Week 9. 2011/11/03.

Today's first goal: the notions in the Proposition on page 24.

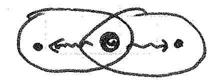
(We start with "kernels" on the previous page.)

Now cokernels.

They are easy in $P\mathrm{Mod}(\mathcal{O})$: for $\varphi: \mathcal{F} \to \mathcal{G}$, $(\mathrm{coker}_p \varphi)\,\mathcal{U} = \mathrm{coker}(\varphi\mathcal{U})$.

But if $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\mathrm{coker}_p\varphi$ need not.

We consider some simple examples.

<u>Example 1</u> (the minimal one). Let $X = \{-1, 0, 1\}$ with minimal open neighborhoods $\{-1,0\}$ of 0, $\{0\}$ of 0, $\{0,1\}$ of 1: 

So: -1 and 1 are closed, 0 is generic. ($X \cong \mathrm{Spec}(S^{-1}\mathbb{Z})$, $S = \{n \in \mathbb{Z} : \gcd(n,6) = 1\}$)

Diagram of opens of X:

$$\begin{tikzcd}[column sep=small] & \{-1,0,1\} & \\ \{-1,0\} \arrow[ur, phantom, "\subset" sloped] & & \{0,1\} \arrow[ul, phantom, "\supset" sloped] \\ & \{0\} \arrow[ul, phantom, "\supset" sloped] \arrow[ur, phantom, "\subset" sloped] & \\ & \emptyset \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

So, a presheaf $\mathcal{F}$ on X is a commutative diagram:

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B \arrow[dr] & & C \arrow[dl] \\ & D \arrow[d] & \\ & E & \end{tikzcd}$$

$\mathcal{F}_{-1} = B$
$\mathcal{F}_1 = C$
$\mathcal{F}_0 = D$.

in $\mathbb{Z}$-mod, say. And this diagram is a sheaf if and only if $E = 0$, and

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B \arrow[dr] & & C \arrow[dl] \\ & D & \end{tikzcd}$$

is cartesian. This statement is very easy to prove, because there are only very few covers of the opens in X.

Now let $Y \xrightarrow{i} X$ be the inclusion of the closed subset $\{-1, 1\}$ in X, and give Y the induced topology (that is, the discrete topology).

And let $\mathcal{U} := \{0\}$, open in X, and $j: \mathcal{U} \to X$ the inclusion (open immersion).

Then we have: $j_!\mathbb{Z}_{\mathcal{U}} \rightarrowtail \mathbb{Z}_X \longrightarrow i_*\mathbb{Z}_Y$ in $\mathbb{Z}_X$-mod.

($j_!$: see the suggested exercise II.1.19 in [H])

$$\begin{tikzcd}[column sep=small] & 0 \arrow[dl] \arrow[dr] & & & & \mathbb{Z} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & & & & \mathbb{Z}\times\mathbb{Z} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ 0 \arrow[dr] & & 0 & \arrow[r, tail, "\varphi"] & \mathbb{Z} \arrow[dr, "\mathrm{id}"'] & & \mathbb{Z} \arrow[dl, "\mathrm{id}"] & \arrow[r, "\psi"] & \mathbb{Z} \arrow[dr] & & \mathbb{Z} \arrow[dl] \\ & \mathbb{Z} \arrow[d] \arrow[ur, leftarrow] & & & & \mathbb{Z} \arrow[d] & & & & 0 \arrow[d] & \\ & 0 & & & & 0 & & & & 0 & \end{tikzcd}$$

<u>Note</u>: ψ is surjective on the stalks, but not on global sections.

Then $\operatorname{coker}_p \varphi = \left(\mathbb{Z}_X \to \left(\begin{tikzcd} & \mathbb{Z} \arrow[dl, "id"'] \arrow[dr, "id"] & \\ \mathbb{Z} \arrow[dr] & & \mathbb{Z} \arrow[dl] \\ & 0 \arrow[d] & \\ & 0 & \end{tikzcd} \right) \right)$, $(\operatorname{coker}_p \varphi)^\# = \psi$.

<u>Example 2.</u> (from algebr. geom.) Let $k = \bar{k}$ (alg. closed field).
Let $X = \mathbb{P}^1(k)$ (alg. var. $/k$), $i : Y \to X$ the closed imm. of $\{0,1\}$ into X, and consider $I_Y \overset{\varphi}{\rightarrowtail} \mathcal{O}_X \xrightarrow{\psi = i^\#} i_* \mathcal{O}_Y$, with $\varphi = \ker(\psi)$.
Then ψ is surjective on stalks, but not on global sections:
$\mathcal{O}_X(X) = k$, constant functions, $I_Y(X) = 0$, $(i_* \mathcal{O}_Y)(X) = \mathcal{O}_Y(Y) = k \times k$.
We will see in a moment that $\psi = \operatorname{coker} \varphi$, but right now we can see that $\operatorname{coker}_p \varphi$ is not a sheaf: cover X with $X - \{0\}$, $X - \{a\}$, $a \notin \{0,1,\infty\}$, for example.

<u>Proposition</u> (see [H], II.1, p. 64–65). Cokernels exist in $\mathcal{O}$-mod.
<u>Proof.</u> For $\varphi : \mathcal{F} \to \mathcal{G}$, take
$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \arrow[rr] \arrow[dr] & & (\operatorname{coker}_p \varphi)^\# \\ & & \operatorname{coker}_p \varphi \arrow[ur] & \end{tikzcd}$$
That this works is formal. $\boxtimes$

<u>Def.</u> $\varphi : \mathcal{F} \to \mathcal{G}$ is <u>surjective</u> if $\operatorname{coker} \varphi = 0$.
<u>Lemma.</u> φ is surjective iff $\forall\, U \subset X$ open, $\forall\, g \in \mathcal{G}(U)$, $\exists$ open cover $(U_i)_{i \in U}$ of U and $g_i \in \mathcal{F}(U_i)$ s.t. $\varphi g_i = g|_{U_i}$ in $\mathcal{G}(U_i)$.
(that is: iff. φ is "locally surjective").
<u>Proof</u>: exercise, or look it up. Note that this does not refer to stalks.
<u>Lemma.</u> $\forall\, x \in X$, $(\operatorname{coker} \varphi)_x = \operatorname{coker}(\varphi_x : \mathcal{F}_x \to \mathcal{G}_x)$, $(\ker \varphi)_x = \ker(\varphi_x)$
(taking stalks is exact). Also: φ surjective $\iff$ φ_x is surjective, $\forall\, x \in X$.
<u>Lemma.</u> Let $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathcal{O}$-mod, such that $\ker \varphi = 0$ and $\operatorname{coker} \varphi = 0$ (that is, φ is injective and surjective). Then φ is an isomorphism.
<u>Proof.</u> Let $U \subset X$, $g \in \mathcal{G}(U)$. As $\varphi : \mathcal{F}U \to \mathcal{G}U$ is injective, g has at most one pre-image. We show that is has one. Let $(U_i)_{i \in I}$ be an open cover of U and $f_i \in \mathcal{F}(U_i)$ s.t. $\varphi f_i = g|_{U_i}$ in $\mathcal{G}(U_i)$. Then $\varphi f_i|_{U_i \cap U_j} = \varphi f_j|_{U_i \cap U_j}$, hence ($\mathcal{F}(U_i \cap U_j) \to \mathcal{G}(U_i \cap U_j)$ injective), $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$.
Hence $\exists!\, f \in \mathcal{F}(U)$ s.t. $\forall i$: $f|_{U_i} = f_i$. Then $\forall i : (\varphi f)|_{U_i} = g|_{U_i}$, hence $\varphi f = g$
$\boxtimes$

Conclusion $\mathcal{O}$-mod is an abelian category:

- Hom's are $\mathbb{Z}$-modules and composition is $\mathbb{Z}$-bilinear
- it has finite direct sums and direct products (final object = empty dir. pr. initial obj. = empty dir. sum)
- it has kernels and cokernels
- f is an isomorphism $\Longleftrightarrow$ ($\ker\varphi = 0$ and $\operatorname{coker}\varphi = 0$)

So, all terminology about and results on abelian categories can be applied to $\mathcal{O}$-mod, for example, additive functors, left-exact, right-exact, exact. (Homological algebra, derived functors, cat's ...).

$\forall x \in X$: $\mathcal{F} \mapsto \mathcal{F}_x$, $\mathcal{O}\text{-mod} \to \mathcal{O}_x\text{-mod}$, is exact.

~~Next thing: $\otimes$ and $\mathcal{H}om$. ([H], II.5, p. 109)~~

In any abelian category:

$$\begin{tikzcd} K \arrow[r, "\ker f"] & A \arrow[r, "f"] \arrow[d, "\operatorname{cok}(\ker f)"'] & B \arrow[r, "\operatorname{cok}(f)"] & C \\ & \operatorname{coim}(f) \arrow[r, "\sim"] & \operatorname{im}(f) \arrow[u, "\ker(\operatorname{cok}(f))"'] & \end{tikzcd}$$

"canonical factorisation of f" into mono $\circ$ iso $\circ$ epi

or: sub $\circ$ iso $\circ$ quot.

Next thing: $\mathcal{H}om$, $\otimes$, f^*. [H], II.5, p. 109–110.

For $\mathcal{F}, \mathcal{G}$ in $Sh(X)$, $\mathcal{H}om(\mathcal{F}, \mathcal{G}) : U \mapsto \operatorname{Hom}_{Sh(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$ is in $Sh(X)$. ("sheaf-Hom").

For $\mathcal{F}, \mathcal{G}$ in $\mathcal{O}$-mod, $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) : U \mapsto \operatorname{Hom}_{\mathcal{O}|_U\text{-mod}}(\mathcal{F}|_U, \mathcal{G}|_U)$ is in $\mathcal{O}$-mod.

For $\mathcal{F}, \mathcal{G}$ in $\mathcal{O}$-mod, $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} := \left(U \mapsto \mathcal{F}U \otimes_{\mathcal{O}U} \mathcal{G}U\right)^{\#}$

facts (exercise):
- $(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{G}_x$.
- $\mathcal{O} \otimes_{\mathcal{O}} \mathcal{G} = \mathcal{G}$ (so if $\mathcal{F}$ is locally free of finite rank, then, locally, $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is easy to understand bec. locally it is a finite direct sum of copies of $\mathcal{G}$).
- if $X = \operatorname{Spec} A$: $\tilde{M} \otimes_{\mathcal{O}} \tilde{N} = (M \otimes_A N)^{\sim}$, so, on schemes, and $\mathcal{F}$ and $\mathcal{G}$ in $QCoh(\mathcal{O})$, $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is "easy" to understand, at least locally.

- if $X = \operatorname{Spec} A$ and M is of finite presentation,
  then $\forall$ mult. system $S \subset A$: $S^{-1}\operatorname{Hom}_A(M,N) \longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$
  is bijective (use a fin. pres. of M + left-exactness of $\operatorname{Hom}(\cdot, N)$ ...)
  hence (use $M \mapsto \tilde{M}$ is equiv.) $\operatorname{Hom}_A(M,N) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\tilde{M}, \tilde{N})$ is isom.

Let $X \xrightarrow{f} Y$ in (RS), and $\mathcal{G}$ in $\mathcal{O}_Y$-mod.

$\mathcal{O}_X \xleftarrow{f^\#} \mathcal{O}_Y$ Then $\underbrace{(f, f^\#)^* \mathcal{G}}_{=\, f^*\mathcal{G}} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$.

This looks horribly complicated, but there is some good news.

- $\forall x \in X$: $(f^*\mathcal{G})_x = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,fx}} \mathcal{G}_{fx}$, so we "understand" the stalks.
- if X & Y affine, $\mathcal{G}$ quasi-coherent, then $f^*\mathcal{G} = \widetilde{\left(\mathcal{O}_X(X) \otimes_{\mathcal{O}_Y(Y)} \mathcal{G}(Y)\right)}$.
- $\forall \mathcal{F}$ in $\mathcal{O}_X$-mod: $\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$.
- if X & Y affine, $\mathcal{F}$ in $QCoh(\mathcal{O}_X)$, then $f_*\mathcal{F}$ is in $QCoh(\mathcal{O}_Y)$,
  in particular, $f_*\mathcal{F} = \widetilde{\mathcal{F}(X)^{\mathcal{O}_Y(Y)}}$. (verify it on the $D(g)$, $g \in \mathcal{O}_Y(Y)$).

To finish, an example "from nature" (non-pathological, I mean), of a ring A, a f.g. A-module M, s.t. $\forall\, p \in \operatorname{Spec} A$, M_p is free, but M is not locally free. (if M is of finite presentation, ~~the~~ and all M_m are free, then M is locally free)

Let $X = S^1$, or any C^∞-manifold, compact, of dim. ≥ 1.

Let $A := C^\infty(X)$. For $x \in X$ we have $m_x \rightarrowtail A \overset{ev_x}{\twoheadrightarrow} \mathbb{R}$, and all max. ideals of A are of this form (exercise; let $I \subset A$ not in any m_x, ...).

The functor (loc. free C^∞_X-mod. fin. rank) $\longrightarrow$ (loc. free A-mod. f.r.), $\mathcal{E} \mapsto \mathcal{E}(X)$ is an equivalence of categories. So far: nice properties.

Let $x \in X$.
$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A_{m_x} \arrow[dl, dashed, "\varphi"] \\ C^\infty_{X,x} & \end{tikzcd}$$
Then: β is surjective (any germ at x can be extended; this could not be done with analytic functions...), $\ker \beta = \{f : f = 0 \text{ on a neighb. of } x\} = \ker \alpha$, bec. $\ker \alpha = \{f : \exists\, g \notin m_x \text{ s.t. } f \cdot g = 0\}$. Hence φ is injective and surjective! Hence α is surjective. Let $M = A_{m_x}$, then M is fin. gen., and $\forall y \in X$: $M_{m_y} = S_y^{-1} S_x^{-1} A = \begin{cases} 0 & \text{if } y \neq x \\ M & \text{if } y = x. \end{cases}$