Advanced Algebraic Geometry, week 1: 2013/09/03.

1. Goal of the course: to state, and partially prove, GHRR for $f: X \to Y$ over a field k, with X and Y nonsingular quasi-projective algebr. var., and f projective.

and DETAILS

As k is not nec. algebr. closed, we use general schemes. We follow Hartshorne's book, but for generality and for precision, it is highly recommended to learn to use the Stacks Project and/or EGA, online at Numdam, at the same time.

GHRR:

$$\begin{tikzcd} K(X) \arrow[r, "ch(\cdot)\cdot T(X)"] \arrow[d, "f_!"'] \arrow[dr, phantom, "\oslash"] & A(X)_{\mathbb{Q}} \arrow[d, "f_*"] \\ K(Y) \arrow[r, "ch(\cdot)\cdot T(Y)"'] & A(Y)_{\mathbb{Q}} \end{tikzcd}$$

$K(X)$ = Groth. ring of coh. $\mathcal{O}_X$-modules,
$A(X)_{\mathbb{Q}}$ = Chow ring of X, with coeff. $\mathbb{Q}$.

For the statement and proof of GHRR we follow Borel-Serre, where $k = \bar{k}$.

Today: Hartshorne, II.1 & II.2 (read it yourself, spend ≥ 8 hrs/week on this course.)

2. Why sheaves, what is geometry?

$\exists$ many kinds of geometries / functions that are studied:
- topology / continuous functions
- diff. topology (manifolds) / smooth functions
- analytic geometry / $\mathbb{C}$ or $\mathbb{R}$-analytic functions
- algebraic geometry / polynomial functions and rat. functions

Common framework: ringed spaces, locally ringed spaces.

<u>Example</u>. Let X be a C^∞-manifold, defined in your favourite way. Then $\forall\, U \subset X$ open, we have $C^\infty_X(U) = \{f : U \to \mathbb{R} : f \text{ is } C^\infty\}$, a sub-$\mathbb{R}$-algebra of $\{f : U \to \mathbb{R}\}$, and $\forall\, V \subset U$ incl. of opens of X, we have $C^\infty_X(U) \to C^\infty_X(V)$, $f \mapsto f|_V$, a morphism of $\mathbb{R}$-algebras. Then C^∞_X is a sheaf of $\mathbb{R}$-alg. on X, and (X, C^∞_X) is a locally ringed space, "over $\mathbb{R}$", and the category of manifolds is a full subcategory of $\mathrm{LRSp}/\mathbb{R}$.

Now I want to give definitions. (SP, Ch. 6...)

<u>Def.</u> Let X be a topological space ($X \in \mathrm{Top}$).
Then $\mathrm{Open}(X) :=$ the cat. of open subsets of X, morphisms: inclusions.
A <u>presheaf</u> (of sets) on X is a functor $\mathcal{F} : \mathrm{Open}(X)^{\mathrm{opp}} \to \mathrm{Set}$.
A <u>morphism of presheaves</u>, $\varphi : \mathcal{F} \to \mathcal{G}$, on X is a morphism of functors.
Notation: $\mathrm{PSet}(X)$ is the <u>category of presheaves</u> on X.
Similarly: $\mathrm{PAb}(X)$, $\mathrm{PRing}(X)$, etc.

<u>Def.</u> Let $X \in \mathrm{Top}$ and $\mathcal{F} \in \mathrm{PSet}(X)$. Then $\mathcal{F}$ is <u>a sheaf</u> iff:
$\forall\, U \in \mathrm{Open}(X)$, $\forall$ open cover $U = \bigcup_{i \in I} U_i$,
$\forall$ collection $s_i \in \mathcal{F}(U_i)$, $i \in I$, s.t.
$\forall\, i,j \in I$: $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ in $\mathcal{F}(U_i \cap U_j)$,
$\exists!\, s \in \mathcal{F}(U)$, such that $\forall\, i \in I$, $s_i = s|_{U_i}$ in $\mathcal{F}(U_i)$.
A <u>morphism</u> $\varphi : \mathcal{F} \to \mathcal{G}$ <u>of sheaves</u> on X is a morphism of presheaves.
Notation: $\mathrm{Sh}(X)$ is the cat. of sheaves of sets on X.
Similarly: $\mathrm{Ab}(X)$, $\mathrm{Ring}(X)$.

<u>Exercises</u>: 1. Show that for $X \in \mathrm{Top}$ and $\mathcal{F} \in \mathrm{Sh}(X)$, $\mathcal{F}(\emptyset)$ is a 1 elmt. set.
2. Show that for $X \in \mathrm{Top}$ and $\mathcal{F} \in \mathrm{PAb}(X)$, $\mathcal{F}$ is a sheaf iff:
$\forall\, U \in \mathrm{Open}(X)$, $\forall$ open cover $U = \bigcup_{i \in I} U_i$, the complex

$$0 \to \mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \longrightarrow \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j) \quad \text{is exact.}$$

$$s \longmapsto (i \mapsto s|_{U_i})$$

$$(s_i)_{i \in I} \longmapsto \big((i,j) \mapsto (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})\big)$$

<u>Def.</u> For $X \in Top$, $\mathcal{F} \in PSet(X)$, $x \in X$: $\mathcal{F}_x := \operatorname{colim}_{x \in U} \mathcal{F}(U)$ is the <u>stalk</u> of $\mathcal{F}$ at x.
Explicitly: $\mathcal{F}_x = \{ (U,s) : U \in Open(X) \text{ s.t. } x \in U,\ s \in \mathcal{F}(U) \} / \sim$,
where $(U_1, s_1) \sim (U_2, s_2) \iff \exists\ (U_3, s_3)$ s.t. $U_3 \subset U_1 \cap U_2$ and
$$s_1|_{U_3} = s_3 = s_2|_{U_3}.$$

<u>Def.</u> Let $f: X \to Y$ in Top, $\mathcal{F}$ in $PSets(X)$, $\mathcal{G} \in PSets(Y)$, then an <u>f-map</u> $\varphi : \mathcal{G} \to \mathcal{F}$ is a collection of maps $\left(\varphi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V) \right)_{V \in Open(Y)}$, s.t. $\forall\ V' \subset V$ with $V, V' \in Open(Y)$:

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r, "\varphi_V"] \arrow[d, "\text{restr. of } \mathcal{G}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}(f^{-1}V) \arrow[d, "\text{restr of } \mathcal{F}"] \\ \mathcal{G}(V') \arrow[r, "\varphi_{V'}"] & \mathcal{F}(f^{-1}V') \end{tikzcd}.$$

<u>Def.</u> <u>The category</u> RSp <u>of ringed spaces</u> has as <u>objects</u> pairs $(X, \mathcal{O}_X)$ with $X \in Top$ and $\mathcal{O}_X \in Ring(X)$, and as <u>morphisms</u> pairs $(f: X \to Y,\ \varphi : \mathcal{O}_Y \to \mathcal{O}_X)$ with $f \in Top$ and φ an f-map of (sheaves of) rings.

<u>Def.</u> A <u>local ring</u> is a ring (A) with a unique maximal ideal (m).
Equivalently: a ring A is local iff $\exists\ A \twoheadrightarrow k$ with k a field, s.t. $A^\times = \{ a \in A \text{ s.t. } a \neq 0 \text{ in } k \}$.
~~For~~ A a local ring, the quotient k is unique up to unique isomorphism, and it is called the <u>residue field</u> of A.

<u>Def.</u> For $A \twoheadrightarrow k_A$ and $B \twoheadrightarrow k_B$ local rings, a ~~map~~ morphism $\varphi : A \to B$ of rings is <u>local</u> if it induces

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, two heads] & B \arrow[d, two heads] \\ k_A \arrow[r, "\bar{\varphi}"] & k_B \end{tikzcd}.$$

<u>Def.</u> The cat. $LRSp$ of locally ringed spaces has as <u>objects</u> pairs $(X, \mathcal{O}_X)$ in RSp, s.t. $\forall\ x \in X$: $\mathcal{O}_{X,x}$ is local, and as <u>morphisms</u> $(f: X \to Y,\ \varphi : \mathcal{O}_Y \to \mathcal{O}_X)$ as in RSp, s.t. $\forall\ x \in X$: $\varphi_x : \mathcal{O}_{Y,fx} \to \mathcal{O}_{X,x}$ is <u>local</u>.

Rem. Let $(X, \mathcal{O}_X) \in LRSp$, $x \in X$, $U \in \mathrm{Open}(X)$ with $x \in U$, $f \in \mathcal{O}_X(U)$.
Then we have the germ/stalk f_x of f in $\mathcal{O}_{X,x}$, and
the value $f(x)$ of f at x in $\mathcal{O}_{X,x}/m_x =: \kappa(x)$.

Exercise. 1. Let $(f, \varphi) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ in $LRSp$, $x \in X$.
Show that (f, φ) induces $\varphi_x : \mathcal{O}_{Y,fx} \to \mathcal{O}_{X,x}$, $\overline{\varphi}_x : \kappa(fx) \to \kappa(x)$.
2. Let $(X, \mathcal{O}_X) \in LRSp$, $f \in \mathcal{O}_X(X)$.
Show that $\{x \in X : f(x) = 0\}$ is a closed subset of X, notation $Z(f)$.
Let $D(f) := X - Z(f)$. Show that $f|_{D(f)} \in \mathcal{O}_X(D(f))^{\times}$.

3. Schemes.

Schemes are the geometric objects in algebr. geom. Just like varieties over a field, they are locally ringed spaces that are locally isomorphic to an affine scheme. So, we must define affine schemes; these are the generalisation of affine variety, and they correspond to arbitrary rings.
"To study rings in geometrical terms, study/use their spectra."

Let A be a ring. Then $\mathrm{Spec}\, A := \{ p \subset A : \text{prime ideal} \}$.
For $p \in \mathrm{Spec}(A)$, we have: $A \to A_p$ localisation, $a \mapsto a_p$ germ

$$A/p \hookrightarrow \mathrm{Frac}(A/p) = \twoheadrightarrow A_p/m_p = \kappa(p). \qquad a_p \mapsto a(p) \text{ value}$$

Concretely: $A_p = \{ (a,f) : a, f \in A, f \notin p \} / \sim$: $(a,f) \sim (a', f') \Leftrightarrow$
$\exists h \in A - p : h \cdot (f'a - fa') = 0$ (think of $[(a,f)]$ as $\frac{a}{f}$).

Zariski topology on $\mathrm{Spec}\, A$: for $T \subset A$ any subset,
$Z(T) := \{ p \in \mathrm{Spec}\, A : \forall f \in T : f(p) = 0 \}$. $(f(p) = 0 \Leftrightarrow f \in p)$.
$= \{ p \in \mathrm{Spec}\, A : T \subset p \} = Z(A \cdot T)$; these are the closed sets.
Indeed: $\bigcap_{i \in I} Z(T_i) = Z\left(\bigcup_{i \in I} T_i \right)$, $Z(T_1) \cup Z(T_2) = Z(T_1 \cdot T_2)$, $T_1 \cdot T_2 = \{ f_1 f_2 : f_1 \in T_1, f_2 \in T_2 \}$.
$\emptyset = Z(\{1\})$.

The structure sheaf $\mathcal{O}$: for $U \subset \operatorname{Spec} A$ open, we define:

$$\mathcal{O}(U) = \left\{ f : U \longrightarrow \coprod_{p\in U} A_p \ : \ \begin{array}{l} 1.\ \forall p\in U :\ f(p)\in A_p \\ 2.\ \forall p \in U\ \exists V\subset U \text{ open s.t. } p\in V,\ \exists g,h\in A \text{ s.t.} \\ \quad a.\ \forall q\in V,\ h(q)\neq \text{ in } \kappa(q) \\ \quad b.\ \forall q\in V,\ f(q) = \dfrac{g_q}{h_q} \text{ in } A_q \end{array}\right\}$$

In words: $f\in\mathcal{O}(U)$ is a collection of $(f(p))_{p\in U}$, locally on U of the form $p \mapsto (g/h)_p$, g, h in A.

The condition 2 is local, hence $\mathcal{O}$ is a sheaf. It is a sheaf of rings.

For $p\in \operatorname{Spec} A$, we have

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & \mathcal{O}_p \\ & A_p \arrow[u, "\text{isomorphism}"'] \end{tikzcd}$$

Hence: $\mathcal{O}_p = A_p$, and $\kappa(p) = A_p/m_p$, and $(\operatorname{Spec} A, \mathcal{O}) \in LRSp$.

<u>Example</u> 1. Let k be a field. Then $(\operatorname{Spec}(k), \mathcal{O})$ is the 1 point space with ring of functions k.

2. $\operatorname{Spec}(\mathbb{Z}) = \{ (0), \underbrace{(2), (3), (5), \dots}_{\text{maximal ideals}} \}$, <u>closed</u>: $\operatorname{Spec}(\mathbb{Z})$ and the finite sets of maximal ideals.

$$\mathcal{O}(D(n)) = \mathbb{Z}[1/n] = \mathbb{Z}[x]/(nx-1). \quad \underline{\text{open}}:\ D(n),\ n\in\mathbb{Z}.$$

3. Let k be a field. Then $\mathbb{A}^1_k := (\operatorname{Spec} k[x], \mathcal{O})$.
Then $\operatorname{Spec}(k[x]) = \operatorname{Gal}(\bar{k}/k)\backslash \bar{k} \cup \{(0)\}$.

<u>Def</u>. A <u>scheme</u> is a locally ringed space $(X, \mathcal{O}_X)$ such that each $x\in X$ has an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to some $(\operatorname{Spec} A, \mathcal{O})$. The <u>category</u> Sch <u>of schemes</u> is the full subcategory of LRSp whose objects are the schemes.

<u>Def.</u> (functoriality of Spec). Let $\varphi: A \to B$ in Ring. For $p \in \operatorname{Spec} B$, $\varphi^{-1}p \subset A$ is a prime ideal, hence a map $f := \operatorname{Spec}\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$.

For $a \in A$, $f^{-1} Z(a) = Z(\varphi a)$, hence f is continuous.

For $p \in \operatorname{Spec} B$:
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A_{\varphi^{-1}p} \arrow[r, dashed, "\exists!"] & B_p \end{tikzcd}$$
, "hence" $\forall\, U \subset \operatorname{Spec} A$ open, a ring morphism $\psi(U): \mathcal{O}_{\operatorname{Spec} A}(U) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}U)$,

hence $\underset{\operatorname{Spec}(\varphi)}{\underset{\parallel}{(f, \psi)}} : (\operatorname{Spec} B, \mathcal{O}) \to (\operatorname{Spec} A, \mathcal{O})$ in LRSp.

<u>Thm.</u> Let A be a ring, $f \in A$. Then $A_f \xrightarrow{\psi} \mathcal{O}_{\operatorname{Spec} A}(D(f))$, $\left(\frac{a}{f^n} \mapsto \left(p \mapsto \underset{\frac{a}{f^n}}{\overset{A_p}{\rotatebox{90}{$\in$}}}\right)\right)$ is an isomorphism. In particular, $\mathcal{O}(\operatorname{Spec} A) = A$.

<u>Proof.</u> Read the details in Hartshorne, II. Prop. 2.2.

You will see that it uses some preliminary results:

- the $D(h)$, $h \in A$, form a basis for the topology of $\operatorname{Spec} A$
- for $T \subset A$ a subset: $I(Z(T)) = \sqrt{A \cdot T}$.
- bijection: $\{\text{closed subsets of } \operatorname{Spec} A\} \underset{Z}{\overset{I}{\rightleftarrows}} \{\text{radical ideals of } A\}$
- $\operatorname{Spec} A$ is quasicompact; $\bigcup_{i \in I} D(h_i) = \operatorname{Spec} A \iff \sum_{i \in I} A \cdot h_i = A$.

<u>Final remarks:</u> $\text{Ring} \underset{\Gamma}{\overset{\operatorname{Spec}}{\rightleftarrows}} \text{AffSch}$ is an anti-equivalence of cat's.

Spec and Γ are right-adjoints of eachother: $\forall$ ~~...~~ $X \in$ LRSp, $\forall$ ring A:

$$\operatorname{Hom}_{\cancel{\text{Sch}}\ \text{LRSp}}(X, \operatorname{Spec} A) = \operatorname{Hom}_{\text{Ring}}(A, \mathcal{O}(X))$$

<u>So:</u> once we have decided to use LRSp, the definition of Spec(-) has no choic.

<u>Organisation:</u> problem session 14:45 – 15:30 ??

<u>Homework:</u> read Hartshorne II §1 and 2 ($\leq$ Example 2.3.6) and compare it with SP Ch. 24. ($\leq$ §6)

Recommended exercises: Hartsh. <u>II</u>. 1.22 and <u>II</u>. 2.12