# AAG Fall 2013, take home assignment

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This is the 'take-home exam' for the advanced algebraic geometry course.

Oral examination on the 28th and 29th of January, where we will ask you questions concerning your solutions.

We want to receive your solutions by **Sunday January 26, 23:59**, either by email (pdf from tex or scan) or in our mailbox in the maths dept.

Grade out of 10. The grade is based on how many of the exercises you solved correctly, and on your answers regarding details during the oral exam.

1. Let $(X, \mathcal{O})$ be a locally ringed topological space (LRTS), and let $f \in \mathcal{O}(X)$.
   (a) Show that $D(f) \stackrel{\text{def}}{=} \{x \in X | f(x) \neq 0 \text{ in } \kappa(x)\}$ is open in $X$;
   (b) Show that $f|_{D(f)}$ is a unit in $\mathcal{O}(D(f))$.
2. Let $(X, \mathcal{O})$ be the ringed space with $X = \mathbb{R}$ (with Archimedean topology), and for all $U \subset X$ open,
$$\mathcal{O}(U) = \{f : U \to R | f \text{ is continuous}\}.$$
   (a) Is this a locally ringed space?
   (b) Are the stalks integral domains?
   (c) What if we replace continuous functions by real analytic functions?
   (d) What if we replace continuous function by smooth (i.e. infinitely differentiable) functions?
3. (a) Let $R$ be a ring and $I = (f_1, \ldots, f_m) \subset R[T_0, \ldots, T_n]$ a homogeneous ideal, where each $f_i$ is a homogeneous polynomial of degree $d_i$. Show that the following two functors are isomorphic:

$$F_1 : R\text{-Alg} \to \text{Sets}$$
$$A \mapsto \text{Hom}_{\text{Sch}_R}\left(\text{Spec}\, A, \text{Proj}\, \frac{R[T_0, \ldots, T_n]}{I}\right)$$

and

$$F_2 : R\text{-Alg} \to \text{Sets}$$

$$A \mapsto \left\{ (L; t_0, \dots, t_n) : \begin{array}{l} L \text{ a locally free } A\text{-module of rank } 1 \\ (t_0, \dots, t_n) \in L^{n+1} \\ t_0 A + \cdots + t_n A = L \\ \forall j \in \{1, \dots m\}, f_j(\underline{t}) = 0 \text{ in } L^{\otimes d_i} \end{array} \right\} / \sim$$

In particular, you should give a precise definition of the symbol '$\sim$' in the definition of $F_2$.

(b) Give an example of a $\mathbb{Z}[\sqrt{-5}]$-valued point of $\mathbb{P}^1_{\mathbb{Z}}$ which cannot be written in the form $(a_0 : a_1)$ with $a_0$ and $a_1$ in $\mathbb{Z}[\sqrt{-5}]$ and $\mathbb{Z}[\sqrt{-5}] = a_0\mathbb{Z}[\sqrt{-5}] + a_1\mathbb{Z}[\sqrt{-5}]$.

4. (a) Let $k$ be a field and $n \geq 2$ an integer. Let $d \geq 2$ be another integer, and $f \in k[x_0, \cdots, x_n]_d$ a homogeneous element of degree $d$. Set $X$ to be the subscheme of zeroes of $f$ in $\mathbb{P}^n_k$. Show that $X$ is connected. [Hint: try to compute $\mathcal{O}_X(X)$, using what you have learned about line bundles on $\mathbb{P}^n_k$ (c.f. Stacks, tag 01XS) ]

   (b) Can you generalise this to a regular sequence $f_1, \cdots, f_r$ of homogeneous elements in $k[x_0, \cdots, x_n]$ of degrees $d_1, \cdots, d_r$ with all $d_i > 0$?

5. Let $k$ be an algebraically closed field. Let $n \geq 1$, let $X = \mathbb{P}^n_k$, and let $T_X = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ be the tangent sheaf. You can use, without proof, Thm. II.8.13 of Hartshorne.

   (a) Show that for all $i \geq 0$, we have

$$c_i(T_X) = \binom{n+1}{i} \epsilon^i$$

   in $A(X)$, where $\epsilon \in A^1(X)$ is the class of a hyperplane: $\epsilon = c_1(\mathcal{O}(1))$.

   (b) Show that

$$\mathrm{td}(X) = \left( \frac{\epsilon}{1 - e^{-\epsilon}} \right)^{n+1}.$$

6. Let $\mathcal{F}$ be a coherent sheaf on $X = \mathbb{P}^2_k$, with $k$ an algebraically closed field. You may use without proof that $A(X) = \mathbb{Z}[\epsilon]/(\epsilon^3)$, with $\epsilon = c_1(\mathcal{O}(1))$. Assume that

   – $\mathrm{rank}(\mathcal{F}) = 7$;

- $c_1(\mathcal{F}) = 2\epsilon$;
- $c_2(\mathcal{F}) = 13\epsilon^2$;

(a) Compute $\chi(\mathcal{F})$;

(b) Can it happen that $\mathrm{H}^1(X, \mathcal{F}) = 0$?

(c) Compute, for all $n \in \mathbb{Z}$, the number $\chi(\mathcal{F}(n))$.