AAG, LECTURE 14

Today: mostly cohomology of quasi-coherent sheaves on schemes. Main reference: Hartshorne III.3, III.4, III.5, but we'll do a few things differently.

1. Vanishing theorem

The following is very useful in sheaf cohomology on schemes. (But not in more "topological" settings like differential geometry).

Theorem 1. *If X is a noetherian topological space of dimension n and $\mathcal{F}$ a sheaf of abelian groups on X then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all $i > n$.*

Proof. See Hartshorne III.2.7. □

2. Flasque sheaves

An abelian sheaf $\mathcal{F}$ on a topological space X is called *flasque* (or *flabby*) if for every inclusion $U \subset V$ the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

Here are some properties of flasque sheaves (see Hartshorne, exercise II.1.15, lemma III.2.4, lemma 3.4):

(1) if $\mathcal{F}$ is flasque and $0 \to \mathcal{F} \to \mathcal{A} \to \mathcal{B} \to 0$ exact then for any open U the sequence $0 \to \mathcal{F}(U) \to \mathcal{A}(U) \to \mathcal{B}(U) \to 0$ is exact;
(2) if $\mathcal{F}$ and $\mathcal{G}$ are flasque and $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0$ is exact then also $\mathcal{Q}$ is flasque;
(3) injective sheaves are flasque;
(4) if I is an injective A-module then $\tilde{I}$ is a flasque sheaf on $\operatorname{Spec} A$.

Theorem 2. *Flasque sheaves are acyclic for $\Gamma(X, -)$.*

Proof. (See Hartshorne, III.2.5) Let $\mathcal{F}$ be a flasque sheaf on X. Choose an injection $\mathcal{F} \hookrightarrow \mathcal{I}$ into an injective sheaf and let $\mathcal{Q}$ be the quotient. Then by the above properties we find that also $\mathcal{I}$ and $\mathcal{Q}$ are flasque, and we find that

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{Q}(X) \to 0$$

is exact.

We now show that for all flasque sheaves $\mathcal{F}$ on X and all $p > 0$ we have $\mathrm{H}^p(X, \mathcal{F}) = 0$, by induction on p. For $p = 1$ use the long exact sequence, the fact that $\mathcal{I}(X) \to \mathcal{Q}(X)$ is surjective, and the fact that $\mathrm{H}^1(X, \mathcal{I}) = 0$ (because $\mathcal{I}$ is injective) to obtain $\mathrm{H}^1(X, \mathcal{F}) = 0$.

For $p > 1$, note that $\mathrm{H}^{p-1}(X, \mathcal{I}) = \mathrm{H}^p(X, \mathcal{I}) = 0$ so we get from the long exact sequence an isomorphism

$$\mathrm{H}^{p-1}(X, \mathcal{Q}) \to \mathrm{H}^p(X, \mathcal{F})$$

but since $\mathcal{Q}$ is also flasque, we find $\mathrm{H}^p(X, \mathcal{F}) = 0$ by the induction hypothesis. □

In particular, we may compute sheaf cohomology by flasque resolutions.

Proposition 1. *If $(X, \mathcal{O})$ is a RS and $\mathcal{I}$ is an injective $\mathcal{O}$-module, then $\mathcal{I}$ is flasque.*

As a corollary we see that the derived functors of global sections on the category of sheaves of abelian groups, and on the category of $\mathcal{O}$-modules coincide.

Corollary 1. *The functors* $\mathrm{H}^n(X,-)$ *are the right derived functors of* $\Gamma(X,-)\colon \mathcal{O}-\mathrm{Mod} \to \mathrm{Ab}$.

3. The Cech complex

Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of a topological space X. We write

$$U_{i_1\cdots i_n} := U_{i_1} \cap U_{i_2} \cdots \cap U_{i_n}.$$

For any $p \geq 0$ and any sheaf $\mathcal{F}$ in $\mathrm{Ab}(X)$ we define

$$\check{\mathfrak{C}}^p(\mathcal{U},\mathcal{F}) = \prod_{v\in I^{p+1}} \mathcal{F}(U_v).$$

Also, define maps

$$d\colon \check{\mathfrak{C}}^p(\mathcal{U},\mathcal{F}) \to \check{\mathfrak{C}}^{p+1}(\mathcal{U},\mathcal{F})$$

as follows:

$$(ds)_{i_0 i_1\cdots i_p} = \sum_{j=0}^{p+1}(-1)^p s_{i_0\cdots\hat{i_j}\cdots i_p}.$$

Then it is a combinatorial exercise to check that $d^2 = 0$ and hence that

$$\check{\mathfrak{C}}(\mathcal{U},\mathcal{F}) = \left[\cdots \to 0 \to \check{\mathfrak{C}}^0(\mathcal{U},\mathcal{F}) \to \check{\mathfrak{C}}^1(\mathcal{U},\mathcal{F}) \to \cdots\right]$$

is a complex of abelian groups.

Proposition 2. $\mathrm{H}^0(\check{\mathfrak{C}}(\mathcal{U},\mathcal{F})) = \mathcal{F}(X)$.

Proof. Exercise: this is just a reformulation of the sheaf property. □

There is a sheaf-theoretic version of the Cech complex. For any open $V \subset X$ denote by $f\colon V \to X$ the inclusion. Then define

$$\mathcal{C}^p(\mathcal{U},\mathcal{F}) = \prod_{v\in I^{p+1}} f_*\mathcal{F}_{|U_v}$$

These form a complex $\mathcal{C}(\mathcal{U},\mathcal{F})$ of sheaves on X with the property that $\Gamma(X,\mathcal{C}) = \check{\mathfrak{C}}$.

Note that there is a natural map $\mathcal{F} \to \mathcal{C}^0(\mathcal{U},\mathcal{F})$ given by sending a section s to its restrictions on all the components.

Proposition 3. $\mathcal{F}[0] \to \mathcal{C}(\mathcal{U},\mathcal{F})$ *is a resolution of* $\mathcal{F}$.

Proof. We need to show that $(*)$

$$0 \to \mathcal{F} \to \mathcal{C}^0 \to \mathcal{C}^1 \to \cdots$$

is exact. Exactness at $\mathcal{F}$ and $\mathcal{C}^0$ follows from the sheaf property of $\mathcal{C}$. To show exactness at $\mathcal{C}^p$ we can restrict to showing exactness on stalks. So take $x \in X$. Choose a $j \in I$ so that $x \in U_j$. Define for all p a map

$$h^p\colon \mathcal{C}^p_x \to \mathcal{C}^{p-1}_x$$

defined by

$$(h^p\alpha)_{i_0 i_1\ldots i_{p-1}} = \alpha_{j i_0\ldots i_{p-1}}.$$

Note that this makes sense: for V a small enough neighborhood of x we have

$$U_{i_0\ldots i_{p-1}} \cap V = U_{j i_0\ldots i_{p-1}} \cap V.$$

Now compute that $hd + dh = \mathrm{id}$, so that the identity map from $(*)$ to itself is homotopic with zero, hence $(*)$ has trivial cohomology groups, hence is exact. □

Corollary 2. *For all p there is a natural map $\mathrm{H}^p(\check{\mathfrak{C}}(\mathcal{U}, \mathcal{F})) \to \mathrm{H}^p(X, \mathcal{F})$, functorial in $\mathcal{F}$.*

Proof. As we have seen last week, there is a morphism of complexes from any resolution to an injective resolution, in particular we have a map from the Cech resolution $\mathcal{C}$ to an injective resolution $\mathcal{I}$ of the sheaf $\mathcal{F}$. Taking global sections and then cohomology gives the desired maps. Since the map $\mathcal{C} \to \mathcal{I}$ is unique up to homotopy, the resulting maps $\mathrm{H}^p(\check{\mathfrak{C}}(\mathcal{U}, \mathcal{F})) \to \mathrm{H}^p(X, \mathcal{F})$ are well-defined.

Functoriality follows from the uniqueness modulo homotopy of $\mathcal{C} \to \mathcal{I}$. □

Another corollary of the proposition is

Corollary 3. *If $\mathcal{F}$ is flasque then for all $p > 0$ we have $\mathrm{H}^p(\check{\mathfrak{C}}(\mathcal{U}, \mathcal{F})) = 0$.*

Proof. Since $\mathcal{F}$ is flasque all the $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ are flasque, so the Čech resolution is an acyclic resolution, so its cohomology is the sheaf cohomology of $\mathcal{F}$, but this is trivial since $\mathcal{F}$ is acyclic. □

4. Alternating and ordered Čech complex

These are variations on the Čech complex that are sometimes more suitable for computations.

An element $s \in \check{\mathfrak{C}}^p(\mathcal{U}, \mathcal{F})$ is called alternating if the following properties hold:

(1) $s_{i_0 \cdots i_p} = 0$ if two indices are equal;
(2) $s_{\sigma(i_0) \cdots \sigma(i_p)} = \epsilon(\sigma) s_{i_0 \cdots i_p}$ for all permutations σ of the indices.

If s is alternating then so is ds, so the alternating s form a sub-complex denoted $\check{\mathfrak{C}}'(\mathcal{U}, \mathcal{F})$. This is sometimes called the *alternating Čech complex*.

Proposition 4. *The natural map of complexes $\check{\mathfrak{C}}'(\mathcal{U}, \mathcal{F}) \to \check{\mathfrak{C}}(\mathcal{U}, \mathcal{F})$ induces an isomorphism on the cohomology groups.*

Instead of working with the alternate one can also choose an ordering on the index set I of the covering $\mathcal{U} = (U_i)_{i \in I}$, and note that s is determined by the $s_{i_0 \cdots i_p}$ with $i_0 < \cdots < i_p$. In other words, the composition

$$\check{\mathfrak{C}}^p(\mathcal{U}, \mathcal{F}) \hookrightarrow \check{\mathfrak{C}}'^p(\mathcal{U}, \mathcal{F}) \twoheadrightarrow \check{\mathfrak{C}}''^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_i)$$

is an isomorphism. One thus obtains a complex $\check{\mathfrak{C}}''$ isomorphic with $\check{\mathfrak{C}}'$. This is sometimes called the *ordered Čech complex*. It is this last version that is used in Hartshorne.

The advantage of $\check{\mathfrak{C}}'$ and $\check{\mathfrak{C}}''$ over $\check{\mathfrak{C}}$ is that for finite covers they form complexes of finite length. For example, if $\mathcal{U} = (U, V)$ then $\check{\mathfrak{C}}''$ is the complex

$$\cdots \to 0 \to \mathcal{F}(U) \times \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0 \to \cdots$$

where the map sends a pair s, t to $s_{|U \cap V} - t_{|U \cap V}$. (Exercise: verify this!)

5. Cohomology of quasi-coherent sheaves

Theorem 3. *Let X be a separated noetherian scheme, $\mathcal{F}$ a quasi-coherent sheaf of $\mathcal{O}_X$-modules and $\mathcal{U}$ a cover of X consisting of affine opens. Then for all p the map $\mathrm{H}^p(\check{\mathfrak{C}}(\mathcal{U},\mathcal{F})) \to \mathrm{H}^p(X,\mathcal{F})$ is an isomorphism.*

We do not need that X is separated, only that all intersections of affine opens in $\mathcal{U}$ are also affine open. Also, the theorem is true without the Noetherian hypothesis, see Stacks project, Coherent cohomology, paragraph 2.

The following is a crucial ingredient:

Proposition 5. *If X is an affine scheme and $0 \to \mathcal{F} \to \mathcal{A} \to \mathcal{B} \to 0$ is an exact sequence of $\mathcal{O}_X$-modules with $\mathcal{F}$ quasi-coherent, then $0 \to \mathcal{F}(X) \to \mathcal{A}(X) \to \mathcal{B}(X) \to 0$ is exact.*

Proof. See Hartshorne, II.5.6. □

Proof of Theorem 3. For $p = 0$ this follows from the sheaf property. Let $\mathcal{F}$ be quasi-coherent and $(U_i)_i$ a finite open affine cover. For every i choose an embedding $\mathcal{F}(U_i) \to I_i$ with I_i an injective $\mathcal{O}_X(U_i)$-module. Then $\mathcal{F} \to \mathcal{G} = \oplus_i f_* \tilde{I}_i$ is an embedding in a quasi-coherent flasque sheaf. Let $\mathcal{Q}$ be the quotient, so that

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0$$

is exact. Note that $\mathcal{Q}$ is itself quasi-coherent. By the lemma we find a short exact sequence

$$0 \to \check{\mathfrak{C}}(\mathcal{U},\mathcal{F}) \to \check{\mathfrak{C}}(\mathcal{U},\mathcal{G}) \to \check{\mathfrak{C}}(\mathcal{U},\mathcal{Q}) \to 0$$

of Cech complexes. Since $\mathcal{G}$ is flasque, the higher cohomology of $\check{\mathfrak{C}}(\mathcal{U},\mathcal{G})$ vanishes, and we find $\mathrm{H}^1(\check{\mathfrak{C}}(\mathcal{U},\mathcal{F})) = 0$ because $\mathcal{G}(X) \to \mathcal{Q}(X)$ is surjective (since $\mathcal{F}$ is quasi-coherent). Moreover, we find isomorphisms $H^{p-1}(\check{\mathfrak{C}}(\mathcal{U},\mathcal{Q})) \to \mathrm{H}^p(\check{\mathfrak{C}}(\mathcal{U},\mathcal{F}))$ which allow us to prove the theorem for $p > 1$ by induction. (Using the fact that $\mathcal{Q}$ is itself quasi-coherent!) □

6. Cohomology of quasi-coherent sheaves on affine schemes

A very important corollary to Theorem 3 is:

Corollary 4. *If X is a noetherian affine scheme and $\mathcal{F}$ a quasi-coherent sheaf on X, then $\mathrm{H}^p(X,\mathcal{F}) = 0$ for all $p > 0$.*

Also true without noetherian hypothesis. Closely related to the theorem (one could also first prove the corollary and then deduce the theorem from it).

Proof. Take the trivial open affine cover (X) of X. □

7. Cohomology of $\mathcal{O}(n)$ on $\mathbf{P}^r$

In this section we do a fundamental computation: we compute the cohomology groups of $\mathcal{O}(n)$ on $\mathbf{P}^r$. For a graded module M we denote by M_n the part which is homogeneous of degree n. In the theorem below, the grading is given by

$$\deg x_0^{e_0} x_1^{e_1} \cdots x_r^{e_r} = e_0 + \cdots + e_r.$$

Theorem 4. *Let A be a ring, $r \geq 1$ and $S := A[x_0, \ldots, x_r]$. Let $n \in \mathbf{Z}$. Then*

$$\mathrm{H}^p(\mathbf{P}^r_A, \mathcal{O}(n)) = \begin{cases} S_n & \text{if } p = 0 \\ \left(\frac{1}{x_0 \cdots x_r} A[\frac{1}{x_0}, \ldots, \frac{1}{x_r}]\right)_n & \text{if } p = r \\ 0 & \text{otherwise} \end{cases}$$

Corollary 5. *If $-r-1 < n$ then $\mathcal{O}(n)$ is an acyclic sheaf on $\mathbf{P}^r_A$.*

Example 7.1. The table below contains the ranks of the free A-modules $\mathrm{H}^p(\mathbf{P}^1, \mathcal{O}(n))$:

n	$\cdots$	-3	-2	-1	0	1	2	$\cdots$
$\mathrm{rk}\,\mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(n))$	$\cdots$	0	0	0	1	2	3	$\cdots$
$\mathrm{rk}\,\mathrm{H}^1(\mathbf{P}^1, \mathcal{O}(n))$	$\cdots$	2	1	0	0	0	0	$\cdots$

Proof of Theorem 4. (See Stacks project, Coherent cohomology, paragraph 10.) The case $p = 0$ was done in Lecture 10. For the general case, we will do a Čech computation, using the ordered variant (but we will drop the double accent from the notation).

Let U_i be the affine open $D^+(x_i)$. Then $\mathcal{U} = (U_i)_{0 \leq i \leq r}$ is an affine open cover of $\mathbf{P}^r_A$. Use the standard ordering on $I = \{0, \ldots, r\}$. Let $\mathcal{C}$ be the ordered Čech complex for the sheaf $\mathcal{O}(n)$ with respect to this cover. Then we have

$$\mathcal{C}^p = \bigoplus_{i_0 < \cdots < i_p} A[x_0, \cdots, x_r, \frac{1}{x_{i_0} \cdots x_{i_p}}]_n$$

Note that all the R-modules that occur have a grading by $\mathbf{Z}^{r+1}$ by declaring the monomial $x^e = x_0^{e_0} \cdots x_r^{e^r}$ to be homogeneous of degree $e \in \mathbf{Z}^{r+1}$. Since the differential respects the grading, our complexes composes as a sum of homogeneous components

$$\mathcal{C} = \bigoplus_e \mathcal{C}(e)$$

where e runs over those $e \in \mathbf{Z}^{r+1}$ with $e_0 + \cdots + e_r = n$. We can now verify the theorem component by component, we just need to show that

$$\mathrm{H}^p(\mathcal{C}(e)) = \begin{cases} S(e) & \text{if } p = 0 \\ \frac{1}{x_0 \cdots x_r} A[\frac{1}{x_0}, \ldots, \frac{1}{x_r}](e) & \text{if } p = r \\ 0 & \text{otherwise.} \end{cases}$$

All modules in this formula are free of rank 1 (generated by x^e) or 0.

We first make $\mathcal{C}(e)$ more explicit. We have

$$\mathcal{C}^p(e) = \bigoplus_{i_0 < \cdots < i_p} \mathcal{C}^p(e, i_0 \cdots i_p)$$

where

$$\mathcal{C}^p(e, i_0 \cdots i_p) = \begin{cases} A \cdot x^e & \text{if } e_j < 0 \Rightarrow j \in \{i_0, \ldots, i_p\} \\ 0 & \text{otherwise} \end{cases}$$

Now one can check that

$$\mathcal{C}^{p-1}(e) \to \mathcal{C}^p(e) \to \mathcal{C}^{p+1}(e)$$

is exact if $0 < p < r$, and that

$$\mathrm{H}^r(\mathcal{C}(e)) = \mathrm{coker}(\mathcal{C}^{r-1}(e) \to \mathcal{C}^r(e))$$

is free of rank 1 (generated by the image of x^e) if all the e_i are negative, and trivial otherwise. □

8. Cohomology of coherent sheaves on projective schemes

Theorem 5 (Serre). *Let A be a noetherian ring, X be a scheme over A and $f\colon X \to \mathbf{P}^r_A$ a closed immersion of A-schemes. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules.*

1. *for all p we have that $\mathrm{H}^p(X, \mathcal{F})$ is a finitely generated A-module;*
2. *there exists an $N \in \mathbf{Z}$ so that for all $n \geq N$ and all $p > 0$ we have $\mathrm{H}^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{O}_{\mathbf{P}^r}(n)) = 0$.*

Note that the projectivity can not be dropped from (1)! Clearly $\mathrm{H}^0(\mathbf{A}^r_A, \mathcal{O}_{\mathbf{A}^r})$ is not a finitely generated A-module if $r > 0$. More general version of (1), due to Grothendieck: if $f\colon X \to Y$ is a proper morphism of schemes and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules then for every p the $\mathcal{O}_Y$-module $\mathrm{R}^p f_*\mathcal{O}_X$ is coherent.

Example: $X = \mathbf{P}^r$, and $\mathcal{F} = \mathcal{O}_X(q)$, then we have seen in the previous section that (i) holds, and that for (ii) we can take $N = -r - q$.

Another example: if X is projective over a field k then $\Gamma(X, \mathcal{O}_X)$ is finite-dimensional.

Proof. $f_*\mathcal{F}$ is a coherent sheaf on $\mathbf{P}^r_A$, and for all p and n we have

$$\mathrm{H}^p(\mathbf{P}^r, f_*\mathcal{F} \otimes \mathcal{O}(n)) = \mathrm{H}^p(X, \mathcal{F} \otimes f^*\mathcal{O}(n)).$$

(For example, because their Čech complexes for the standard affine open cover of $\mathbf{P}^r$, resp. the intersection of this open cover with X coincide). So we may assume $X = \mathbf{P}^r$ and $f = \mathrm{id}$.

By the computation from the previous section, we know that the theorem holds for any sheaf which is a finite direct sum of line bundles of the form $\mathcal{O}_{\mathbf{P}^r}(q)$.

To prove (i), use descending induction on p. For $p > r$ the cohomology group is trivial (use the standard affine open cover of $\mathbf{P}^r$, and the ordered Cech complex to see this). In particular it is finitely generated.

Assume the theorem holds for some p and for all coherent sheaves $\mathcal{F}$ on $\mathbf{P}^r$. For every coherent sheaf $\mathcal{F}$ there is a short exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{F} \to 0$$

where $\mathcal{B}$ is a finite direct sum of various $\mathcal{O}(q)$'s. It follows that also $\mathcal{A}$ is coherent. The long exact sequence gives an exact sequence of A-modules

$$\cdots \to \mathrm{H}^{p-1}(\mathbf{P}^r, \mathcal{B}) \to \mathrm{H}^{p-1}(\mathbf{P}^r, \mathcal{F}) \to \mathrm{H}^p(\mathbf{P}^r, \mathcal{A}) \to \cdots.$$

The module on the right is finitely generated by the induction hypothesis, the module on the left is finitely generated because $\mathcal{B}$ is a finite sum of $\mathcal{O}(q)$'s, so since the ring A is noetherian the module in the middle is also finitely generated.

To prove (ii) we use the same kind of induction. Twisting the short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{F} \to 0$ gives an exact sequence of A-modules

$$\cdots \to \mathrm{H}^{p-1}(\mathbf{P}^r, \mathcal{B} \otimes \mathcal{O}(n)) \to \mathrm{H}^{p-1}(\mathbf{P}^r, \mathcal{F} \otimes \mathcal{O}(n)) \to \mathrm{H}^p(\mathbf{P}^r, \mathcal{A} \otimes \mathcal{O}(n)) \to \cdots.$$

For n sufficiently large the module on the left is trivial because $\mathcal{B}$ is a finite sum of $\mathcal{O}(q)$'s, and for n sufficiently large the module on the right is trivial by the induction hypothesis. □