but with $(X, \mathcal{O}_X)$ a LRS.

Exercise II 2.4. This is in fact Prop. I.1.6.3 of EGA. (Actually the way I present things is very much like there!)

Let $(X, \mathcal{O}_X)$ be a LRS, and A a ring.

Then $\mathrm{Hom}_{LRS}(X, \operatorname{Spec} A) \longrightarrow \mathrm{Hom}_{Rings}(A, \mathcal{O}_X(X))$ is bijective.

$$(f, f^\#) \longmapsto A = \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \xrightarrow{f^\#} \mathcal{O}_X(X)$$

Proof: Let us construct the inverse. So let $\varphi : A \to \mathcal{O}_X(X)$. ~~...~~

For $x \in X$, $A \xrightarrow{\varphi} \mathcal{O}_X(X) \to \kappa(x)$, $f(x) :=$ kernel of this.

Then for $a \in A$: $f^{-1} D(a) = D(\varphi(a))$ $\quad$ ($x \in f^{-1}(D(a)) \iff f(x) \in D(a) \iff$

$\iff a \not\mapsto 0$ in $\kappa(x) \iff (\varphi(a))(x) \neq 0 \iff x \in D(\varphi(a))$.

Also: for $g \in \mathcal{O}_X(X)$: $D(g) \subset X$ is open: let $x \in D(g)$, $\quad \begin{matrix} g_x & \longmapsto & g(x) \\ \cap & & \cap \\ \mathcal{O}_{X,x} & \twoheadrightarrow & \kappa(x) \end{matrix}$, $g(x) \neq 0$

hence $g_x \in \mathcal{O}_{X,x}^\times$, hence $\exists\, h_x \in \mathcal{O}_{X,x}$ s.t. $g_x h_x = 1$, take $U \ni x$ s.t.

$\exists\, h \in \mathcal{O}_X(U)$ with $h \mapsto h_x$ in $\mathcal{O}_{X,x}$, then $(g \cdot h)_x = 1$ in $\mathcal{O}_{X,x}$, hence

after shrinking U, $g \cdot h = 1$ in $\mathcal{O}_X(U)$, $x \in U \subset D(g)$.

So f is continuous. Now we must construct $f^\#$.

For $a \in A$ we have:

$$\begin{tikzcd} \mathcal{O}_{\operatorname{Spec} A}(D(a)) = A_a \arrow[rr, dashed, "f^\#(D(a))"] & & \mathcal{O}_X(f^{-1}D(a)) = \mathcal{O}_X(D(\varphi a)) \\ A \arrow[u] \arrow[rr, "\varphi"] & & \mathcal{O}_X(X) \arrow[u] \end{tikzcd}$$

φa is a unit here. (exercise LRS: $g \in \mathcal{O}_X(D(g))^\times$.)

Compatible with restrictions: $D(b) \subset D(a) \iff V(b) \supset V(a) \iff \sqrt{Ab} \subset \sqrt{Aa} \iff$

$\iff \exists\, n \in \mathbb{Z}_{\geq 1}, \exists\, c \in A,\ b^n = c \cdot a$. Let b, a in A with $D(b) \subset D(a)$.

Take n, c s.t. $b^n = c \cdot a$. Then $\varphi(b)^n = \varphi(c) \cdot \varphi(a)$ hence $D(\varphi b) \subset D(\varphi(a))$

Gives:

$$\begin{tikzcd} A_b \arrow[r] & \mathcal{O}_X D(\varphi b) \\ A_a \arrow[u] \arrow[r] \arrow[ur] & \mathcal{O}_X D(\varphi a) \arrow[u] \\ A \arrow[u] \arrow[r] & \mathcal{O}_X(X) \arrow[u] \end{tikzcd}$$

this is localisation w.r.t. c, and therefore the top square is commutative.

This construction gives $(f, f^\#)$ in LRS.

We also use the principle: X a top. space, $\mathcal{B}$ a basis for the topology, then sheaves (and morphisms of) are determined by their restrictions to $\mathcal{B}$. (I agree, ~~...~~ when we apply this it is convenient to use $f_* \mathcal{O}_X$.)

Exercise: the map we just constructed is the inverse $\boxtimes$ $\quad \circlearrowleft = \mathrm{id}_A$ clear.

Well, let me still write a bit. So let $(f, f^\#)$ be given.

That gives $\varphi = f^\# : A \longrightarrow \mathcal{O}_X(X)$.

For $x \in X$: $\quad X \longrightarrow \mathrm{Spec}\,A$, $\quad x \longmapsto fx$

$$\begin{tikzcd} \mathcal{O}_X(X) \arrow[d] & A \arrow[l, "f^\#"'] \arrow[d] \arrow[dl] \\ \mathcal{O}_{X,x} \arrow[d] & A_{fx} \arrow[l] \arrow[d] \arrow[dl] \\ \kappa(x) & \kappa(f(x)) \arrow[l] \end{tikzcd}$$

$A \supset fx$ = inv. image of m_{fx} in A; property of localisation. $\quad \leftarrow m_{fx}$

$(X, \mathcal{O}_X) \longmapsto \mathcal{O}_X(X)$

~~Other~~ <u>Consequence</u>: $(LRS) \underset{\mathrm{Spec}}{\overset{\Gamma}{\rightleftarrows}} (Ring)$

$$\mathrm{Hom}_{LRS}((X, \mathcal{O}_X), \mathrm{Spec}\,A) = \mathrm{Hom}_{Ring}(A, \mathcal{O}_X(X)), \qquad \mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X)$$

Γ is the right-adjoint of Spec.
Spec is the right-adjoint of Γ.
(I'll come back to this after Yoneda's lemma)

From varieties: $\mathcal{O}(\mathbb{A}^n_k) = k[x_1, \ldots, x_n]$, $\quad \mathbb{A}^n_{\mathbb{Z}} := \mathrm{Spec}\,\mathbb{Z}[x_1, \ldots, x_n]$.

underlying set of $\mathbb{A}^n_k$ is k^n is $\mathrm{Hom}_{k\text{-}Sp}(\cdot, \mathbb{A}^n_k)$, $\quad \cdot = \mathrm{Spec}(k)$

$\forall$ k-~~Space~~ Var X: $\mathrm{Hom}_{k\text{-}Var}(X, \mathbb{A}^n_k) = \mathrm{Hom}_{k\text{-}alg}(k[x_1, \ldots, x_n], \mathcal{O}(X)) = \mathcal{O}(X)^n$

$$\varphi \longmapsto (\varphi x_1, \ldots, \varphi x_n)$$

Now we see it is the same for $\mathbb{A}^n_{\mathbb{Z}}$:

$\forall\, (X, \mathcal{O}_X)$ LRS: $\mathrm{Hom}_{LRS}(X, \mathbb{A}^n_{\mathbb{Z}}) = \mathrm{Hom}_{Ring}(\mathbb{Z}[x_1, \ldots, x_n], \mathcal{O}_X(X)) = \mathcal{O}_X(X)^n$.

set of <u>points</u> of $\mathbb{A}^n_{\mathbb{Z}}$ with values in X, very close to the naive interpretation of affine n-space. $A \mapsto A^n$.

I introduced the notation $\mathbb{A}^n_{\mathbb{Z}}(X)$ for this, and $\mathbb{A}^n_{\mathbb{Z}}(A)$ if $X = \mathrm{Spec}\,A$. $\quad A^n$ = set of pts of $\mathbb{A}^n_{\mathbb{Z}}$ with val. in $\mathrm{Spec}(A)$.

{affine schemes} $\underset{\mathrm{Spec}}{\overset{\Gamma}{\rightleftarrows}}$ ~~Ring~~ (Ring) $\quad$ <u>anti-equivalence</u>.

~~$(X, \mathcal{O}_X) \to \mathrm{Spec}\,\mathcal{O}_X(X)$ ... $A \to \mathcal{O}_{\mathrm{Spec}A}(\mathrm{Spec}\,A)$ iso ...~~

corresp. to $\mathcal{O}_X(X) \xrightarrow{id} \mathcal{O}_X(X)$

$\mathrm{Hom}_{LRS}(\mathrm{Spec}\,B, \mathrm{Spec}\,A) = \underline{\mathrm{Hom}_{Rings}(A, B)}$

$$\begin{tikzcd} \mathrm{Spec}\,A & A \arrow[l, mapsto] \arrow[d, "\text{isom.}"] \\ & \mathcal{O}(\mathrm{Spec}\,A) \arrow[ul, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \text{affine} \to X \arrow[r, mapsto] \arrow[d, "\text{isom}"'] & \mathcal{O}(X) \arrow[dl, bend left] \\ \mathrm{Spec}\,\mathcal{O}(X) & \end{tikzcd}$$

2.6. $\operatorname{Spec}(0) = \emptyset$ 0 = final obj. in Ring, $\emptyset$ = initial obj. in LRS.

2.8. Nice! $\operatorname{Spec}(k[\varepsilon]/(\varepsilon^2)) = \longrightarrow$, really a tangent vector.

Tangent bundle of ~~a variety~~ a scheme X over k: $\operatorname{Hom}_{\mathrm{Sch}/k}\left(\operatorname{Spec}\left(k[\varepsilon]/\varepsilon^2\right), X\right)$
$= X(k[\varepsilon]/(\varepsilon^2))$.

Explain! Sch/S.

2.9. Good to know.

2. In $(\mathrm{LRS}/\mathbb{R})$ $\operatorname{Spec}(\mathbb{R}[\varepsilon])$ is useful for tangent vectors of smooth manifolds. And topol. manif?

2.10. $\operatorname{Spec} \mathbb{R}[x] = \{\eta\} \cup \mathbb{C}/\langle\sigma\rangle$ complex conj.

2.12 Formal, dull, but important. We've done it for varieties.

2.15. just do this for affine varieties, then glue.

2.16 Very useful for 2.17. Remark: Not true for all LRS. Just think of $\mathbb{R}$ as manifold, $f = x$ ...

For c: Let $(U_i)_{i\in I}$ be an finite ~~open~~ affine open cover, and $\forall i,j \in I$, let $(U_{i,j,k})_{k\in K_{i,j}}$ be a finite open affine cover of $U_i \cap U_j$.

Then
$$\begin{tikzcd} \mathcal{O}(X) \arrow[r] \arrow[d] & \prod\limits_{i\in I} \mathcal{O}(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod\limits_{\substack{i,j\in I\\ k\in K_{i,j}}} \mathcal{O}(U_{i,j,k}) \arrow[d] \\ \mathcal{O}(X_f) \arrow[r] & \prod\limits_{i\in I} \mathcal{O}(U_i)_f \arrow[r, shift left] \arrow[r, shift right] & \prod\limits_{\substack{i\in I\\ j\in I\\ k\in K_{i,j}}} \mathcal{O}(U_{i,j,k})_f \end{tikzcd}$$
$\mathcal{O}(U_i)_f = \mathcal{O}(U_i \cap X_f)$ — by a

Use sheaf property of $\mathcal{O}$.

d: Follows from b and c.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d] & \mathcal{O}(X) \arrow[d] \\ A_f \arrow[r, hook] & \mathcal{O}(X_f) \end{tikzcd}$$

2.17. (a) Trivial/formal. (More for "topos theory"?)

(b) Suppose $f_1, \dots, f_r \in A = \mathcal{O}(X)$, with the X_{f_i} affine and $Af_1 + \dots + Af_r = A$.

Then $\bigcup_i X_{f_i} = X$. Consider $X \xrightarrow{(g,g^\#)} \operatorname{Spec} A$ given by $A \underset{=!}{\rightarrow} \mathcal{O}(X)$.

Then ~~[illegible]~~ $g^{-1} D(f_i) = X_{f_i}$. By 2.16: (use the cover $X = \bigcup (X_{f_i})$, and note that $X_{f_i} \cap X_{f_j} = D(f_j)$ in X_{f_i} is affine): $\mathcal{O}(X_{f_i}) = A_{f_i}$.

So $(g, g^\#)$ is an isomorphism over ~~Spec A~~ $D(f_i)$, for all i. $\boxed{X}$ Then a.

2.18 Do it!

2.19. Instructive.