In other words: an action of $\mathbb{G}_{m,A}$ on M is a set of orthogonal idempotents in $\mathrm{End}_A(M)$, or, in yet other words, a $\mathbb{Z}$-grading of M: $M = \bigoplus_{i\in\mathbb{Z}} M_i$ $\quad (M_i = \alpha_i M)$

This decomposition is the one in eigenspaces for the $\mathbb{G}_{m,A}$-action!

Namely: $(\alpha u)\cdot(b\otimes(\alpha_j m)) = \sum_i (u^i b)\otimes(\alpha_i \alpha_j m) = (u^j b)\otimes(\alpha_j m)$
$= u^j\cdot(b\otimes \alpha_j m)$.

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Consequence: now you know how to grade $\mathrm{Hom}_A(M,N)$, $M\otimes_A N$, etc. if M & N are graded! And you understand why one speaks of "weights". Exercise: prove that $\mathrm{Hom}(\mathbb{G}_{m,A}, \mathbb{G}_{m,A}) =$

$= \mathrm{Hom}_{(\mathrm{Top})}(\mathrm{Spec}\, A, (\mathbb{Z}, \text{discrete}))$.

$= (\mathbb{Z}_{\mathrm{Spec}A})(\mathrm{Spec}(A))$, global sections of const. sheaf $\mathbb{Z}$ on $\mathrm{Spec}A$.

Let $A \to S$ in (Ring). What is a $\mathbb{G}_{m,A}$-action on S:

well: it is an action on the A-module S such that $\forall B, \forall u \in B^\times$ αu is a B-algebra automorphism of $B\otimes_A S$.

Hence: it is a $\mathbb{Z}$-grading of S as A-algebra:

$$S = \bigoplus_{i\in\mathbb{Z}} S_i, \quad \text{and} \quad \begin{cases} S_i\cdot S_j \subset S_{i+j}. \\ 1 \in S_0. \end{cases} \quad \left(\alpha u: \begin{array}{l} s_i \mapsto u^i s_i \\ s_j \mapsto u^j s_j \\ \text{so } s_i s_j \mapsto u^{i+j} s_i s_j \\ 1 \mapsto u^0\cdot 1 \end{array} \right).$$

We say S is positively graded if $\begin{cases} \forall i<0: S_i = \{0\}. \\ S_0 \end{cases}$

For $f\in S_d$, $S \to S_f$, $\forall B$: $B\otimes_A(S_f) = (B\otimes_A S)_f$,

$\forall u \in B^\times$ $\quad f \mapsto u^d\cdot f$

$$\begin{tikzcd} B\otimes_A S \arrow[r] \arrow[d, "\alpha u"', "\wr"] & (B\otimes_A S)_f \arrow[d, "\exists!"', "\wr"] \\ B\otimes_A S \arrow[r] & (B\otimes_A S)_f = (B\otimes_A S)_{u^d\cdot f} \end{tikzcd}$$

So we get a unique $\mathbb{Z}$-grading on S_f, such that $\frac{1}{f} \in (S_f)_{-d}$.

(Another way to see this: $S_f = S[x]/(x.f-1)$ homogeneous of degree 0 if $x \in (S[x])_{-d}$)

Now we are ready for the Proj-construction.

<u>$\mathrm{Proj}(S)$ for S a graded ring</u>, II.2 in [H], §20.7 in [Stacks], edix/talks/2010-04-12.pdf. EGA II §2–3.

I follow my own text (which can be seen as an extract from EGA).

Let S be a positively $\mathbb{Z}$-graded ring: $S = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} S_i$.

Let $d \in \mathbb{Z}_{>0}$ and $f \in S_d$, $\varphi_f : S \to S_f$ the localisation morph. in (Ring). Then S_f is $\mathbb{Z}$-graded, $\frac{1}{f} \in (S_f)_{-d}$. Let $S_{(f)} := (S_f)_0$, and put $D_+(f) := \mathrm{Spec}(S_{(f)})$. The idea is now to glue all $D_+(f)$.

So, let $e \in \mathbb{Z}_{>0}$, $g \in S_e$. Then $f^e/g^d \in S_{(g)}$, $g^d/f^e \in S_{(f)}$, hence we have natural isomorphisms:

$$(S_{(f)})_{g^d/f^e} = S_{(fg)} = (S_{(g)})_{f^e/g^d} \quad (\text{all inside } S_{fg}).$$

Hence $D_+(fg)$ is a principal open in $D_+(f)$ and in $D_+(g)$.

This gives indeed a glueing datum, where S is doing our administration: for $c \in \mathbb{Z}_{>0}$, $h \in S_c$: $(S_{(f)})_{\frac{g^d}{f^e}\cdot\frac{h^d}{f^c}} = S_{(fgh)} = (S_{(g)})_{\frac{f^e}{g^d}\cdot\frac{h^e}{g^c}} = \cdots$

So we can define: $\mathrm{Proj}(S) :=$ the scheme obtained by this.

$$\boxed{\mathcal{O}(D_+(f)) = S_{(f)}}$$

<u>Example</u> $\mathbb{P}^n_{\mathbb{Z}} := \mathrm{Proj}\, \mathbb{Z}[x_0, \ldots, x_n]$, with $\mathbb{Z}[x_0, \ldots, x_n]$ graded by total degree of monomials.

Then $\mathbb{P}^n = \bigcup_{0 \leq i \leq n} D_+(x_i)$, and $D_+(x_i) = \mathrm{Spec}\, \mathbb{Z}\left[\left\{ \frac{x_j}{x_i} : j \neq i \right\}\right]$, exactly as when we discussed $\mathbb{P}^n(k)$ as k-variety, $k = \bar{k}$.

$$\begin{array}{ccccc} \mathbb{A}^{n+1} - 0(\mathrm{Spec}\,\mathbb{Z}) & = & U & \subset & \mathbb{A}^{n+1} \\ & & \downarrow & \nwarrow \exists\ \text{section.} & \\ & & \mathbb{P}^n & \supset & D_+(x_i) \end{array}$$

Let $U := \bigcup_{f \in \bigcup_{d>0} S_d} D(f) \subset \mathrm{Spec}(S)$

Then $U = \mathrm{Spec}(S) - V(S_{>0})$.

for $T \subset \bigcup_{d>0} S_d$ s.t. $V(T) = V(S_{>0})$: $U = \bigcup_{f \in T} D(f)$

And $\forall\, d, f$:

$$\begin{array}{ccc} D(f) & \subset & U \\ \downarrow & & \downarrow \\ D_+(f) & \subset & \mathrm{Proj}(S) \\ & & \\ S_f & \hookleftarrow & S_{(f)} \end{array}$$

is quotient for $\mathbb{G}_m$-action. The action is <u>free</u>!, <u>provided</u> $f \in S_1$. (to be explained) assume $f \in S_1$.

Indeed: $f \cdot \circlearrowright S_f$ is an isomorphism of $S_{(f)}$-modules, giving isoms $(S_f)_i \xrightarrow{\sim} (S_f)_{i+1}$

Hence:
$$\begin{array}{ccc} S_{(f)}[x, x^{-1}] & \longrightarrow & S_f \\ x & \longmapsto & f \end{array}$$
is an isomorphism of $S_{(f)}$-alg's.

In terms of schemes:
$$\begin{tikzcd} \mathbb{G}_m \times D_+(f) \arrow[rr, leftrightarrow, "\sim"] \arrow[dr] & & D(f) \arrow[dl] \\ & D_+(f) & \end{tikzcd}$$
, and this is compatible with the $\mathbb{G}_m$-action.

Claim For every A in (Ring), the $A^\times$-action on $(D(f))(A)$ is free, and the quotient is $(D(f))A \to (D_+(f))A$.

Proof
$$\begin{tikzcd} S_f \arrow[r, "\varphi"] & A \\ S_{(f)} \arrow[u, hook] & \end{tikzcd}$$
, $u \in A^\times$: $u \cdot \varphi = \varphi$ on $S_{(f)}$, $u \cdot \varphi : f \mapsto u \cdot \varphi(f)$ $(f \in S_1)$.

Given φ_1 and φ_2 s.t. $\varphi_1 = \varphi_2$ on $S_{(f)}$, $\exists!\, u \in A^\times$ s.t. $\varphi_2 = u \cdot \varphi_1$ $(\varphi_2 f = u \cdot (\varphi_1 f)$, indeed$)$.

And: given $\psi : S_{(f)} \to A$, we have $\tilde{\psi} : S_f \to A$, $f \mapsto 1$.

To do list: 1. $\mathbb{P}^n(A)$, 2. $\mathbb{P}^n/\mathbb{Z}$ is proper, 3. "functoriality" of $\mathrm{Proj}(\cdot)$.

1. Let S be in (LRS). As $\mathbb{P}^n = \bigcup_i D_+(x_i)$, we get:

$$\mathrm{Hom}_{LRS}(S, \mathbb{P}^n) = \left\{ ((S_i)_i, (P_i)_i) : \begin{array}{l} S_i \subset S \text{ open}, \ \bigcup_i S_i = S, \\ P_i \in \mathcal{O}(S_i)^{n+1}, \ P_{i,i} = 1, \\ \forall i,j : \exists u \in \mathcal{O}(S_i \cap S_j)^\times : u \cdot P_i = P_j \text{ in } \mathcal{O}(S_i \cap S_j)^{n+1} \\ \forall i,j : S_i \cap S_j = \text{the open subset of } S_j \text{ where } (P_i)_j \text{ is invertible.} \end{array} \right\}$$

$$P \longmapsto \left((P^{-1} D_+(x_i))_i, \left(\text{standard lift}\right)_i \right)$$

2. Finite type and separated: exercise.

Universal closedness. Let A be a ring, $Z \subset \mathbb{P}^n_A$ closed.

Consider this:
$$\begin{tikzcd} \mathbb{A}^{n+1}_A & U \arrow[l, phantom, "\supset"] \arrow[d, "q"] & q^{-1}Z \arrow[l, hook'] \arrow[d] \\ & \mathbb{P}^n_A \arrow[d, "f"] & Z \arrow[l, hook'] \\ & \mathrm{Spec}\, A \arrow[uul, bend left, "0"] & \end{tikzcd}$$

$\mathrm{Cone}(Z) := 0(\mathrm{Spec}\, A) \cup q^{-1}Z$, closed in $\mathbb{A}^{n+1}_A$.

$I := I(\mathrm{Cone}(Z)) \subset A[x_0, \ldots, x_n]$,

I is graded: $I = \bigoplus_{d \geq 0} I_d$.

Let now $p \in \operatorname{Spec} A$, $p \notin fZ$. We want to show: $\exists\, g \in A - p$ s.t. $D(g) \cap fZ = \phi$.

We have $Z_p = \phi$, $\operatorname{Cone}(Z)_p = \{0\} \subset \mathbb{A}^{n+1}_{\kappa(p)}$.

$I_{\kappa(p)} := I(\operatorname{Cone}(Z)_p) \subset \kappa(p)[x_0, \dots, x_n]$ is generated by the image of I.

Hilbert's NS: $(x_0, \dots, x_n) = \sqrt{I_{\kappa(p)}}$, hence $\exists\, d \geq 1$ s.t. $\kappa(p)[x_0, \dots, x_n]_d = (I_{\kappa(p)})_d$.

Nakayama's lemma, applied to $M = A_p[x_0, \dots, x_n]_d$ and the submodule $N = (I_p)_d$, says that $A_p[x_0, \dots, x_n]_d = (I_p)_d$.

Hence $\exists\, g \in A - p$ s.t. $\forall i$, $x_i^d \in (I_g)_d \subset A_g[x_0, \dots, x_n]_d$ $\quad (A_g = A[g^{-1}])$.

Hence over $D(g)$: $\operatorname{Cone}(Z) = 0(\operatorname{Spec} A)$. $\boxtimes$.

<u>Lemma (Nakayama)</u>. Let A be a local ring, M a finitely generated A-module, $N \subset M$ a submodule such that $k \otimes_A N \twoheadrightarrow k \otimes_A M$, then $N = M$.

<u>Proof</u>. Let $N \subset M$ be a subm. Suppose that $N \neq M$. Then N is contained in a maximal submodule $N' \subsetneq M$ (here we use that M is finitely generated: let M be generated by $m_1, \dots, m_n$, then a submodule $M' \subset M$ is equal to M iff $\forall i$ $m_i \in M'$, then do Zorn, just like for maximal ideals.).

Now M/N' is a <u>simple</u> A-module, hence is iso. to k.

(let $0 \neq x \in M/N'$, then $A \cdot x = M/N'$, hence $M/N' \cong A/I$ for some ideal I, but then $I = m$). Hence

$$\begin{tikzcd} N' \arrow[r, phantom, "\subset"] \arrow[d] & M \arrow[r, two heads] \arrow[d] & M/N' \cong k \\ V \arrow[r, phantom, "\underset{\text{codim}\,1}{\subset}"] & k \otimes_A M = M/mM \arrow[ur, two heads] & \end{tikzcd}$$

<u>Conclusion</u>: $N \longrightarrow k \otimes_A M$ is not surjective.

$\boxtimes$

3. $\operatorname{Proj}(\cdot)$ is <u>not</u> functorial in S, just as linear maps between k-vector spaces $f: V \to W$ do not induce $f: \frac{V - 0}{k^\times} \longrightarrow \frac{W - 0}{k^\times}$.

One can consider $\frac{V - \ker f}{k^\times} \longrightarrow \frac{W - 0}{k^\times}$.

For us: $S' \overset{\varphi}{\twoheadrightarrow} S$ gives $\operatorname{Proj}(S') \xleftarrow{\operatorname{Proj}\varphi} \operatorname{Proj}(S)$. Check it <u>yourself</u>.