Rem. There is also a notion of Cartier divisors. On integral schemes this also leads to a Cartier divisor class group that is naturally isomorphic to $\mathrm{Pic}(X)$. We skip it.

Examples. 1. Let A be a noetherian unique fact. domain, and $X = \mathrm{Spec}(A)$. Then X satisfies the conditions, and $\mathrm{Pic}(X) = 0$, because $\forall\, Y \in P(X)$, I_Y is generated by one element.

2. Let X satisfy the hypotheses, and let $n \in \mathbb{Z}_{\geq 1}$.

Then
$$\begin{tikzcd} \mathbb{P}^n_X \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & \mathbb{P}^n \arrow[d] \\ X \arrow[r] & \mathrm{Spec}\,\mathbb{Z} \end{tikzcd}$$
satisfies the hypotheses, and
$$\mathbb{Z} \times \mathrm{Pic}(X) = \mathrm{Pic}(\mathbb{P}^n) \times \mathrm{Pic}(X) \xrightarrow[p_1^* + p_2^*]{} \mathrm{Pic}(\mathbb{P}^n_X)$$
is an isomorphism. (See [H] II. 6.4 + 6.5 & 6.6)

3. Let k be a field, E/k an elliptic curve

for example $V(-y^2z + x^3 + axz^2 + bz^3) \subset \mathbb{P}^2_k$ with $4a^3 + 27b^2 \neq 0$ if $6 \in k^\times$.

Then $E(k) \rightarrowtail \mathrm{Pic}(E) \overset{\deg}{\twoheadrightarrow} \mathbb{Z}$. to be explained.

$P \longmapsto [\mathcal{O}_E(P - 0)]$

recall: on a projective nonsing connected curve, principal divisors have degree zero.

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Some examples of morphisms to $\mathbb{P}^n$

1. Let k be a field, $n \in \mathbb{Z}_{\geq 1}$. Then $GL_{n+1}(k)$ acts on $\mathbb{A}^{n+1}_k$, as follows: for $g \in GL_{n+1}(k)$ and for $k \to A$ any k-algebra, g acts on $\mathbb{A}^{n+1}_k(A) = A^{n+1}$ by $(g, P) \mapsto g \cdot P$ (matrix times columnvector).

This action commutes with the scalar multiplication $\mathbb{G}_{m,k} \circlearrowright \mathbb{A}^{n+1}_k - \{0\}$, hence we get a faithful action $GL_{n+1}(k)/k^\times \circlearrowright \mathbb{P}^n_k$.

Thm. $\mathrm{Aut}_k(\mathbb{P}^n_k) = GL_{n+1}(k)/k^\times$.

Proof. [H], II.7.1.1. Idea. Let $\varphi : \mathbb{P}^n_k \xrightarrow{\sim} \mathbb{P}^n_k$ be an isom. of k-schemes. Then φ is given by $(\varphi^*\mathcal{O}(1), \varphi^* x_0, \ldots, \varphi^* x_n)$. As $\mathrm{Pic}(\mathbb{P}^n_k) \underset{\sim}{\overset{\deg}{\to}} \mathbb{Z}$, we see that $\varphi^*\mathcal{O}(1) \cong \mathcal{O}(1)$. Take an isom. $\psi : \varphi^*\mathcal{O}(1) \xrightarrow{\sim} \mathcal{O}(1)$.

Then $\varphi \sim (\mathcal{O}(1), \psi(\varphi^* x_0), \ldots, \psi(\varphi^* x_n))$, and use that $(\mathcal{O}(1))(\mathbb{P}^n_k) =$
$= k[x_0, \ldots, x_n]_1 = k \cdot x_0 \oplus \cdots \oplus k \cdot x_n$. $\boxtimes$ **Rem:** $\mathrm{Aut}_k(\mathbb{A}^2_k)$ is more complicated.

2. On $\mathbb{P}^n$, $(\mathcal{O}(2), x_0^2, \ldots, x_n^2)$ gives
$f\colon \mathbb{P}^n \longrightarrow \mathbb{P}^n$, $(x_0^2 : \cdots : x_n^2)$, such that $\forall A$,
$\forall (a_0 : \cdots : a_n) \in \mathbb{P}^n(A)$, $f\colon (a_0 : \cdots : a_n) \mapsto (a_0^2 : \cdots : a_n^2)$.
$\Big\}$ f is ~~locally~~ finite free of degree 2^n.

3. On $\mathbb{P}^1$, $(\mathcal{O}(2), x_0^2, x_0x_1, x_1^2)$: $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^2$, $(a_0 : a_1) \mapsto (a_0^2 : a_0a_1 : a_1^2)$
and f is a closed immersion with image $V(x_0x_2 - x_1^2)$ $\quad (\frac{x_1}{x_0})^2 \longleftarrow\!\shortmid \frac{x_2}{x_0}$
To prove it: $f^{-1}D_+(x_0) = D_+(x_0)$, $\quad \mathbb{Z}\left[\frac{x_1}{x_0}\right] \twoheadleftarrow \mathbb{Z}\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right]$, $\frac{x_1}{x_0} \longleftarrow\!\shortmid \frac{x_1}{x_0}$.
(etc.)
$\frac{a_1}{a_0} = (a_0 : a_1) \longmapsto (a_0^2 : a_0a_1 : a_1^2) = \left(\frac{a_1}{a_0}, \left(\frac{a_1}{a_0}\right)^2\right)$

4. On $\mathbb{P}^1$, $(\mathcal{O}(4), x_0^4, x_0^3x_1, x_0^2x_1^2, x_0x_1^3)$: $\mathbb{P}^1 \xrightarrow{f_1} \mathbb{P}^3$, $(a_0 : a_1) \mapsto (a_0^4 : a_0^3a_1 : a_0^2a_1^2 : a_0a_1^3)$
$(\mathcal{O}(4), x_0^3x_1, x_0^2x_1^2, x_0x_1^3, x_1^4)$: $\mathbb{P}^1 \xrightarrow{f_2} \mathbb{P}^3$, $(a_0 : a_1) \mapsto (a_0^3a_1 : a_0^2a_1^2 : a_0a_1^3 : a_1^4)$
both are closed immersions, but they do <u>not</u> differ by an automorphism of $\mathbb{P}^3$. (but they differ by an automorphism of $\mathbb{P}^1$).

5. $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$, $((a_0 : a_1), (b_0 : b_1)) \longmapsto (a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1)$
is a closed immersion.
It corresponds to: $(p_1^*\mathcal{O}(1) \otimes_{\mathcal{O}} p_2^*\mathcal{O}(1),\ p_1^*x_0 \otimes p_2^*x_0,\ p_1^*x_0 \otimes p_2^*x_1, \text{etc.})$
So, in this case, "Yoneda" gives a somewhat simpler description than via line bundles. "local".

<u>Remark</u>. We see that to know generating spaces of sections of invertible $\mathcal{O}$-modules is very good for us. Cohomology and duality is a good tool to be able to do so on more general spaces than just projective spaces and products of them. For example, think about curves of higher genus, to start with.

So, next time we go to cohomology, in [H], III.
(Actually, this will be done on December 1 & 8 by Lenny Taelman. In those two weeks, the course will end at 16:00, because he is one of the organisers of the colloquium.

Some classes of morphisms in (Sch).

We have already seen: open immersions, closed immersions, immersions, of finite type, locally of finite type, of finite presentation, locally of f. pres., finite morphisms, separated, and proper morphisms. We add a few.

Projective morphisms. [H], II.7, II.4 definition on page 103; see also [Stack], §22.40. Let $f: X \to S$ in (Sch).

- f is H-projective: $\exists\, n \in \mathbb{Z}_{\geq 0}$ and a closed immersion
$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{P}^n_S \arrow[d] \\ & S \end{tikzcd}$$
- f is locally projective: $\exists$ an open covering $S = \bigcup_{i \in I} U_i$ s.t. $\forall i \in I$, $f^{-1}U_i \to U_i$ is H-projective (see Lemma 22.40.4 [Stacks]).

Examples. Let A be a ring, $n \in \mathbb{Z}_{\geq 0}$, $(f_i)_{i\in I}$ in $A[x_0,\ldots,x_n]_{\mathrm{hom}}$, then $V_+(\{f_i : i \in I\})$ in $\mathbb{P}^n_A$ is projective over A.

[Stacks], §13.17, [H] II.9 and §7.32

Flat morphisms. Let $f: X \to Y$ be in (RS) and $x \in X$. Then f is flat at x if $f^\#: \mathcal{O}_{Y,fx} \to \mathcal{O}_{X,x}$ makes $\mathcal{O}_{X,x}$ into a flat $\mathcal{O}_{Y,fx}$-module, that is, if $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,fx}} - : \mathcal{O}_{Y,fx}\text{-mod} \to \mathcal{O}_{X,x}\text{-mod}$ is exact. Also: f is flat if it is so at all $x \in X$.

If f is flat, then $f^*: \mathcal{O}_Y\text{-mod} \to \mathcal{O}_X\text{-mod}$ is exact.

Examples. 1. For $A \to B$ in (Ring) with B flat as A-module, $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat. For example: if B is free as A-module.

2. If A is a discrete valuation ring, and M an A-module, then M is flat $\iff$ M is torsion free.

3. $\forall A$, $\forall S \subset A$ mult. syst., $A \to S^{-1}A$ is flat.

4. $\forall A$, $\forall M$ in A-mod: (M is flat and of finite presentation) $\iff$ M is locally free of finite rank. ([Stacks] Lemma 7.67.2)

5. $\mathbb{Z}^{\mathbb{N}}$ is flat as $\mathbb{Z}$-module, but not free.

6. For $A \to B$ in (Ring), M a flat A-module, $B\otimes_A M$ is flat as B-module: $\forall B$-module N: $N \otimes_B (B \otimes_A M) = N \otimes_A M$.

Smooth morphisms.

[H] III.10 only considers schemes of finite type over fields. That is really insufficient for too many things a number theorist wants to do. So we follow [Stacks] § 22.32 and § 30.7.

Def. Let $f: X \to S$ in (Sch). Then f is <u>smooth</u> if and only if $\forall x \in X$, $\exists\, U \subset X$ affine open containing x, and $\exists\, V \subset S$ affine open containing fx, and $\exists$ a presentation $\mathcal{O}(V)[x_1, \dots, x_n]/(f_1, \dots, f_c) \xrightarrow{\sim} \mathcal{O}(U)$ such that

$$g := \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_c}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_c} & \cdots & \frac{\partial f_c}{\partial x_c} \end{pmatrix} \text{ is a unit in } \mathcal{O}(U),$$

(Jacobian matrix, Jacobian criterion)

(equivalently: $\mathcal{O}(V)[x_1, \dots, x_n]/(f_1, \dots, f_c, g) = 0$).

Intuitively: locally on X and S, c (= <u>codim</u>) equations, whose gradients are linearly independent. Also: $\mathbb{A}^n_{\mathcal{O}(V)} \xrightarrow{f} \mathbb{A}^c_{\mathcal{O}(V)}$ is a <u>submersion</u>, and $U = f^{-1}0$:

$$\begin{array}{ccc} \mathbb{A}^n_{\mathcal{O}(V)} & \xrightarrow{f} & \mathbb{A}^c_{\mathcal{O}(V)} \\ \uparrow & \square & \uparrow 0 \\ U & \longrightarrow & \operatorname{Spec} \mathcal{O}(V) \end{array}$$

(<u>Remark</u>: of course, in alg. geom, all morphisms are locally given by regular functions, and therefore are differentiable. So, smoothness means something else than differentiable. It implies for example that geometric fibres are nonsingular varieties.)

<u>Remark</u> The number $n-c$ is independent of the presentation, and is called the relative dimension.

<u>Remark</u>. Smooth $\Rightarrow$ local complete intersection $\Rightarrow$ flat.

Def. Let $f: X \to S$ in (Sch). Then f is <u>formally smooth</u> if and only if $\forall\; I \rightarrowtail A \twoheadrightarrow \bar{A}$ with $I^2 = 0$, $\forall$

$$\begin{array}{ccc} \operatorname{Spec}(\bar{A}) & \longrightarrow & X \\ \downarrow & \exists \tilde{P} \nearrow & \downarrow \\ \operatorname{Spec}(A) & \xrightarrow[P]{} & S. \end{array}$$

(infinitesimal lifting property).

Thm ([Stacks], lemma 30.7.7). Let $f: X \to S$ in (Sch).
Then f is smooth $\Leftrightarrow$ (f is loc. of finite pres. & f is formally smooth).

Depending on the situation, it is advantageous to use either the Jacobian criterion or the formal smoothness to prove that an f is smooth.
If one has a presentation of f one can use the J.C.; if one understands well the functor of points of X/S, one uses formal smoothness.

Rem. It is now very simple to define <u>étale morphisms</u>: smooth of rel. dim. zero, or also: in formal smoothness $\tilde{p}$ is unique

<u>Example 1</u>. Let $k \to K$ be a finite separable field extension. Then $\operatorname{Spec} K \to \operatorname{Spec} k$ is smooth. <u>Proof</u>: $K = k[x]/(f)$ (prim. element), s.t. f has no multiple root in $\bar{k}$; $\gcd(f, f') = 1$. That's it! $\boxtimes$

2. Finite inseparable field extensions are <u>not</u> smooth.
Consider $a \in k \supset \mathbb{F}_p$, a not a pth power in k.
Then $x^p - a \in k[x]$ is irreducible (over $\bar{k}$, $a = b^p$, $x^p - a = (x-b)^p$, etc.)
Let $K := k[x]/(x^p - a)$, $k \to K$ is an insep. field ext. degree p.

$$\begin{tikzcd}
\operatorname{Spec}\left(\bar{k}[x]/(x-b)^{p}\right) \arrow[r] \arrow[d] & \operatorname{Spec} \bar{k}[x]/(x-b)^p \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K \arrow[d] \\
\operatorname{Spec} \bar{k}[x]/(x-b)^{2p} \arrow[r] \arrow[ur, dashed, "\times"] & \operatorname{Spec}(\bar{k}) \arrow[r] & \operatorname{Spec} k.
\end{tikzcd}$$

3. 

$x \mapsto \varepsilon$
$y \mapsto \varepsilon$

$$\begin{tikzcd}
k[x,y]/(xy) \arrow[r] \arrow[dr, dashed, "\times"] & k[\varepsilon]/(\varepsilon^2) \\
k \arrow[u] \arrow[r] & k[\varepsilon]/(\varepsilon^3) \arrow[u]
\end{tikzcd}$$

<u>Rem.</u> Smooth and proper (or projective) morphisms $f: X \to S$ are very important.
They combine: finiteness properties for f_* of finitely presented quasi-coherent $\mathcal{O}_X$-modules, flatness (exactness of f^*), nonsingular fibres.