AAG, LECTURE 13

If
$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$
is a short exact sequence of sheaves on a topological space X then we have an exact sequence
$$0 \to \mathcal{F}_1(X) \to \mathcal{F}_2(X) \to \mathcal{F}_3(X)$$
but it is not necessarily the case that $\mathcal{F}_2(X) \to \mathcal{F}_3(X)$ is surjective. (The surjectivity of $\mathcal{F}_2 \to \mathcal{F}_3$ implies something weaker: that for any $f \in \mathcal{F}_3(X)$ and $x \in X$ there is an open $x \in U \subset X$ and a $g \in \mathcal{F}_2(U)$ so that $g \mapsto f_{|U}$. These g's are not unique, since we can change them by sections of $\mathcal{F}_1$, so one should not expect them to glue to a global section of $\mathcal{F}_2$.) In this lecture I will define groups $\mathrm{H}^n(X, \mathcal{F})$ indexed by non-negative integers n and depending functorially on $\mathcal{F}$, called *sheaf cohomology groups*. They have the following properties. First of all $\mathrm{H}^0(X, \mathcal{F}) = \mathcal{F}(X)$ and to any short exact sequence as above there is an associated exact sequence
$$0 \to \mathcal{F}_1(X) \to \mathcal{F}_2(X) \to \mathcal{F}_3(X) \to \mathrm{H}^1(X, \mathcal{F}_1) \to \mathrm{H}^1(X, \mathcal{F}_2) \to \mathrm{H}^1(X, \mathcal{F}_3) \to \mathrm{H}^2(X, \mathcal{F}_1) \to \cdots$$
called *the long exact sequence*.

Today: generalities (homological algebra in the framework of abelian categories) and definition of sheaf cohomology groups (roughly: Hartshorne III.1, III.2). Next week: cohomology of quasi-coherent sheaves on schemes (roughly: Hartshorne III.3, III.4, III.5). In the final lecture Bas Edixhoven will prove *Serre duality*, a fundamental result about cohomology of quasi-coherent sheaves on smooth projective schemes.

Todays generalities are useful not only in algebraic geometry but also in topology, analytic geometry, differential geometry, number theory ...

Will skip many proofs, but strongly recommend you look them up and verify them. This is theory you might be using a lot in the future.

1. Complexes

Let $\mathcal{A}$ be an abelian category. Typical examples: $\mathrm{Ab}(X)$ for a topological space X and $\mathcal{O}_X - \mathrm{Mod}$ for a ringed space $(X, \mathcal{O}_X)$. A *complex* in $\mathcal{A}$ is a sequence
$$\cdots \xrightarrow{d^{-1}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots$$
such that for all n we have $d^n \circ d^{n-1} = 0$. Short notation: $d \circ d = 0$.

More precisely, our notion of complex is that of a *cochain complex*. There is also a notion of *chain complex* in which the maps go the other way, and the indices are usually written by subscripts in stead of superscripts.

The complexes in $\mathcal{A}$ form a category $\mathrm{Kom}(\mathcal{A})$. A map f from a complex A to a complex B is a commutative diagram
$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & A^{-1} & \xrightarrow{d} & A^0 & \xrightarrow{d} & A^1 & \xrightarrow{d} & \cdots \\ & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \\ \cdots & \xrightarrow{d} & B^{-1} & \xrightarrow{d} & B^0 & \xrightarrow{d} & B^1 & \xrightarrow{d} & \cdots \end{array}$$

The category $\mathrm{Kom}(\mathcal{A})$ is itself abelian.

By the "$d \circ d = 0$" property we have $\operatorname{im} d^{n-1} \subset \ker d^n$ for all n. (Better: a factorization $A^{n-1} \to \operatorname{im} d^{n-1} \to \ker d^n \to A^n$.) We define

$$\mathrm{H}^n(A) = \frac{\ker d^n}{\operatorname{im} d^{n-1}}.$$

The objects $\mathrm{H}^n(A)$ are called the *cohomology objects* of the complex A. The complex A is exact (forms an exact sequence) if and only if $\mathrm{H}^n(A) = 0$ for all n.

Warning. Do not confuse $\mathrm{H}^n(A)$ for a complex of sheaves A with the still-to-be-defined $\mathrm{H}^n(X, \mathcal{F})$ for a sheaf $\mathcal{F}$. The former is a sheaf itself (or an object of $\mathcal{A}$), the latter is an abelian group.

For every n we have that H^n is a functor from $\mathrm{Kom}(\mathcal{A})$ to $\mathcal{A}$.

The following theorem is a kind of super-snake-lemma.

Theorem 1 (The long exact sequence)**.** *To a short exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathrm{Kom}(\mathcal{A})$ *one can associate an exact sequence*

$$\cdots \to \mathrm{H}^{n-1}(C) \to \mathrm{H}^n(A) \to \mathrm{H}^n(B) \to \mathrm{H}^n(C) \to \mathrm{H}^{n+1}(A) \cdots$$

in $\mathcal{A}$*, which is functorial in the short exact sequence.*

Proof. Exercise. The snake lemma is the case where the complexes A, B, C are "concentrated in degrees 0 and 1", meaning that $A^n = B^n = C^n = 0$ unless $n \in \{0, 1\}$. The general case can be proved similarly. $\square$

2. Homotopies

Let A and B be complexes and f, g maps from A to B:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & A^{n-1} & \xrightarrow{d} & A^n & \xrightarrow{d} & A^{n+1} & \xrightarrow{d} & \cdots \\ & & {\scriptstyle f^{n-1}}\downarrow{\scriptstyle g^{n-1}} & & {\scriptstyle f^{n}}\downarrow{\scriptstyle g^{n}} & & {\scriptstyle f^{n+1}}\downarrow{\scriptstyle g^{n+1}} & & \\ \cdots & \xrightarrow{d} & B^{n-1} & \xrightarrow{d} & B^n & \xrightarrow{d} & B^{n+1} & \xrightarrow{d} & \cdots \end{array}$$

A *homotopy* h from f to g is a collection of maps $(h^n \colon A^n \to B^{n-1})_n$ such that for all n we have $f^n - g^n = dh^n + h^{n+1}d$ (or shorter: "$f - g = dh + hd$"). If there exists a homotopy from f to g then we say that f and g are homotopic. (This is an equivalence relation). Note that we do *not* demand that h commutes with the differentials.

Theorem 2. *If f and g are homotopic, then for all n they induce the same map* $\mathrm{H}^n(A) \to \mathrm{H}^n(B)$.

Proof. We will pretend the objects of $\mathcal{A}$ are modules over some ring, and therefore have "elements" with which we can do "diagram chasing". One can justify this using the Freyd embedding theorem which says that any abelian category is a full subcategory of the category of modules over some ring, or one can try to do the same argument without referring to elements. (Good luck!) For all a in

$$\mathrm{H}^n(A) = \frac{\ker(d \colon A^n \to A^{n+1})}{\operatorname{im}(d \colon A^{n-1} \to A^n)}$$

we have $(f_n - g_n)(a) = dh_n a + h_n da$. But $da = 0$ since $a \in \ker d$ and $dh_n a = 0$ in $\mathrm{H}^n(B)$ since it is in $\operatorname{im} d$. We conclude that $f_n(a) = g_n(a)$ in $\mathrm{H}^n(B)$. $\square$

A map $f\colon A \to B$ is called a *homotopy equivalence* if there is a $g\colon B \to A$ so that fg is homotopic with id_B and gf is homotopic with id_A. The above theorem now immediately implies

Corollary 1. *If f is a homotopy equivalence then f induces an isomorphism $\mathrm{H}^n(A) \to \mathrm{H}^n(B)$ for all n.*

The converse is not always true, being a homotopy equivalence is a very strong condition!

A and B are called *homotopy equivalent* if there exists a homotopy equivalence $A \to B$. (Again, this is an equivalence relation.)

3. Injective resolutions

Let M be an object of $\mathcal{A}$. A *resolution* of M is an exact sequence

$$0 \to M \to A^0 \to A^1 \to \cdots$$

in $\mathcal{A}$. It is often convenient to see this as a map of complexes. Denote by $M[0]$ the complex with

$$M[0]^n = \begin{cases} M & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

It is concentrated in degree 0. Then the above exact sequence gives rise to a map of complexes $M[0] \to A$.

Note that

$$\mathrm{H}^n(M[0]) = \mathrm{H}^n(A) = \begin{cases} M & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

An object I of $\mathcal{A}$ is called *injective* if the following equivalent conditions hold:

(1) given an injective map $f\colon M \hookrightarrow N$ in $\mathcal{A}$ and a map $h\colon M \to I$ there is a $g\colon N \to I$ such that $h = gf$;
(2) the functor $\mathrm{Hom}(-, I)\colon \mathcal{A} \to \mathrm{Ab}$ is exact.

Examples 3.1. (Exercise: verify these!)

(1) in the category of vector spaces over a field k all objects are injective;
(2) $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are injective in the category Ab of abelian groups;
(3) if R is a commutative ring then $\mathrm{Hom}_{\mathrm{Ab}}(R, \mathbf{Q}/\mathbf{Z})$ is an injective R-module.

Note that the abelian groups $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are not finitely generated, and the R-module $\mathrm{Hom}(R, \mathbf{Q}/\mathbf{Z})$ can be really huge! This is a general theme: "injective" and "finite generation" don't mix very well.

An *injective resolution* of an object M in $\mathcal{A}$ is a resolution $M[0] \to A$ where for all n we have that A^n is an injective object of $\mathcal{A}$.

Theorem 3 (Maps to injective resolutions)**.** *Let $f\colon M \to N$ be a map in $\mathcal{A}$. Let $M[0] \to A$ be a resolution and $N[0] \to I$ an injective resolution. Then there is a map of complexes $g\colon A \to I$ such that the diagram*

$$\begin{array}{ccc} M[0] & \longrightarrow & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ N[0] & \longrightarrow & I \end{array}$$

commutes and any two such maps g_1 and g_2 are homotopic.

Proof. We'll do existence, for uniqueness modulo homotopy, see Gelfand-Manin, Introduction to homological algebra, Theorem 1.1.3 and the following remarks.

We construct g inductively. In the diagram

$$\begin{array}{ccc} M & \longrightarrow & A^0 \\ \Big\downarrow f & & \\ N & \longrightarrow & I^0 \end{array}$$

the map $M \to A^0$ is injective, so from the definition of injective object we find there is a map $g^0 \colon A^0 \to I^0$ making the square commute. Now consider the induced diagram

$$\begin{array}{ccc} A^0/M & \longrightarrow & A^1 \\ \Big\downarrow g^0 & & \\ I^0/N & \longrightarrow & I^1 \end{array}$$

Again, the top map is injective so we find a map $g^1 \colon A^1 \to I^1$ making the square commute. Repeating this we construct the desired g. □

Corollary 2 (Injective resolutions are unique up to homotopy)**.** *If $M[0] \to I$ and $M[0] \to J$ are injective resolutions of M then I and J are homotopy equivalent.*

Proof. Apply the theorem to id_M in both directions. We obtain $f \colon I \to J$ and $g \colon J \to I$. But the uniqueness implies that fg is homotopic with id_J and gf with id_I. □

4. Left exact functors

A functor $F \colon \mathcal{A} \to \mathcal{B}$ of abelian categories is *additive* if for all objects M, N of $\mathcal{A}$ the induced map

$$\mathrm{Hom}_{\mathcal{A}}(M,N) \to \mathrm{Hom}_{\mathcal{B}}(FM,FN)$$

is a homomorphism of abelian groups. An additive functor F is *left exact* if for all short exact sequences

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

in $\mathcal{A}$ the induced sequence

$$0 \to FM_1 \to FM_2 \to FM_3$$

is exact.

Examples 4.1. (Exercise: verify that these are indeed left exact!)

(1) (X a topological space). $\mathrm{Ab}(X) \to \mathrm{Ab} \colon \mathcal{F} \mapsto \mathcal{F}(X)$;
(2) ($(X, \mathcal{O})$ a ringed space). $\mathcal{O} - \mathrm{Mod} \to \mathrm{Ab} \colon \mathcal{F} \mapsto \mathcal{F}(X)$;
(3) (R a ring and I an ideal). $R{-}\mathrm{Mod} \to \mathrm{Ab}$, $M \mapsto M[I] = \{m \in M \colon Im = 0\}$;
(4) (N is an object in $\mathcal{A}$). $\mathcal{A} \to \mathrm{Ab} \colon M \mapsto \mathrm{Hom}_{\mathcal{A}}(N, M)$.

This looks like a lot of examples, but really the first three are special cases of the last: take N to be $\mathbf{Z}_X$, $\mathcal{O}$, R/IR respectively.

5. Derived functors and cohomology

From now on we assume that all objects of the abelian category $\mathcal{A}$ have an injective resolution. This is the case for all the examples we have considered so far. It is however *not* the case for category of finitely generated abelian groups (the category of coherent sheaves on $\operatorname{Spec} \mathbf{Z}$). In general one should be careful not to impose too many "finiteness conditions".

Let $F\colon \mathcal{A} \to \mathcal{B}$ a left exact functor and assume that every object of $\mathcal{A}$ has an injective resolution. We will construct functors $\mathrm{R}^n F\colon \mathcal{A} \to \mathcal{B}$ for all $n \geq 0$. These are called the *derived* functors of F.

For every object M of $\mathcal{A}$ choose an injective resolution $M[0] \to I_M$. If A is a complex in $\mathcal{A}$ define FA to be the complex

$$\cdots \to FA^{-1} \to FA^0 \to FA^1 \to \cdots$$

in $\mathcal{B}$. Now consider for every M in $\mathcal{A}$ and $n \in \mathbf{Z}_{\geq 0}$ the object $\mathrm{H}^n(FI_M)$ of $\mathcal{B}$.

Proposition 1.

1. *$M \mapsto \mathrm{H}^n(FI_M)$ is a functor;*
2. *if $(M[0] \to J_M)_M$ is a second collection of injective resolutions then the functors $M \mapsto \mathrm{H}^n(FI_M)$ and $M \mapsto \mathrm{H}^n(FJ_M)$ are isomorphic.*

Proof. Let $M \to N$ be a map in $\mathcal{A}$. Then by Theorem 3 we can choose a compatible map $I_M \to I_N$. This induces a map $FI_M \to FI_N$ which in turn induces maps $\mathrm{H}^n(FI_M) \to \mathrm{H}^n(FI_M)$. The map $I_M \to I_N$ is unique up to homotopy, so also the induced map $FI_M \to FI_N$ is unique up to homotopy, and we conclude that the map $\mathrm{H}^n(FI_M) \to \mathrm{H}^n(FI_N)$ only depends on the map $M \to N$. It is straightforward to check that this indeed defines a functor.

For the second part, apply Theorem 3 to id_M and the resolutions $M[0] \to I_M$ and $M[0] \to J_M$ to obtain a *canonical* isomorphism $\mathrm{H}^n(FI) \to \mathrm{H}^n(FJ)$. □

We will from now on suppose a choice of $(M \to I_M)_M$ has been made, write $\mathrm{R}^n F(M) = \mathrm{H}^n(FI_M)$ and simply call $\mathrm{R}^n F$ *the n-th derived functor of F.*

The most important properties are the following:

Theorem 4.

1. *the functor $\mathrm{R}^0 F$ is isomorphic with the functor F;*
2. *if F is exact then $\mathrm{R}^n F(M) = 0$ for all $n > 0$ and all M;*
3. *if M is injective then $\mathrm{R}^n F(M) = 0$ for all $n > 0$;*
4. *for every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ there is a* long exact sequence

$$0 \to FM_1 \to FM_2 \to FM_3 \to \mathrm{R}^1 F(M_1) \to \mathrm{R}^1 F(M_2) \to \mathrm{R}^1 F(M_3) \to \mathrm{R}^2 F(M_1) \to \cdots$$

which depends functorially on the short exact sequence.

We can now define the cohomology groups mentioned in the introduction. For a topological space X the functor

$$\Gamma(X, -)\colon \mathrm{Ab}(X) \to \mathrm{Ab}\colon \mathcal{F} \mapsto \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$$

is left exact, and one defines $\mathrm{H}^n(X, \mathcal{F}) = \mathrm{R}^n \Gamma(X, \mathcal{F})$.

6. Acyclic resolutions

Because of their uniqueness modulo homotopy, injective resolutions are very useful for theoretical purposes. But because injective objects are typically huge, they are usually completely useless for computations. Fortunately, one can often compute derived functors using more manageable resolutions.

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. Then an object N of $\mathcal{A}$ is called F*-acyclic* if $\mathrm{R}^i F(N) = 0$ for all $i \geq 1$. An F*-acyclic* resolution of an object M in $\mathcal{A}$ is a resolution

$$M[0] \to A$$

where A^n is F-acyclic for all n.

Theorem 5. *If $F\colon \mathcal{A} \to \mathcal{B}$ is left exact, M an object of $\mathcal{A}$ and $M[0] \to A$ an F-acyclic resolution of M then there are natural isomorphisms*

$$\mathrm{R}^n F(M) = \mathrm{H}^n(FA)$$

for all n.

Proof of special case. Just to give some idea, assume A is concentrated in degree 0 and 1. In other words, we have a short exact sequence

$$0 \to M \to A^0 \to A^1 \to 0$$

and the associated long exact sequence becomes

$$0 \to FM \to FA^0 \to FA^1 \to \mathrm{R}^1 F(M) \to \mathrm{R}^1 FA^0 = 0$$

and therefore we indeed have $\mathrm{R}^n F(M) = \mathrm{H}^n(FA)$. □

Proposition 2. *If $(X, \mathcal{O})$ is a RS and M an injective $\mathcal{O}$-module then M is $\Gamma(X, -)$-acyclic as object in $\mathrm{Ab}(X)$.*

Proof. See Hartshorne III.2.4 and III.2.5. □

Corollary 3. *The right derived functors of*

$$\mathcal{O} - \mathrm{Mod} \to \mathrm{Ab}\colon \mathcal{F} \mapsto \Gamma(X, \mathcal{F})$$

are the sheaf cohomology groups $\mathrm{H}^n(X, \mathcal{F})$.