

More on Cartesian diagrams.

Let \mathcal{C} be a category. The def'n of "Cartesian diagram" says:

Then $Z \xrightarrow{g'} Y$ is Cartesian iff $\forall T$ in $\text{Ob}(\mathcal{C})$:

$$\begin{array}{ccc} Z & \xrightarrow{g'} & Y \\ \downarrow f' & & \downarrow g \\ X & \xrightarrow{f} & S \end{array}$$

$$\begin{array}{ccc} Z(T) & \xrightarrow{g'} & Y(T) \\ \downarrow f' & & \downarrow g \\ X(T) & \xrightarrow{f} & S(T) \end{array} \text{ is Cartesian}$$

iff $\forall T$ in \mathcal{C} :

$$Z(T) \rightarrow X(T) \times_{S(T)} Y(T) \text{ is a bijection.}$$

Let us now prove again the lemma of last time (transitivity...)

Lemma 1 Let $A \rightarrow B \rightarrow C$ be a comm. diagram in \mathcal{C} .

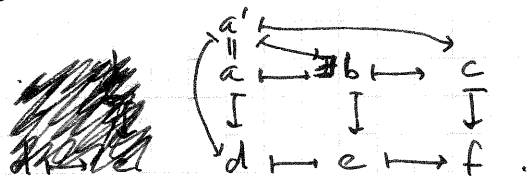
$$\begin{array}{ccc} A & \rightarrow & B & \rightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \rightarrow & E & \rightarrow & F \end{array} \text{ Then Assume } B \rightarrow C \text{ is Cartesian.}$$

$$\begin{array}{ccc} B & \rightarrow & C \\ \downarrow & & \downarrow \\ E & \rightarrow & F \end{array}$$

Then $A \rightarrow B$ is Cart. $\Leftrightarrow A \rightarrow C$ is.

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ D & \rightarrow & E \end{array} \Leftrightarrow \begin{array}{ccc} A & \rightarrow & C \\ \downarrow & & \downarrow \\ D & \rightarrow & F \end{array}$$

Proof \Rightarrow : let $T \in \mathcal{C}$, then we have $A(T) \rightarrow B(T) \rightarrow C(T)$



$$\begin{array}{ccc} A(T) & \rightarrow & B(T) & \rightarrow & C(T) \\ \downarrow & \square & \downarrow & \square & \downarrow \\ D(T) & \rightarrow & E(T) & \rightarrow & F(T) \end{array}$$

"it suffices to prove the statement in Set "

\Leftarrow : (don't write the T anymore, just say "as sets")

$$\begin{array}{ccc} a & \rightarrow & b & \rightarrow & c \\ \downarrow & & \downarrow & & \downarrow \\ d & \rightarrow & e & \rightarrow & f \end{array} \text{ uniqueness: } \begin{array}{ccc} a & \rightarrow & b & \rightarrow & c \\ \downarrow & & \downarrow & & \downarrow \\ d & \rightarrow & e & \rightarrow & f \end{array} \text{ hence } a = a' \quad \square$$

Lemma 2. "Graph & diagonal". Let $X \xrightarrow{f} Y$ in \mathcal{C} and suppose that

$Y \times_S Y$ and $X \times_S Y$ exist. (Ex. II.4.8) \downarrow_S

Then $X \xrightarrow{(id_X, f)} X \times_S Y$ is Cartesian.

$$\begin{array}{ccc} X & \xrightarrow{(id_X, f)} & X \times_S Y \\ \downarrow f & & \downarrow f \times id_Y \\ Y & \xrightarrow{(id_Y, id_Y)} & Y \times_S Y \end{array}$$

$$\begin{array}{ccc} (x, y') \\ \downarrow \\ (f(x), y') \\ Y \rightarrow (Y, Y) \end{array} \quad \begin{array}{l} y' = y \text{ and} \\ y = fx \end{array}$$

Proof. ~~Why is this Cartesian?~~

$$\begin{array}{ccc} \text{Let } T \in \mathcal{C}. & X(T) & \xrightarrow{(id_X, f)} & X(T) \times_{S(T)} Y(T) \\ & \downarrow f & & \downarrow f \times id_Y \\ & Y(T) & \xrightarrow{(id_Y, id_Y)} & Y(T) \times_{S(T)} Y(T) \end{array}$$

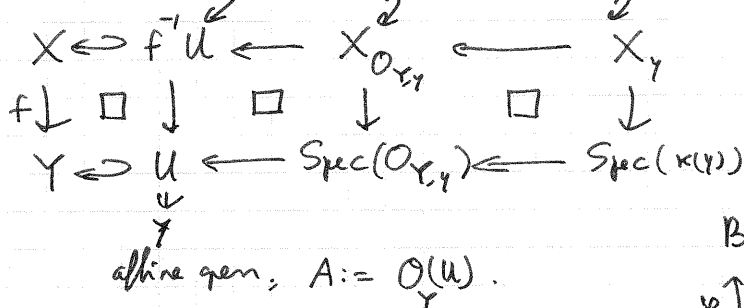
$\Sigma X(T) \rightarrow$ fibre product is a bijection. \square

Exercise II.3.10 Let $f: X \rightarrow Y$ be in (Sch), $y \in Y$.

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Then $X_y \rightarrow X$ induces $X_y \rightarrow f^{-1}(y)$ in (Top), homeomorphism.
 $\downarrow \square \downarrow f$, gen affine, $B := \mathcal{O}(V)$ \uparrow induced top. from X .
 $\text{Spec}(k(y)) \rightarrow Y \leftarrow V \leftarrow W \leftarrow X$

Proof



affine gen, $A := \mathcal{O}(U)$.

$$A_y := \mathcal{O}_{Y,y} = \tilde{S}'A$$

$$C \rightarrow C/C \cdot \varphi m_y$$

$$\varphi \uparrow$$

$$\uparrow$$

$$m_y \hookrightarrow \mathcal{O}_{Y,y} \rightarrow k(y)$$

$$B \rightarrow \tilde{S}'B$$

$$\varphi \uparrow$$

$$A \rightarrow \tilde{S}'A$$

$$\text{Spec } C \hookrightarrow \text{Spec } C/C \cdot \varphi m_y$$

injective, homeo on image.

on image. \square

injective, homeo on image.

Corollary

Let $\begin{array}{c} Y \\ \downarrow \varphi \\ X \xrightarrow{f} S \end{array}$ be in (Sch). Then $\text{set}(X \times_S Y) = \{(x, s, y, p) : x \in X, y \in Y, s \in S, \text{ s.t. } fx = s = sy, \text{ and } p \in \text{Spec}(k(x) \otimes_{k(s)} k(y))\}$.

Exercise II.3.4 Let $f: X \rightarrow Y$ be finite, and $U \subset Y$ open affine.

To prove: $f^{-1}U \subset X$ is affine, and $\mathcal{O}(f^{-1}U)$ is f.g. as $\mathcal{O}(U)$ -module.

Proof Given: $\forall y \in Y \exists U_y \subset Y$ open affine s.t. $f^{-1}U_y$ is affine and

$\mathcal{O}(f^{-1}U_y)$ is f.g. as $\mathcal{O}(U_y)$ -module.

Let $y \in U$. Let $g \in \mathcal{O}(U)$ s.t. $D(g) \subset U_y$. Then $f^{-1}D(g) \subset f^{-1}U_y$ is

affine and $\mathcal{O}(f^{-1}D(g))$ is f.g. as $\mathcal{O}(D(g))$ -module: $\square \downarrow$

$$\mathcal{O}(f^{-1}(D(g))) = \mathcal{O}(D(g)) \otimes_{\mathcal{O}(U)} \mathcal{O}(f^{-1}U_y).$$

$$D(g) \subset U_y$$

As U is q.c., cover U with finitely many $D(g_i)$.

Then the $(f^{-1}U_{g_i})$ cover $f^{-1}U$ and are affine.

By Exercise II.2.17, $f^{-1}U$ is affine.

The f.g. as $\mathcal{O}(U)$ -module: Let A be a ring, g_1, \dots, g_r in A

s.t. $Ag_1 + \dots + Ag_r = A$, M an A -module s.t. $\forall i M_{g_i}$ is f.g. as A_{g_i} -mod.

Then M is f.g. as A -module. Proof $\forall i$, let $\{ \frac{m_{i,j}}{s_{i,j}} : j \in J_i \}$ be a finite

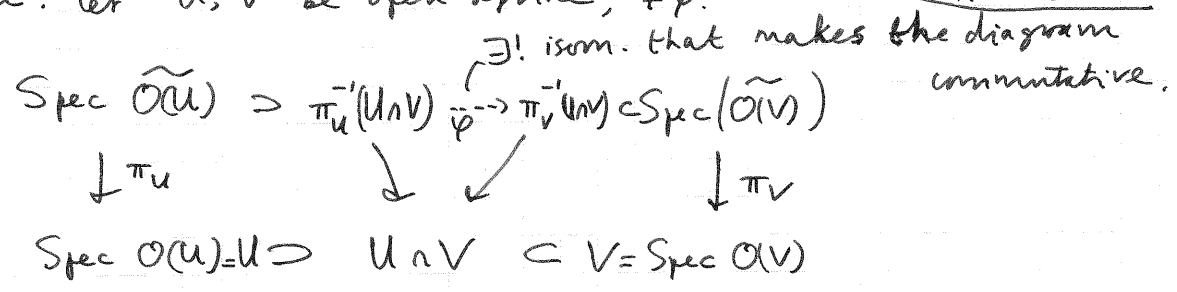
set of gen's. Prove that $\{ m_{i,j} : 1 \leq i \leq r, j \in J_i \}$ generate M .

\square

Exercise II.3.8. Let X be an integral scheme. Let $\eta \in X$ be the generic point.
 $\triangleright U \subset X$ open affine let $\tilde{\mathcal{O}}(U) \subset \mathbb{L}$ be the integral closure of $\mathcal{O}(U)$ in \mathbb{L} .
 Then $\mathcal{O}_{X,\eta} =: K$ is a field. (function field of X)
 Let $K \rightarrow L$ be a field ext'n.

Claim. The $\text{Spec } \tilde{\mathcal{O}}(U)$ can be sheafed naturally.

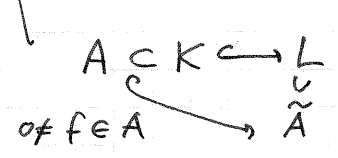
More precise: let U, V be open affine, $U \neq \emptyset$.



We construct φ locally on $U \cap V$ (as $U \cap V$ is not nec. affine...).

Let $x \in U \cap V$. Take $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$ s.t. $D_U(f) = D_V(g) \ni x$.
 Then ("localisation commutes with normalisation"):

$$\begin{aligned} \pi_U^{-1} D(f) &= \text{Spec} \left(\tilde{\mathcal{O}}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U)_f \right) = \text{Spec} \left(\tilde{\mathcal{O}}(U)_f \right) = \\ &= \text{Spec} \left(\tilde{\mathcal{O}}(V)_g \right) = \pi_V^{-1} D(g). \end{aligned}$$



then $(\tilde{A}_f) = (\tilde{A})_f$: " \supset ": $\frac{b}{f^n}$ $b \in \tilde{A}$ $b^m + a_{m-1}b^{m-1} + \dots + a_0 = 0$
 divide by f^{nm} : b/f^n int. over A_f .
 " \subset ": Let $b \in L$ be integral over A_f : $b^n + \frac{a_{n-1}}{f} b^{n-1} + \dots + \frac{a_0}{f^n} = 0$ in L .
 multiply by f^{nm} . Then $f^m \cdot b$ int. / A . \square
 Hence $b \in (\tilde{A})_f$.

Application: construction of curves / fields in terms of their function fields.

$$\begin{array}{ccc} & k(x)[y]/(f) & \\ & \uparrow & \parallel \\ \mathbb{P}_k^1 & \sim k(x) & y^n + f_{n-1}y^{n-1} + \dots + f_0, f_i \in k(x). \\ & & \text{irreducible.} \end{array}$$

Yoneda's Lemma

We follow Stacks chapter 4, up to § 4.3.

Give the example $\mathcal{C} = \text{Open}(X), X \text{ in } (\text{Top})$.

The important point of lemma 4.3.5 is that to construct a morphism from X to Y in \mathcal{C} it is equivalent to construct a morphism from h_X to h_Y in $\text{PSh}(\mathcal{C})$, that is, to construct maps, $\forall T \text{ in } \mathcal{C}$, $X(T) \rightarrow Y(T)$, functorial in T .

Example 1. $X = \text{Spec } \mathbb{Z}[x,y,z] / (x^6 + y^6 - z^6)$

$$Y = \text{Spec } \mathbb{Z}[x,y,z] / (x^3 + y^3 - z^3)$$

$$\text{Then, } \forall \text{ ring } A, X(A) = \{(a,b,c) \in A^3 : a^6 + b^6 = c^6\}$$

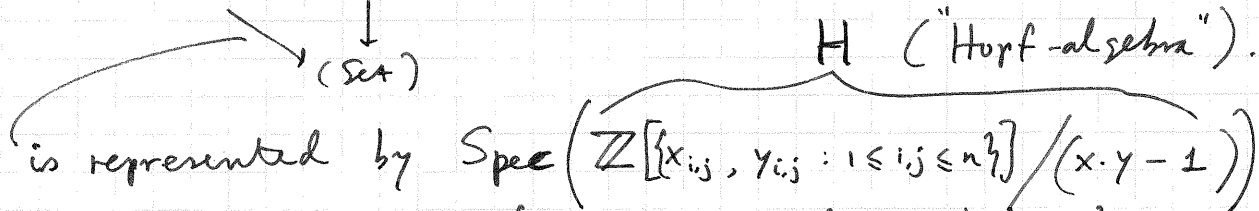
$$Y(A) = \{(a,b,c) \in A^3 : a^3 + b^3 = c^3\}$$

So we have $X(A) \rightarrow Y(A)$, $(a,b,c) \mapsto (a^2, b^2, c^2)$, and this is functorial in A . Hence $\exists!$ $X \xrightarrow{f} Y$ giving this.

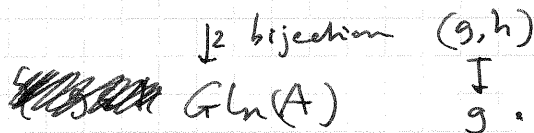
It corresp. to $x^2 \mapsto x, y^2 \mapsto y, z^2 \mapsto z$, indeed $\checkmark \leftarrow x^2 + y^2 - z^2 \dots$

Example 2. let $n \in \mathbb{Z}_{\geq 1}$.

$$\text{GL}_n : (\text{Ring}) \rightarrow (\text{Grp}), A \mapsto \text{GL}_n(A) = \{g \in M_n(A) : g \text{ invertible}\}$$



$$\text{Indeed: Hom}_{\text{Ring}}(H, A) = \{(g,h) \in M_n(A)^2 : g \cdot h = 1\}$$



$$\text{We have } \text{GL}_n(A) \times \text{GL}_n(A) \rightarrow \text{GL}_n(A), (g_1, g_2) \mapsto g_1 g_2$$

" functional in A .

$$(\text{GL}_n \times_{\text{Spec } \mathbb{Z}} \text{GL}_n)(A)$$

Clearly, it is easier to think about $\text{GL}_n \times_{\text{Spec } \mathbb{Z}} \text{GL}_n \rightarrow \text{GL}_n$ in terms of the functor than in terms of LRS.

Question for audience to think about: what is $\mathbb{P}^n(A)$? We'll see it soon.