Week 5, 2011/10/06.

More on Cartesian diagrams.

Let $\mathcal{C}$ be a category. The def'n of "Cartesian diagram" says:

Then $\begin{array}{ccc} Z & \xrightarrow{g'} & Y \\ \downarrow f' & & \downarrow g \\ X & \xrightarrow{f} & S \end{array}$ is Cartesian iff $\forall T$ in $Ob(\mathcal{C})$: $\begin{array}{ccc} Z(T) & \xrightarrow{g'} & Y(T) \\ \downarrow f' & & \downarrow g \\ X(T) & \xrightarrow{f} & S(T) \end{array}$ is Cartesian

iff $\forall T$ in $\mathcal{C}$:

$$Z(T) \to X(T) \times_{S(T)} Y(T) \text{ is a bijection.}$$

Let us now prove again the lemma of last time (transitivity ...)

Lemma ① Let $\begin{array}{ccccc} A & \to & B & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \to & E & \to & F \end{array}$ be a comm. diagram in $\mathcal{C}$. ~~Then~~ Assume $\begin{array}{ccc} B & \to & C \\ \downarrow & & \downarrow \\ E & \to & F \end{array}$ is Cartesian.

Then $\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ D & \to & E \end{array}$ is Cart. $\Longleftrightarrow$ $\begin{array}{ccc} A & \to & C \\ \downarrow & & \downarrow \\ D & \to & F \end{array}$ is.

Proof $\Rightarrow$: Let $T \in \mathcal{C}$, then we have $\begin{array}{ccccc} A(T) & \to & B(T) & \to & C(T) \\ \downarrow & \square & \downarrow & \square & \downarrow \\ D(T) & \to & E(T) & \to & F(T). \end{array}$

$$\begin{array}{ccccc} a' = a & \mapsto & \exists!\, b & \mapsto & c \\ \updownarrow & & \updownarrow & & \updownarrow \\ d & \mapsto & e & \mapsto & f. \end{array}$$

$\Leftarrow$: (don't write the T anymore, just say "it suffices to prove the statement in (Set)")

$$\begin{array}{ccccc} a & \mapsto & b' = b & \mapsto & c \\ \updownarrow & & \updownarrow & & \updownarrow \\ d & \mapsto & e & \mapsto & f \end{array} \qquad \text{uniqueness: } \begin{array}{ccccc} a, a' & \mapsto & b & \mapsto & c \\ \updownarrow & & \updownarrow & & \updownarrow \\ d & \mapsto & e & \mapsto & f \end{array} \quad \text{hence } a = a'. \ \boxtimes$$

Lemma 2. "Graph & diagonal". Let $X \xrightarrow{f} Y$ in $\mathcal{C}$ (both over S) and suppose that $Y \times_S Y$ and $X \times_S Y$ exist. (Ex. II.4.8)

Then $\begin{array}{ccc} X & \xrightarrow{(id_X, f)} & X \times_S Y \\ f \downarrow & & \downarrow f \times id_Y \\ Y & \xrightarrow{(id_Y, id_Y)} & Y \times_S Y \end{array}$ is Cartesian.

Proof. Let $T \in \mathcal{C}$. $\begin{array}{ccc} X(T) & \xrightarrow{(id_X, f)} & X(T) \times_{S(T)} Y(T) \\ f \downarrow & & \downarrow f \times id_Y \\ Y(T) & \xrightarrow{(id_Y, id_Y)} & Y(T) \times_{S(T)} Y(T) \end{array}$

$$\begin{array}{ccc} & & (x, y') \\ & & \downarrow \\ & & (f(x), y') \\ y & \longmapsto & (y, y) \end{array} \qquad y' = y \text{ and } y = fx$$

So $X(T) \to$ fibre product is a bijection. $\boxtimes$

Exercise II.3.10 Let $f : X \to Y$ be in (Sch), $y \in Y$.

Then $X_y \longrightarrow X$ induces $X_y \longrightarrow f^{-1}\{y\}$ in (Top), homeomorphism.

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \downarrow & \square & \downarrow f \\ \mathrm{Spec}(\kappa(y)) & \longrightarrow & Y \end{array}$$

open affine, $B := \mathcal{O}(V)$; $Y \supseteq V \leftarrow W \leftarrow Z$; induced top. from X.

Proof

$$\begin{array}{ccccccc} X & \hookleftarrow & f^{-1}U & \longleftarrow & X_{\mathcal{O}_{Y,y}} & \longleftarrow & X_y \\ f\downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow \\ Y & \hookleftarrow & U & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{Y,y}) & \longleftarrow & \mathrm{Spec}(\kappa(y)) \end{array}$$

$U \ni y$ affine open, $A := \mathcal{O}_Y(U)$.

$A_y := \mathcal{O}_{Y,y} = S^{-1}A$

$$\begin{array}{ccc} B & \longrightarrow & S^{-1}B \\ \varphi\uparrow & & \varphi\uparrow \\ A & \longrightarrow & S^{-1}A \end{array}$$

$\mathrm{Spec}\, B \leftarrow \mathrm{Spec}(S^{-1}B)$ injective, homeo on image.

$$\begin{array}{ccc} C & \twoheadrightarrow & C/C\cdot\varphi m_y \\ \varphi\uparrow & & \uparrow \\ m_y \hookrightarrow \mathcal{O}_{Y,y} & \twoheadrightarrow & \kappa(y) \end{array}$$

$\mathrm{Spec}\, C \hookleftarrow \mathrm{Spec}\, C/C\cdot\varphi m_y$ injective, homeo on image. ⊠.

Corollary. Let $\begin{array}{c} Y \\ \downarrow g \\ X \xrightarrow{f} S \end{array}$ be in (Sch). Then $\mathrm{set}(X \times_S Y) =$

$= \{(x, s, y, p) : x \in X, y \in Y, s \in S,$ s.t. $fx = s = gy$, and $p \in \mathrm{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y))\}$.

Exercise II.3.4. Let $f : X \to Y$ be finite, and $U \subset Y$ open affine.

To prove: $f^{-1}U \subset X$ is affine, and $\mathcal{O}(f^{-1}U)$ is f.g. as $\mathcal{O}(U)$-module.

Proof Given: $\forall y \in Y\ \exists U_y \subset Y$ open affine s.t. $f^{-1}U_y$ is affine and $\mathcal{O}(f^{-1}U_y)$ is f.g. as $\mathcal{O}(U_y)$-module.

Let $y \in U$. Let $g \in \mathcal{O}(U)$ s.t. $y \in D(g) \subset U \cap U_y$. Then $f^{-1}D(g) \subset f^{-1}U_y$ is affine and $\mathcal{O}(f^{-1}D(g))$ is f.g. as $\mathcal{O}(D(g))$-module: $\square \downarrow$, $D(g) \subset U_y$.

$$\mathcal{O}(f^{-1}(D(g))) = \mathcal{O}(D(g)) \otimes_{\mathcal{O}(U_y)} \mathcal{O}(f^{-1}U_y).$$

As U is q.c., cover U with finitely many these $D(g_i)$.

Then the $(f^{-1}U)_{g_i}$ cover $f^{-1}U$ and are affine.

By Exercise II.2.17, $f^{-1}U$ is affine.

The f.g. as $\mathcal{O}(U)$-module: Let A be a ring, $g_1, \ldots, g_r$ in A s.t. $Ag_1 + \cdots + Ag_r = A$, M an A-module s.t. $\forall i$ M_{g_i} is f.g. as A_{g_i}-mod. Then M is f.g. as A-module. Proof. $\forall i$, let $\{\frac{m_{i,j}}{g_i^{n_{i,j}}} : j \in J_i\}$ be a finite set of gen's. Prove that $\{m_{i,j} : 1 \le i \le r, j \in J_i\}$ generates M.

⊠

Exercise II.3.8. Let X be an integral scheme. Let $\eta \in X$ be the generic point. Then $\mathcal{O}_{X,\eta} =: K$ is a field. (function field of X)

Let $K \hookrightarrow L$ be a field ext'n.

$\forall\, U \subset X$ open affine ($U \neq \emptyset$) let $\widetilde{\mathcal{O}(U)} \subset L$ be the integral closure of $\mathcal{O}(U)$ in L.

Claim. The Spec $\widetilde{\mathcal{O}(U)}$ can be glued naturally.

More precise: let U, V be open affine, $\neq \emptyset$.

∃! isom. that makes the diagram commutative.

$$\begin{array}{ccccc} \operatorname{Spec}\widetilde{\mathcal{O}(U)} \supset & \pi_U^{-1}(U\cap V) & \overset{\varphi}{\dashrightarrow} & \pi_V^{-1}(U\cap V) & \subset \operatorname{Spec}(\widetilde{\mathcal{O}(V)}) \\ \downarrow \pi_U & \searrow & & \swarrow & \downarrow \pi_V \\ \operatorname{Spec}\mathcal{O}(U) = U \supset & & U\cap V & & \subset V = \operatorname{Spec}\mathcal{O}(V) \end{array}$$

We construct φ locally on $U \cap V$ (as $U\cap V$ is not nec. affine ...).

Let $x \in U \cap V$. Take $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$ s.t. $D_U(f) = D_V(g) \ni x$.

Then ("localisation commutes with normalisation"):

$$\pi_U^{-1} D(f) = \operatorname{Spec}\left(\widetilde{\mathcal{O}(U)} \otimes_{\mathcal{O}(U)} \mathcal{O}(U)_f\right) = \operatorname{Spec}\left(\widetilde{\mathcal{O}(U)_f}\right) =$$

$$= \operatorname{Spec}\left(\widetilde{\mathcal{O}(V)_g}\right) = \pi_V^{-1} D(g).$$

$A \subset K \hookrightarrow L \supset \tilde{A}$ (integral closure of A in L), $f \in A$

then $\widetilde{(A_f)} = (\tilde{A})_f$: "$\supset$": $\frac{b}{f^n}$, $b \in \tilde{A}$, $b^m + a_{m-1} b^{m-1} + \dots + a_0 = 0$; divide by f^{nm}: b/f^n int. over A_f.

"$\subset$": Let $b \in L$ be integral over A_f: $b^n + \frac{a_{n-1}}{f^m} b^{n-1} + \dots + \frac{a_0}{f^m} = 0$ in L.

multiply by f^{nm}. Then $f^m b$ int. / A. ⊠ Hence $b \in (\tilde{A})_f$.

Application: construction of curves / fields in terms of their function fields.

$$\begin{array}{ccc} & k(x)[y]/(f) & \\ & \uparrow & \\ \mathbb{P}^1_k \sim & k(x) & \end{array} \qquad f = y^n + f_{n-1} y^{n-1} + \dots + f_0, \; f_i \in k(x).$$

irreducible.

Yoneda's lemma.

We follow Stacks chapter 4, up to § 4.3.

Give the example $\mathcal{C} = \mathrm{Open}(X)$, X in (Top).

The important point of lemma 4.3.5 is that to construct a morphism from X to Y in $\mathcal{C}$ it is equivalent to construct a morphism from h_X to h_Y in $PSh(\mathcal{C})$, that is, to construct maps, $\forall T$ in $\mathcal{C}$, $X(T) \to Y(T)$, functorial in T.

Example 1. $X = \mathrm{Spec}\ \mathbb{Z}[x,y,z]/(x^6+y^6-z^6)$

$Y = \mathrm{Spec}\ \mathbb{Z}[x,y,z]/(x^3+y^3-z^3)$

Then, $\forall$ ring A, $X(A) = \{(a,b,c) \in A^3 : a^6+b^6 = c^6\}$

$Y(A) = \{(a,b,c) \in A^3 : a^3+b^3 = c^3\}$.

So we have $X(A) \to Y(A)$, $(a,b,c) \mapsto (a^2,b^2,c^2)$, and this is functorial in A. Hence $\exists!\ X \xrightarrow{f} Y$ giving this.

It corresp. to $x^2 \leftarrow x$, $y^2 \leftarrow y$, $z^2 \leftarrow z$, indeed $x^6+y^6-z^6 \ldots \leftarrow x^3+y^3-z^3$

Example 2. Let $n \in \mathbb{Z}_{\geq 1}$.

$GL_n : (\text{Ring}) \longrightarrow (\text{Grp})$, $A \mapsto GL_n(A) = \{g \in M_n(A) : g \text{ invertible}\}$

$(\text{Grp}) \downarrow (\text{Set})$

is represented by $\mathrm{Spec}\Big(\underbrace{\mathbb{Z}[\{x_{i,j}, y_{i,j} : 1 \leq i,j \leq n\}]/(x \cdot y - 1)}_{H \text{ ("Hopf-algebra")}}\Big)$.

Indeed: $\mathrm{Hom}_{\text{Ring}}(H, A) = \{(g,h) \in M_n(A)^2 : g \cdot h = 1\}$

$\downarrow \wr$ bijection $\quad (g,h) \mapsto g$.

$GL_n(A)$

We have $GL_n(A) \times GL_n(A) \longrightarrow GL_n(A)$, $(g_1, g_2) \mapsto g_1 g_2$

$\quad\quad\quad\quad \| \quad\quad\quad\quad\quad$ functorial in A.

$(GL_n \times_{\mathrm{Spec}\,\mathbb{Z}} GL_n)(A)$

Clearly, it is easier to think about $GL_n \times_{\mathrm{Spec}\,\mathbb{Z}} GL_n \to GL_n$ in terms of the functor than in terms of LRS.

Question for audience to think about: what is $\mathbb{P}^n(A)$? <u>We'll see it soon.</u>