<u>Week 10, November 10.</u>

<u>First topic today</u>: <u>quasi-coherent $\mathcal{O}$-modules on $\mathrm{Proj}(S)$</u>. ([H], II.5)

Let S be a positively $\mathbb{Z}$-graded ring: $S = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} S_i$.

Let M be a graded S-module: $M = \bigoplus_{i \in \mathbb{Z}} M_i$, $S_i \times M_j \to M_{i+j}$.

(in terms of $\mathbb{G}_m$-actions: $\mathbb{G}_{m,\mathbb{Z}}$ acts on (S, M).)

Recall that $\mathrm{Proj}(S) = \bigcup D_+(f)$, union over (f,d), $d \in \mathbb{Z}_{>0}$, $f \in S_d$,

where $D_+(f) = \mathrm{Spec}(S_{(f)})$; we glued them $D_+(f) \cap D_+(g) = D_+(fg)$.

Now, for each f we have the $S_{(f)}$-module $M_{(f)}$, hence the q.coh. $\mathcal{O}$-module $\widetilde{M_{(f)}}$ on $D_+(f)$. Just like the $D_+(f)$, these $\widetilde{M_{(f)}}$ glue, to an $\mathcal{O}$-module $\widetilde{M}$ on $\mathrm{Proj}(S)$, by construction q.coherent.

<u>Example</u>: Serre's twisting sheaves $\mathcal{O}(n)$, $n \in \mathbb{Z}$.

For $n \in \mathbb{Z}$, let $S(n)$ be the graded S-module: $S(n)_i := S_{i+n}$

(the grading has been "shifted"). Then $\mathcal{O}(n) := \widetilde{S(n)}$.

<u>Proposition</u> [H, II 5.12(a)]. If S is generated by S_1 as algebra over $A := S_0$, then all $\mathcal{O}(n)$ are locally free of rank 1, and $\mathcal{O}(n) \otimes_{\mathcal{O}} \mathcal{O}(m) = \mathcal{O}(n+m)$.

<u>Proof</u>. We have $\mathrm{Proj}(S) = \bigcup_{f \in S_1} D_+(f)$.

Let $n \in \mathbb{Z}$.

For $f \in S_1$: $\mathcal{O}(n)|_{D_+(f)} = \widetilde{S(n)_{(f)}}$, $S(n)_{(f)} = (S_f)_n \xleftarrow[\sim]{f^n} (S_f)_0 = \underset{\mathcal{O}(D_+(f))}{\overset{\shortparallel}{S_{(f)}}}$

So: $\mathcal{O}(n)|_{D_+(f)}$ is free of rank 1, generated by f^n. So, $\mathcal{O}(n)$ is loc. free of rank 1. Moreover, on each $D_+(f)$ we have a unique isomorphism $\mathcal{O}(n) \otimes_{\mathcal{O}} \mathcal{O}(m) \longrightarrow \mathcal{O}(n+m)$ (both are free of rank 1), $f^n \otimes f^m \mapsto f^{n+m}$, and these isomorphisms glue. $\boxtimes$

$$\left(\text{on } D_+(fg): \quad \begin{tikzcd} f^n \otimes f^m \arrow[r, mapsto] \arrow[d, "(g/f)^{n+m}"'] \arrow[dr, phantom, "/\!/"] & f^{n+m} \arrow[d, "(g/f)^{n+m}"] \\ g^n \otimes g^m \arrow[r, mapsto] & g^{n+m} \end{tikzcd} \right)$$

<u>Definition</u>. For $\mathcal{F} \in \mathcal{O}\text{-Mod}$ on $\mathrm{Proj}(S)$, S gen. by S_1 as S_0-algebra,

For $n \in \mathbb{Z}$, $\mathcal{F}(n) := \mathcal{O}(n) \otimes_{\mathcal{O}} \mathcal{F}$. (<u>this</u> operation is called twisting; locally, $\mathcal{F}$ and $\mathcal{F}(n)$ are isomorphic, but not necessarily globally).

One can also start with an $\mathcal{O}$-module $\mathcal{F}$ on $\mathrm{Proj}(S)$, and produce a graded S-module as follows:

$$\begin{tikzcd} \mathbb{G}_m \circlearrowright U \arrow[r, hook, "j"] \arrow[d, "q"'] & \mathrm{Spec}(S) \\ \mathrm{Proj}(S) & \end{tikzcd}$$

$\mathcal{F} \longmapsto (j_* q^* \mathcal{F})(\mathrm{Spec}\, S)$, the $\mathbb{G}_m$-action gives the grading. This is denoted $\Gamma_*(\mathcal{F})$ in [H], II.5 (p. 118).

We can also view $\widetilde{M}$ on $\mathrm{Proj}(S)$ in this way: $\left(q_*(\widetilde{M}|_U)\right)^{\mathbb{G}_m}$.

Proposition II.5.15 of [H] says that for S generated by S_1 as S_0-algebras one has, for all $\mathcal{F}$ in $\mathrm{QCoh}(\mathcal{O}_{\mathrm{Proj}\, S})$: $\widetilde{\Gamma_*(\mathcal{F})} = \mathcal{F}$.

But it is not always true (usually not) that $M \longrightarrow \Gamma_*(\widetilde{M})$ is an isomorphism.

So, there is not such a simple equivalence as for $\mathrm{QCoh}(\mathrm{Spec}\, A)$.

I will give an example. But first, $\Gamma(\mathbb{P}^r_A, \mathcal{O}(n))$.

<u>Prop</u> ([H], II.5.13). Let A be a ring, $r \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}$, $S := A[x_0, \dots, x_r]$.
Then $\Gamma(\mathbb{P}^r_A, \mathcal{O}(n)) = S_n = A[x_0, \dots, x_r]_n$.

<u>Proof.</u> We use the sheaf property of $\mathcal{O}(n)$ for the covering $\mathbb{P}^r_A = \bigcup_{i \in \{0,\dots,r\}} D_+(x_i)$:

$$(f_i)_i \longmapsto (f_i - f_j)_{(i,j)}$$

$$0 \longrightarrow (\mathcal{O}(n))(\mathbb{P}^r_A) \longrightarrow \prod_i S(n)_{(x_i)} \longrightarrow \prod_{(i,j)} S(n)_{(x_i x_j)}$$

(exact!)

Now $S(n)_{(x_i)}$ is free as A-module with basis: $\left\{ x_0^{d_0} \cdots x_r^{d_r} : \begin{array}{l} \forall k \neq i : d_k \geq 0 \\ d_0 + \dots + d_r = n \end{array} \right\}$

$S(n)_{(x_i x_j)}$ free, basis: $\{ x^d : \forall k \notin \{i,j\},\ d_k \geq 0,\ d_0 + \dots + d_r = n \}$.

Let $(f_i)_i$ be in the kernel. Consider the condition that $f_0 - f_r = 0$.

Write $f_0 = \sum_d f_{0,d} x^d$, with $f_{0,d} = 0$ unless $\begin{cases} d_0 + \dots + d_r = n \\ \forall i \neq 0 : d_i \geq 0 \end{cases}$

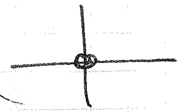
$f_r = \sum_d f_{r,d} x^d$ with $f_{r,d} = 0$ unless $\begin{cases} d_0 + \dots + d_r = n \\ \forall i \neq r : d_i \geq 0. \end{cases}$

Then we see: $f_{0,d} = 0$ unless $\begin{cases} d_0 + \dots + d_r = n \\ \forall i : d_i \geq 0. \end{cases}$

But that means: $f_0 \in A[x_0, \dots, x_r]_n$ and $\forall j : f_j = f_0$. $\boxtimes$

Now the example

Let $S = \mathbb{Q}[x_0, x_1]/(x_0 x_1)$. Then $\dim_{\mathbb{Q}}(S_n) = \begin{cases} 0 & \text{if } n<0 \\ 1 & \text{if } n=0 \\ 2 & \text{if } n \geq 1 . \end{cases}$

But $\mathrm{Proj}(S) = \mathrm{Spec}(\mathbb{Q}) \amalg \mathrm{Spec}(\mathbb{Q})$: $\Big($  $\Big) = U = \mathrm{Spec}\left(k[x_0, x_0^{-1}] \times k[x_1, x_1^{-1}]\right)$

So $\mathcal{O}(\mathrm{Proj}(S)) = \mathbb{Q} \times \mathbb{Q}$,

hence $\dim_{\mathbb{Q}} \mathcal{O}(n)(\mathrm{Proj}\, S) = 2$, for $\underline{\underline{\text{all}}}$ $n \in \mathbb{Z}$.

Second topic of today: $\mathbb{P}^n(A)$. See [H], II Thm. 7.1. But we generalise it.

Let $n \in \mathbb{Z}_{\geq 0}$, $(X, \mathcal{O})$ in (LRS).

Then $\mathrm{Hom}_{(LRS)}((X,\mathcal{O}), \mathbb{P}^n) = \left\{ (\mathcal{L}, s_0, \ldots, s_n) : \begin{array}{l} \mathcal{L} \text{ is an invertible } \mathcal{O}\text{-module}, \\ s_0, \ldots, s_n \in \mathcal{L}(X), \text{ generating:} \\ \mathcal{L}: \mathcal{O}^{n+1} \twoheadrightarrow \mathcal{L},\ f \mapsto \sum f_i s_i \end{array} \right\} \Big/ \sim$

where $(\mathcal{L}, s_0, \ldots, s_n) \sim (\mathcal{L}', s_0', \ldots, s_n') \Leftrightarrow \exists$ isom. $\varphi: \mathcal{L} \to \mathcal{L}'$ s.t. $\forall i$, $\varphi(s_i) = s_i'$.

For $(f, f^\#)$ in $\mathrm{Hom}_{(LRS)}((X,\mathcal{O}), \mathbb{P}^n)$: $\mathcal{L} := f^* \mathcal{O}(1)$, $s_i := f^* x_i$

(for $(f, f^\#): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, $\mathcal{G}$ in $\mathcal{O}_Y$-mod, $g \in \mathcal{G}(Y)$, we have $\mathcal{G}(Y) \to (f^{-1}\mathcal{G})(Y) \to (f^*\mathcal{G})(Y)$, $g \longmapsto f^* g$

for $x \in X$: $(f^*\mathcal{G})_x = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,fx}} \mathcal{G}_{fx} \longleftarrow \mathcal{G}_{fx}$

$\qquad\qquad (f^*g)_x \longleftarrow g_{fx}$)

For $(\mathcal{L}, s_0, \ldots, s_n)$ as on RHS, $X = \bigcup_i X_{s_i}$ $\left(X_{s_i} = \{ x \in X : s_i(x) \neq 0 \} \right)$, $s_i(x) \in \kappa(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x$

On X_{s_i} there are unique $f_{ij} \in \mathcal{O}(X_{s_i})$ s.t. $s_j = f_{ij} s_i$ in $\mathcal{L}(X_{s_i})$

(it is in fact nice to write $f_{ij} = \frac{s_j}{s_i}$, this makes sense, as s_i is a basis of $\mathcal{L}(X_{s_i})$ as $\mathcal{O}(X_{s_i})$-module).

Then we have:

$$\begin{tikzcd} & \mathbb{A}^{n+1} - V(x_0, \ldots, x_n) \subset \mathbb{A}^n \arrow[d, "q"] \\ X_{s_i} \arrow[ur, "f_i"] \arrow[r, "\varphi_i"'] & \mathbb{P}^n \end{tikzcd}$$

and the φ_i glue, bec. on $X_{s_i} \cap X_{s_j}$, f_i and f_j differ by a unit.

Notation for φ_i: $(s_0 : \ldots : s_n)$, this also makes sense.

The projective space $\mathbb{P}^n$ as Grassmannian: debatable if to be taken up to isomorphism. Well, not. (Aut of a quotient is trivial).

$\mathbb{P}^n(X,\mathcal{O}) = \{$ loc. free rank 1 quotients of $\mathcal{O}^{n+1}\}$

$= \{$ loc. free rank n $\mathcal{O}$-submodules M of $\mathcal{O}^{n+1}$ s.t. $\mathcal{O}^{n+1}/M$ is loc. free rank 1 $\}$.

Remark For $0 \le d \le n$, the contra-functor $(LRS) \longrightarrow (Sets)$,

$(X,\mathcal{O}) \longmapsto \{$ loc. free rank d quotients of $\mathcal{O}^n\}$ is representable by a projective scheme $/\mathbb{Z}$, $Gr_{n,d} = GL_n / \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$.

This can be proved rather easily.

[H], II.6, in particular the end of that.

Third topic of today. Picard group, effective Cartier divisors.

For $(X,\mathcal{O})$ a ringed space, let $\mathrm{Pic}(X)$ be the set of isomorphism classes of loc. free $\mathcal{O}$-modules of rank 1.

The tensor product over $\mathcal{O}$ makes it into a monoid, and, even a group.

Namely:
$$\begin{tikzcd} \mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}}(\mathcal{L},\mathcal{O}) \times \mathcal{L} \arrow[r] \arrow[d] & \mathcal{O} : (\varphi, l) \mapsto \varphi(l) \\ \mathcal{L}^\vee \otimes_{\mathcal{O}} \mathcal{L} \arrow[ur, "\sim"'] & \end{tikzcd}$$

Notation: for $n \in \mathbb{Z}$, $\mathcal{L}^{\otimes n} := \begin{cases} \mathcal{L} \otimes \cdots \otimes \mathcal{L} & n \text{ factors if } n \ge 0 \\ \mathcal{L}^\vee \otimes \cdots \otimes \mathcal{L}^\vee & -n \text{ factors if } n \le 0. \end{cases}$

In the course in the Spring of 2011 we defined, for X a smooth connected quasi-projective variety over an alg. closed field,

$\mathrm{Pic}(X) = \mathrm{Div}(X) / \mathrm{div}(k(X)^*)$.

$I_Y \rightarrowtail \mathcal{O}_X \xrightarrow{i^*} i_*\mathcal{O}_Y$

Def. Let $i : Y \to X$ be a closed immersion in (Sch). Then i is called locally principal if I_Y is, as $\mathcal{O}_X$-module, loc. free of rank 1, that is, if I_Y is locally generated by a non-zero-divisor.

This is a very useful way to produce elements of $\mathrm{Pic}(X)$.

Example. $X := \mathbb{P}^n$, $n \ge 1$, $Y := V(x_0)$. Then $\mathcal{O}_X(-1) \xrightarrow[\sim]{x_0 \cdot} I_Y \subset \mathcal{O}_X$.

Functoriality of Pic. For $f : X \longrightarrow Y$, we have

$f^* : \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X)$,
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$
$(gf)^* = f^* g^*$.