Week 8, 2011/10/27.

Sheaves of modules, [H] II.5, see also [Stacks 6.6, 6.10, 13, 19.7 etc.]

There's a lot of material. So it is important to understand what it is good for, all this. We have seen: geometry is anti-equiv. to rings, both are non-linear, "difficult". Modules are the linear representations of rings (compare with group theory), hence they are a tool for studying rings. Vice versa, we often get rings from a module. Good example: for k a field, V a fin. dim. k-vect. sp. and $u: V \to V$ k-linear, we get the ~~subring~~ k-algebra of $\mathrm{End}_k(V)$ generated by u, very useful. Sheaves of modules are input for the machinery of homological algebra, cohomology. Therefore, "coefficients" we also have to discuss "additive categories", "abelian categories".

Let $(X, \mathcal{O})$ be a ringed space.

A presheaf of $\mathcal{O}$-modules is a pair $(\mathcal{F}, \cdot)$ with $\mathcal{F}$ a presheaf of abelian groups on X, and $\cdot : \mathcal{O} \times \mathcal{F} \to \mathcal{F}$ a morphism of presheaves that makes every $\mathcal{F}(U)$ into an $\mathcal{O}(U)$-module.

Morphisms: the obvious notion. Category: $\mathrm{PMod}(\mathcal{O})$.

A sheaf of $\mathcal{O}$-modules is a $(\mathcal{F}, \cdot)$ in $\mathrm{PMod}(\mathcal{O})$ with $\mathcal{F}$ a sheaf.

Category: $\mathrm{Mod}(\mathcal{O})$, or $\mathcal{O}$-mod, or ... .

Example 1. Let X be in (Top), $\mathbb{Z}_X$ the constant sheaf associated to $\mathbb{Z}$. Then every sheaf of ab. grps is naturally a $\mathbb{Z}_X$-module:

$\mathrm{Ab}(X) = \mathbb{Z}_X\text{-mod}.$

Example 2. Let X be a smooth manifold, with sheaf of functions $\mathcal{O} = C^\infty_{X,\mathbb{R}}$.

Let $(E \xrightarrow{p} X,\ E\times_X E \xrightarrow{+} E,\ \mathbb{R}\times E \xrightarrow{\cdot} E)$ be a smooth vector bundle on X.

Then we get what is called the $\mathcal{O}$-module of local sections $\mathcal{E}$ associated to E/X as follows: $(\underset{\text{open}}{U\subset X}) \longmapsto \mathcal{E}(U) = E(U) = \left\{ \begin{array}{c} s \nearrow E \\ \downarrow \\ U \overset{\text{incl.}}{\hookrightarrow} X \end{array},\ s \text{ smooth} \right\}$

As E/X can locally be trivialised, $\mathcal{E}$ is a locally trivial $\mathcal{O}$-module:

$\forall x\in X,\ \exists\ \underset{x}{\underset{\Downarrow}{U}}\subset X,\ \exists\ n\in\mathbb{Z}_{\geq 0},\ \exists\ \varphi : (\mathcal{O}|_U)^n \xrightarrow{\sim} \mathcal{E}|_U$.

In fact, the category of vector bundles on X is equivalent to that of loc. free $\mathcal{O}$-modules of finite rank.

To get E back from $\mathcal{E}$: $E_x = p^{-1}\{x\} = \mathbb{R}\otimes_{\mathcal{O}_x}\mathcal{E}_x$, ~~etc~~.

Of course, this also works for topological manifolds and for complex analytic ones. Not all $\mathcal{O}$-modules are locally free (unless $\dim_X = 0$). For $i: Y \to X$ a closed immersion (of manifolds) we have $I_Y \rightarrowtail \mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_Y$, $i_*\mathcal{O}_Y$ and I_Y are $\mathcal{O}_X$-modules.

Example 3. (p. 110 of [H]). Let A be a ring, M an A-module.

We define $\widetilde{M}$ on $\operatorname{Spec} A$, a presheaf, at first:

$$\widetilde{M}(U) = \left\{ s : U \to \coprod_{p\in U} M_p \ :\ \forall p\in U,\ s(p)\in M_p,\ \begin{array}{l}\text{locally, } s \text{ is a fraction} \\ m/f,\ m\in M \\ \quad f\in A-p\end{array} \right\}$$

Then $\widetilde{M}$ is a sheaf, of $\mathcal{O}$-modules.

Proposition. (a) $\forall\, p\in\operatorname{Spec}A$, $\widetilde{M}_p = M_p$.

(b) $\forall\, f\in A$, $\widetilde{M}(D(f)) = M_f = A_f\otimes_A M$

Proof Just as we did for $\mathcal{O}$ on $\operatorname{Spec} A$. $\boxtimes$.

Def Let $(X,\mathcal{O})$ be a scheme, $\mathcal{F}$ in $\mathcal{O}$-mod. Then $\mathcal{F}$ is called quasi-coherent if $\forall x\in X$, $\exists\, U\subset X$ affine open such that $x\in U$, and $\exists$ an $\mathcal{O}(U)$-module M s.t. $\mathcal{F}|_U \cong \widetilde{M}$ as $\mathcal{O}_U$-modules. Category: $\mathrm{QCoh}(\mathcal{O})$.

Remark $\exists$ notion of quasi-coherent $\mathcal{O}$-modules for ringed spaces, see [Stacks, 13.10], or [EGA, ~~I~~ 0.5].

Proposition ([H] II.5.2, 5.5) Let A be a ring, $X = \operatorname{Spec} A$.

Then $A\text{-mod} \rightleftarrows Q\mathrm{Coh}(\mathcal{O}_X)$ is an equivalence of

$$M \longmapsto \tilde{M}$$
$$\mathcal{F}(X) \longleftarrow\!\shortmid \mathcal{F}$$

categories, compatible with: structure of abelian category, tensor product, colimits, pushforward & pullback for $(A \xrightarrow{\varphi} B) \leadsto (\operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A)$.

But to make sense of this, we have to introduce these notions. That we do in the context of ringed spaces, or LRS if that is better.

Most general morphisms.

$$\mathcal{O}_X \xleftarrow{f^\#} \mathcal{O}_Y \qquad (RS) \qquad \mathcal{F} \text{ in } \mathcal{O}_X\text{-mod}$$
$$X \xrightarrow{f} Y \qquad\qquad \mathcal{G} \text{ in } \mathcal{O}_Y\text{-mod}$$
$$\mathcal{F} \xleftarrow{\varphi} \mathcal{G}$$

we want: $\forall\, U \subset Y$, $\forall\, s \in \mathcal{G}(U)$, $\forall\, g \in \mathcal{O}_Y(U)$: $\varphi(g \cdot s) = (f^\# g) \cdot (\varphi s)$ in $\mathcal{F}(f^{-1}U)$. The set of such φ, given $(f, f^\#)$: $\operatorname{Hom}_{(f,f^\#)}(\mathcal{G}, \mathcal{F})$.

It is the same as $\operatorname{Hom}_{\mathcal{O}_Y\text{-mod}}(\mathcal{G}, f_* \mathcal{F})$.

We go back for a moment to [H] II.1, for $\boxed{f_* \text{ and } f^{-1}}$.

Let $f: X \to Y$ in (Top).

For $\mathcal{F}$ in $Sh(X)$, (sheaves of sets) we have $f_*\mathcal{F}: \underset{\cup}{Y} \atop U \mapsto \mathcal{F}(f^{-1}U)$ in $Sh(Y)$ (check that $f_*\mathcal{F}$ is a sheaf) (very easy).

For $\mathcal{G}$ on Y, pulling back is a bit more difficult.

Let $\mathcal{G}$ be in $PSh(Y) = PSh(\mathrm{Opens}(Y))$. We define a presheaf $f_p^{-1}\mathcal{G}$ on X by: $(f_p^{-1}\mathcal{G})\,U := \operatorname{colim}_{V \supset fU} \mathcal{G}(V)$

this is a good example of a filtered colimit of a system indexed by a partially ordered set $\begin{matrix} V_1 \supset fU \\ V_2 \supset fU \end{matrix}$ $\quad V_1 \cap V_2 \supset fU \quad$ $\begin{matrix} \mathcal{G}(V_1) \searrow \\ \mathcal{G}(V_2) \nearrow \end{matrix} \mathcal{G}(V_1 \cap V_2)$.

General notion of limit and colimit:

[Stacks] 4.13. In our case above: I ("index category") $= \begin{pmatrix} \text{Opens in } Y \text{ containing} \\ fU \end{pmatrix}^{\mathrm{opp}}$, the functor is $\mathcal{G}$ and the target category is (Sets).

Special case: let $y \in Y$, $X = \{y\}$, f = inclusion, $U = X$.

Then $(f_p^{-1}\mathcal{G})\,U = \mathcal{G}_y$, the stalk of $\mathcal{G}$ at y.

Now, even if $\mathcal{G} \in Sh(Y)$, $f_p^{-1}\mathcal{G}$ need not be in $Sh(X)$.

Example: Y is a one point space, X a 2 point space with discrete top.

So, one "sheafifies" $f_p^{-1}\mathcal{G}$. Let us discuss this briefly.

[Stacks, 6.17]

Sheafification. Let X be in (Top), $\mathcal{F}$ in $PSh(X)$.

Then $\exists$ $\mathcal{F} \xrightarrow{\theta} \mathcal{F}^\#$ with $\mathcal{F}^\#$ in $Sh(X)$, s.t.

$$\begin{tikzcd} & \mathcal{F}^\# \arrow[d, dashed, "\exists!"] \\ \mathcal{F} \arrow[ur, "\theta"] \arrow[r] & \mathcal{G} \text{ in } Sh(X) \end{tikzcd}$$

In other words: $\mathrm{Hom}_{PSh(X)}(\mathcal{F}, \mathcal{G}) \underset{\text{via } \theta}{=} \mathrm{Hom}_{Sh(X)}(\mathcal{F}^\#, \mathcal{G})$

In terms of adjoints: $i: Sh(X) \to PSh(X)$ the inclusion.

then $\mathrm{Hom}_{PSh(X)}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Hom}_{Sh(X)}(\mathcal{F}^\#, \mathcal{G})$; $\#$ is a left-adjoint of i.

$(\forall \mathcal{F},\ \mathcal{F}^\#$ represents $\mathrm{Hom}(\mathcal{F}, i(\cdot)))$.

See [H] II.1 or [Stacks] for ~~the~~ a construction of $\mathcal{F} \mapsto \mathcal{F}^\#$.

Good to know: $\forall x \in X$: $(\mathcal{F}^\#)_x = \mathcal{F}_x$.

Back to $f: X \to Y$, $\mathcal{G}$ in $Sh(Y)$.

Then we define $f^{-1}\mathcal{G} := (f_p^{-1}\mathcal{G})^\#$.

This looks horribly complicated, but for $\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$ and $\mathcal{G}$ on Z a sheaf we have $(gf)^{-1}\mathcal{G} = f^{-1}g^{-1}\mathcal{G}$,

and therefore (take $X = \{y\}$, $y \in Y$) $(g^{-1}\mathcal{G})_y = \mathcal{G}_{gy}$. (or do it directly).

<u>More adjointness</u>: $\mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_f(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F})$.

f^{-1} = left adj. of f_*, f_* = right adjoint of f^{-1}.

<u>Kernels & cokernels in $\mathcal{O}$-mod</u>. Let $(X, \mathcal{O})$ be a ringed space.

For $\varphi: \mathcal{F} \to \mathcal{G}$ in $\mathcal{O}$-mod we define:

$\ker\varphi : U \mapsto \ker(\varphi(U): \mathcal{F}U \to \mathcal{G}U)$, it is in $\mathcal{O}$-mod,

and it <u>is</u> a category-theoretical kernel:

$$\begin{tikzcd} \ker\varphi \arrow[r, "i"] \arrow[rr, bend left, "0"] & \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \\ & \mathcal{H} \arrow[ul, dashed, "\exists!\psi'"] \arrow[u, "\psi"] \arrow[ur, "0"'] & \end{tikzcd}$$

Note: $\mathrm{Hom}_{\mathcal{O}\text{-mod}}(\mathcal{F}, \mathcal{G})$ is a $\mathbb{Z}$-module (actually a $\mathcal{O}(X)$-module)

composition: $\mathrm{Hom}_{\mathcal{O}\text{-mod}}(\mathcal{G}, \mathcal{H}) \times \mathrm{Hom}_{\mathcal{O}\text{-mod}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{O}\text{-mod}}(\mathcal{F}, \mathcal{H})$ is $\mathbb{Z}$-bilinear,

$\mathcal{O}$-mod has finite direct sums (and they're the same as direct products).

This all together: $\mathcal{O}$-mod is additive and has kernels.