

1. Recall: for $A \in \text{Ring}$, $\forall f \in A$, $\mathcal{O}_{\text{Spec} A}(D(f)) = A_f$

• $\forall f_1, \dots, f_n \in A$, $D(f_1) \cup \dots \cup D(f_n) = \text{Spec}(A) \iff \exists g_1, \dots, g_n \in A$,
 $g_1 f_1 + \dots + g_n f_n = 1$.

• for $S = \bigoplus_{d \geq 0} S_d$ a graded ring; $\text{Proj}(S) = \bigcup_{\substack{f \in \cup S_d \\ d > 0}} D_+(f)$,

$D_+(f)$ affine, $\mathcal{O}(D_+(f)) = S_{(f)}$.

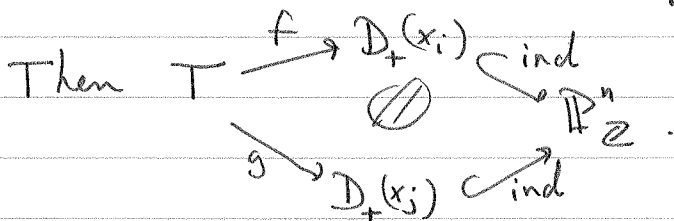
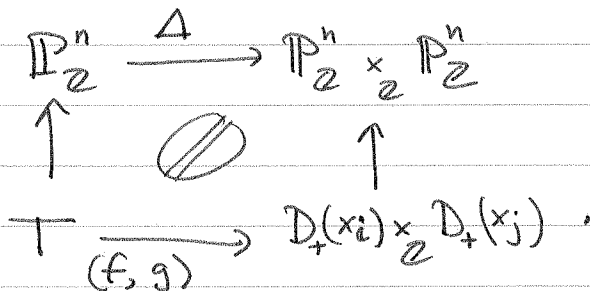
Example: $S = \mathbb{Z}[x_0, \dots, x_n]$, $\mathbb{P}_{\mathbb{Z}}^n = \text{Proj}(S) = \bigcup_{i=0}^n D_+(x_i)$,

$D_+(x_i) = \text{Spec } \mathbb{Z}[\{\frac{x_j}{x_i} : j \neq i\}] \cong \mathbb{A}_{\mathbb{Z}}^n$.

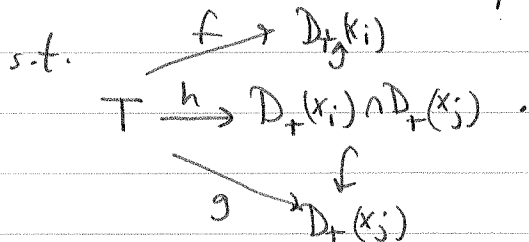
David has shown: $\mathbb{P}_{\mathbb{Z}}^n \rightarrow \text{Spec } \mathbb{Z}$ is universally closed.

2. Proper morphisms. We follow David's § 8, with a little correction to his proof of lemma 8.3:

let $0 \leq i, j \leq n$, and consider:



That means: $\exists! h: T \rightarrow D_+(x_i) \cap D_+(x_j)$



Hence: $\mathbb{P}_{\mathbb{Z}}^n \xrightarrow{\Delta} \mathbb{P}_{\mathbb{Z}}^n \times_{\mathbb{Z}} \mathbb{P}_{\mathbb{Z}}^n$

$D_+(x_i x_j) \rightarrow D_+(x_i) \times_{\mathbb{Z}} D_+(x_j)$

on rings: $\mathbb{Z}[x_0, \dots, x_n, \frac{1}{x_i x_j}] \leftarrow \mathbb{Z}[x_0, \dots, x_n, \frac{1}{x_i}] \otimes_{\mathbb{Z}} \mathbb{Z}[x_0, \dots, x_n, \frac{1}{x_j}]$

The ring map is surj; $ab \leftarrow a \otimes b$
 the scheme map is a closed immersion. \square

3. Projective and quasi-projective schemes. (SP 01W7) (27.43)
01VV (27.41)

Def. Let $f: X \rightarrow S$ be a morphism of schemes

1. We say f is H -projective if $\exists n \in \mathbb{Z}_{\geq 0}$ and a closed immersion $X \rightarrow \mathbb{P}_S^n$ over S . (H for Hartshorne).
2. We say f is H -quasi-projective if $\exists n \in \mathbb{Z}_{\geq 0}$ and a quasi-compact immersion $X \rightarrow \mathbb{P}_S^n$ over S .

Rem. 1 So an S -scheme is H -projective if it is isomorphic to a closed subscheme of some \mathbb{P}_S^n .

For A a ~~not~~ noetherian ring all closed subschemes of \mathbb{P}_A^n are of the form $V_+(f_1, \dots, f_r)$, $f_i \in A[x_0, \dots, x_n]$.

2. An S -scheme is H -quasi-projective if it is isomorphic to an open subscheme of ~~an~~ a closed subscheme of ~~some~~ \mathbb{P}_S^n for some n , with the open immersion quasi-compact.

Exercise: give an example of an open imm. that is not q -compact.

Exercise: show that for S locally noetherian, and $f: X \rightarrow S$ locally of finite type, every open immersion $j: U \rightarrow X$ is quasi-compact.

Prop. Let S be a scheme, $X \xrightarrow{f} Y$ a morphism of H -quasi-projective S -schemes.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ h \downarrow & & \downarrow g \\ S & & S \end{array}$$

Then: f is proper $\Leftrightarrow f$ is H -projective.

We need a few lemma's.

Lemma Let $X \xrightarrow{f} Y \xrightarrow{g} S$ in (Sch). If f & g are separated, then so is $g \circ f$.

Proof. $X \times_S X \xrightarrow{(f \circ p_1, f \circ p_2)} Y \times_S Y$ is Cartesian (use a test-scheme $T \rightarrow S$),

$$\begin{array}{ccc} X \times_S X & \xrightarrow{(f \circ p_1, f \circ p_2)} & Y \times_S Y \\ \uparrow (id_X \circ p_1, id_X \circ p_2) & & \uparrow (id_Y, id_Y) \\ X \times_Y X & \xrightarrow{f \circ p_1 = f \circ p_2} & Y \end{array}$$

hence:

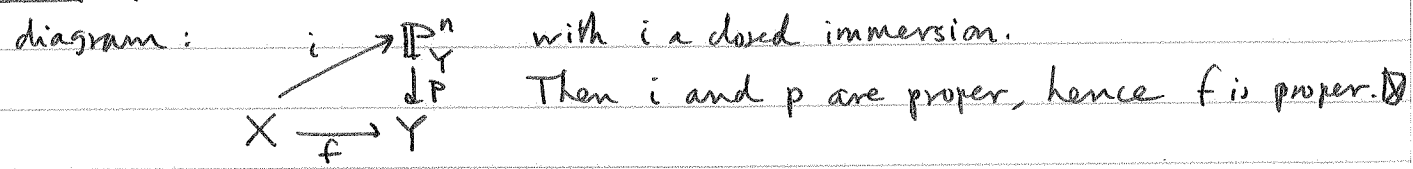
$$\begin{array}{ccc} & & Y \times_S Y \\ \Delta_{X/S} \nearrow & & \uparrow \\ X & \xrightarrow{\Delta_{X/Y}} & X \times_Y X \end{array}$$

- closed imm. stable by base change,
- cl. imm. stable under composition

Cor. Let $X \xrightarrow{f} Y \xrightarrow{g} S$ in (Sch) with f & g proper. Then $g \circ f$ is proper.

Proof. It is universally closed, of finite type, and separated. \square

Proof of " \Leftarrow " in the Proposition. We have f H-projective, hence a comm.

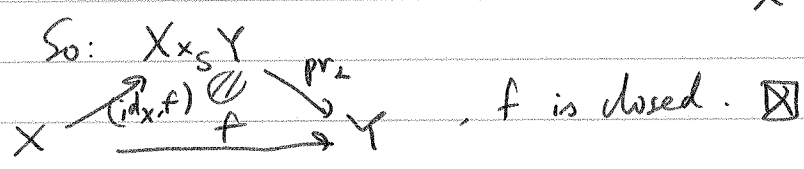
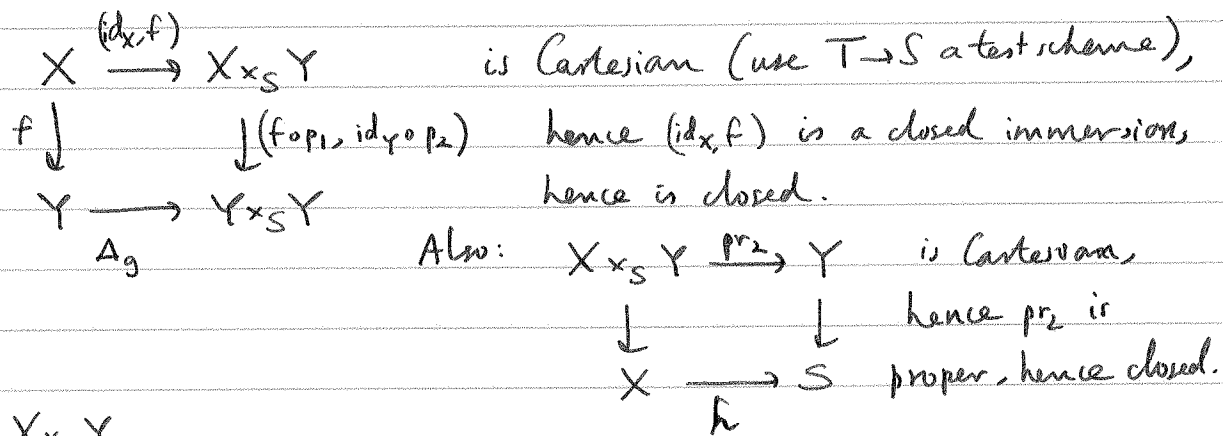


The other implication of the Proposition is proved using the following very useful Lemma, which is the analog of: $f: X \rightarrow Y$ in Top with X compact and Y Hausdorff, then f is closed.

Lemma Let $X \xrightarrow{f} Y$ in Sch with h proper and g separated.

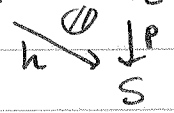


Proof

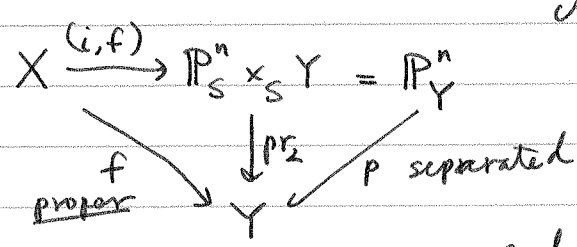


Proof of " \Rightarrow " of the Proposition. $h: X \rightarrow S$ is H-quasi-projective, so we

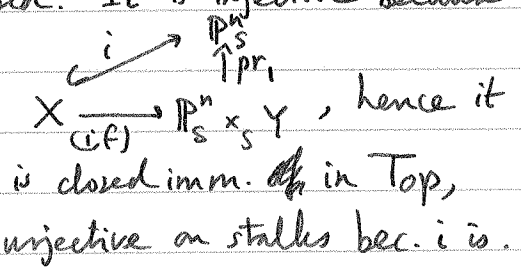
can take $X \xrightarrow{i} \mathbb{P}_S^n$ with i a quasi-compact immersion.



This gives:



Hence (previous lemma) (i, f) is closed. It is injective because



\square