, "s^2"'] \\ A \arrow[r] & \prod_{i \in I} A \arrow[r] & \prod_{i,j \in I} A \end{tikzcd}$$

We want an $a \in A$ s.t. $\forall i$: $a_i = h_ia$ in A.

Note: $\sum_i Ah_i = A$. Take $b_i \in A$ s.t. $\sum_i b_ih_i = 1$ in A.

Put $s^1((a_i)_{i \in I}) := \sum_i b_ia_i$. Then $h_ia = \sum_j h_ib_ja_j = \sum_j b_jh_ja_i = a_i$ $\boxtimes$.

a!!