

Exercise II 2.4. ^{but with (X, \mathcal{O}_X) a LRS.} This is in fact Prop. I.1.6.3 of EGA. (Actually the way I present things is very much like there)

Let (X, \mathcal{O}_X) be a LRS, and A a ring.

Then $\text{Hom}_{\text{LRS}}(X, \text{Spec } A) \rightarrow \text{Hom}_{\text{Rings}}(A, \mathcal{O}_X(X))$ is bijective.

$$(f, f^\#) \longmapsto A = \mathcal{O}_{\text{Spec } A}(\text{Spec } A) \xrightarrow{f^\#} \mathcal{O}_X(X)$$

Proof: Let us construct the inverse. So let $\varphi: A \rightarrow \mathcal{O}_X(X)$.

For $x \in X$, $A \xrightarrow{\varphi} \mathcal{O}_X(X) \rightarrow \kappa(x)$, $f(x) := \text{kernel of this}$.

Then for $a \in A$: $f^{-1}D(a) = D(\varphi(a))$ ($x \in f^{-1}D(a) \Leftrightarrow f(x) \in D(a) \Leftrightarrow$

$$\Leftrightarrow a \mapsto 0 \text{ in } \kappa(x) \Leftrightarrow (\varphi(a))(x) \neq 0 \Leftrightarrow x \in D(\varphi(a)).$$

Also: for $g \in \mathcal{O}_X(X)$: $D(g) \subset X$ is open: let $x \in D(g)$, $\mathcal{O}_{X,x} \xrightarrow{g_x} \kappa(x)$,

hence $g_x \in \mathcal{O}_{X,x}^\times$, hence $\exists h_x \in \mathcal{O}_{X,x}$ s.t. $g_x h_x = 1$, take $U \ni x$ s.t.

$\exists h \in \mathcal{O}_X(U)$ with $h \mapsto h_x$ in $\mathcal{O}_{X,x}$, then $(g \cdot h)_x = 1$ in $\mathcal{O}_{X,x}$, hence

after shrinking U , $g \cdot h = 1$ in $\mathcal{O}_X(U)$, $x \in U \subset D(g)$.

So f is continuous. Now we must construct $f^\#$.

For $a \in A$ we have:

$$\begin{array}{ccc} \mathcal{O}_{\text{Spec } A}(D(a)) = A_a & \xrightarrow{f^\#(D(a))} & \mathcal{O}_X(f^{-1}D(a)) = \mathcal{O}_X(D(\varphi(a))) \\ \uparrow & & \uparrow \\ A & \xrightarrow{\varphi} & \mathcal{O}_X(X) \end{array}$$

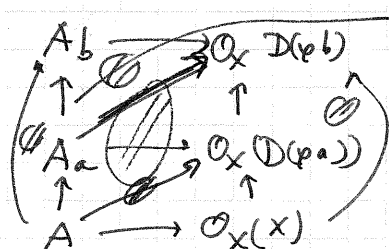
(exercise LRS: $g \in \mathcal{O}_X(D(g))^\times$.)

Compatible with restrictions: $D(b) \subset D(a) \Leftrightarrow V(b) \supset V(a) \Leftrightarrow \sqrt{A_b} \subset \sqrt{A_a} \Leftrightarrow$

$\Leftrightarrow \exists n \in \mathbb{Z}_{>0}, \exists c \in A, b^n = c \cdot a$. Let b, a in A with $D(b) \subset D(a)$.

Take n, c s.t. $b^n = c \cdot a$. Then $\varphi(b)^n = \varphi(c) \cdot \varphi(a)$ hence $D(\varphi(b)) \subset D(\varphi(a))$

Gives:



this is localisation w.r.t. c , and therefore the top square is commutative.

This construction gives $(f, f^\#)$ in LRS.

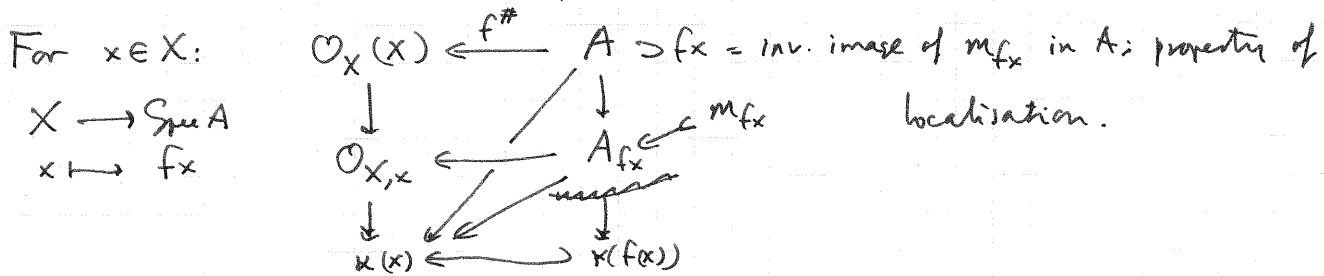
We also use the principle: X a top. space, B a basis for the topology, then sheaves (and morphisms of) are determined by their restrictions to B .

(I agree, ~~know~~ when we apply this it is convenient to use $f_* \mathcal{O}_X$)

Exercise: the map we just constructed is the inverse $\square \hookrightarrow = \text{id}_A$ clear.

Well, let me still write a bit. So let $(f, f^\#)$ be given.

That gives $\varphi = f^\# : A \rightarrow \mathcal{O}_X(X)$.



Key Consequence: $(\text{LRS}) \xrightarrow{\Gamma} (\text{Ring})$ $\xleftarrow{\text{Spec}}$ $(\text{Ring}) \xrightarrow{\Gamma} (\text{LRS})$

$\text{Hom}_{\text{CRS}}(X, \mathcal{O}_X) \xrightarrow{\Gamma} \text{Spec } A$
 \parallel
 $\text{Hom}_{\text{Ring}}(A, \mathcal{O}_X(X)) \xrightarrow{\Gamma} \Gamma(X, \mathcal{O}_X)$

Γ is the right-adjoint of Spec . (I'll come back to $\Gamma(X, \mathcal{O}_X)$ this after Yoneda's lemma)
 Spec is the right-adjoint of Γ .

From varieties: $\mathcal{O}(A_k^n) = k[x_1, \dots, x_n]$, $A_{\mathbb{Z}}^n := \text{Spec } \mathbb{Z}[x_1, \dots, x_n]$.

underlying set of A_k^n is k^n is $\text{Hom}_{k\text{-Sp}}(\cdot, A_k^n)$

$\forall k\text{-var } X$: $\text{Hom}_{k\text{-var}}(X, A_k^n) = \text{Hom}_{k\text{-alg}}(k[x_1, \dots, x_n], \mathcal{O}(X)) = \mathcal{O}(X)^n$
 $\varphi \mapsto (\varphi x_1, \dots, \varphi x_n)$

Now we see it is the same for $A_{\mathbb{Z}}^n$:

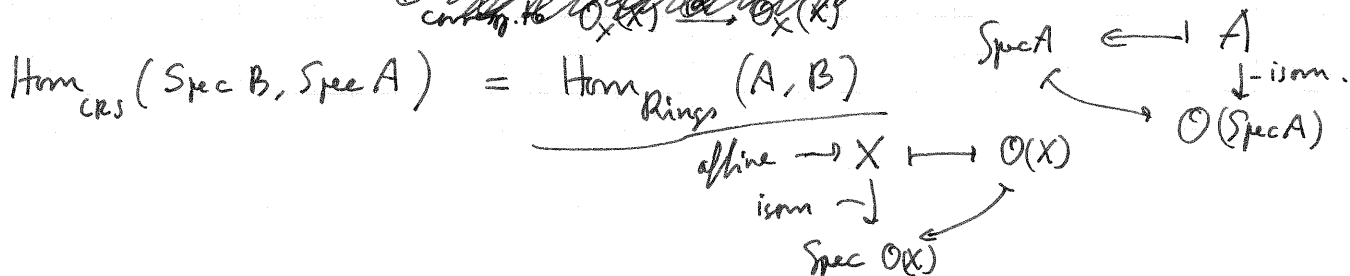
$\forall (X, \mathcal{O}_X)$ LRS: $\text{Hom}_{\text{CRS}}(X, A_{\mathbb{Z}}^n) = \text{Hom}_{\text{Ring}}(\mathbb{Z}[x_1, \dots, x_n], \mathcal{O}_X(X)) = \mathcal{O}_X(X)^n$.

set of points of $A_{\mathbb{Z}}^n$ with values in X , \checkmark very close to the naive interpretation of affine n -space. $A \mapsto A^n$

I introduced the notation $A_{\mathbb{Z}}^n(X)$ for this, $\left[\begin{array}{l} \text{set of pts of } A_{\mathbb{Z}}^n \\ \text{with val. in } \text{Spec}(A) \end{array} \right]$
 and $A_{\mathbb{Z}}^n(A) \mapsto X = \text{Spec } A$.

{affine schemes} $\xrightarrow{\Gamma} (\text{Ring})$ $\xleftarrow{\text{Spec}}$ anti-equivalence.

~~... $\text{Spec } \mathcal{O}_X(X) \rightarrow \text{Spec } \mathcal{O}_X(x) \rightarrow \text{Spec } k(x)$...~~



2.6. $\text{Spec}(0) = \emptyset$ $0 = \text{final obj. in Rings}$, $\emptyset = \text{initial obj. in LRS}$.

2.8. Nice! $\text{Spec}(k[\epsilon]/(\epsilon^2)) = \rightarrow$, really a tangent vector.

Tangent bundle of ~~algebraic~~ ^{a scheme} X over k : $\text{Hom}_{\text{Sch}/k}(\text{Spec}(k[\epsilon]/(\epsilon^2)), X)$
 $= X(k[\epsilon]/(\epsilon^2))$.

Explain: (Sch/S) .

2.9. Good to know.

2.10. $\text{Spec } \mathbb{R}[x] = \{ \eta \} \cup \mathbb{C}/\langle \sigma \rangle$ ^{complex conj.}

2. In (LRS/\mathbb{R}) $\text{Spec}(\mathbb{R}[\epsilon])$ is useful for tangent vectors of smooth manifolds. And

2.12. Formal, dull, but important. We've done it for varieties. ^{topol. manit?}

2.15. just do this for affine varieties, then glue.

2.16. Very useful for 2.17. Remark: Not true for all LRS. Just think of \mathbb{R} as manifold $\mathbb{R} = x \dots$

For c : let $(U_i)_{i \in I}$ be a ^{finite} open affine cover, and $\forall i, j \in I$, let $(U_{i,j,k})_{k \in K_{ij}}$ be a finite open affine cover of $U_i \cap U_j$.

$$\begin{array}{ccc} \mathcal{O}(X) & \rightarrow & \prod_{i \in I} \mathcal{O}(U_i) \rightrightarrows \prod_{\substack{i, j \in I \\ k \in K_{ij}}} \mathcal{O}(U_{i,j,k}) \\ \downarrow & & \downarrow \\ \mathcal{O}(X_f) & \rightarrow & \prod_{i \in I} \mathcal{O}(U_i)_f \rightrightarrows \prod_{\substack{i, j \in I \\ k \in K_{ij}}} \mathcal{O}(U_{i,j,k})_f \end{array}$$

Use sheaf property of \mathcal{O} .

d: Follows from b and c. $\mathcal{O}(U_i \cap X_f) \xrightarrow{\cong} \mathcal{O}(U_i)_f$ $\mathcal{O}(U_i \cap X_f) \xrightarrow{\cong} \mathcal{O}(U_i)_f$ $A = \mathcal{O}(X)$
 \downarrow \downarrow
 $A_f \hookrightarrow \mathcal{O}(X_f)$

2.17. (a) Trivial/formal. (More for "topos theory"?)

(b) Suppose $f_1, \dots, f_r \in A = \mathcal{O}(X)$, with X_{f_i} affine and $A_{f_1} + \dots + A_{f_r} = A$.

Then $\bigcup_i X_{f_i} = X$. Consider $X \xrightarrow{(g, g^*)} \text{Spec } A$ given by $A \xrightarrow{=} \mathcal{O}(X)$.

Then $\tilde{g}^{-1} D(f_i) = X_{f_i}$. By 2.16: (use the cover $X = \bigcup X_{f_i}$)

and note that $X_{f_i} \cap X_{f_j} = D(f_j)$ in X_{f_i} is affine: $\mathcal{O}(X_{f_i}) = A_{f_i}$.

So (g, g^*) is an isomorphism over $D(f_i)$, for all i . \square
 Then a.

2.18. Do it!

2.19. Instructive.