Adv. Alg. Geom. week 3: 2013/09/17. (Bas)

1. Recall: • for $A \in \mathrm{Ring}$, $\forall f \in A$, $\mathcal{O}_{\mathrm{Spec} A}(D(f)) = A_f$

• $\forall f_1, \dots, f_n \in A$, $D(f_1) \cup \dots \cup D(f_n) = \mathrm{Spec}(A) \iff \exists g_1, \dots, g_n \in A$, $g_1 f_1 + \dots + g_n f_n = 1$.

• for $S = \bigoplus_{d \geq 0} S_d$ a graded ring; $\mathrm{Proj}(S) = \bigcup_{f \in \bigcup_{d>0} S_d} D_+(f)$,

$D_+(f)$ affine, $\mathcal{O}(D_+(f)) = S_{(f)}$.

Example: $S = \mathbb{Z}[x_0, \dots, x_n]$, $\mathbb{P}^n_{\mathbb{Z}} = \mathrm{Proj}(S) = \bigcup_{i=0}^n D_+(x_i)$,

$D_+(x_i) = \mathrm{Spec}\, \mathbb{Z}\left[\left\{ \frac{x_j}{x_i} : j \neq i \right\}\right] \cong \mathbb{A}^n_{\mathbb{Z}}$.

David has shown: $\mathbb{P}^n_{\mathbb{Z}} \to \mathrm{Spec}\, \mathbb{Z}$ is universally closed.

2. Proper morphisms. We follow David's §8, with a little correction to his proof of Lemma 8.3:

let $0 \leq i, j \leq n$, and consider:

$$\begin{tikzcd} \mathbb{P}^n_{\mathbb{Z}} \arrow[r, "\Delta"] & \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}} \\ T \arrow[u] \arrow[r, "{(f, g)}"] \arrow[ur, phantom, "\circlearrowleft"] & D_+(x_i) \times_{\mathbb{Z}} D_+(x_j) \arrow[u] \end{tikzcd}$$

Then

$$\begin{tikzcd} & D_+(x_i) \arrow[dr, hook, "\mathrm{ind}"] & \\ T \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\circlearrowleft"] & & \mathbb{P}^n_{\mathbb{Z}} \\ & D_+(x_j) \arrow[ur, hook, "\mathrm{ind}"'] & \end{tikzcd}$$

That means: $\exists!\, h : T \to D_+(x_i) \cap D_+(x_j)$ s.t.

$$\begin{tikzcd} & & D_+(x_i) \\ T \arrow[urr, "f"] \arrow[r, "h"] \arrow[drr, "g"'] & D_+(x_i) \cap D_+(x_j) \arrow[ur, hook] \arrow[dr, hook] & \\ & & D_+(x_j) \end{tikzcd}$$

Note: $D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j)$

Hence:

$$\begin{tikzcd} \mathbb{P}^n_{\mathbb{Z}} \arrow[r, "\Delta"] & \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}} \\ D_+(x_i x_j) \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & D_+(x_i) \times_{\mathbb{Z}} D_+(x_j) \arrow[u] \end{tikzcd}$$

on rings: $\mathbb{Z}\left[x_0, \dots, x_n, \frac{1}{x_i x_j}\right]_0 \longleftarrow \mathbb{Z}\left[x_0, \dots, x_n, \frac{1}{x_i}\right]_0 \otimes_{\mathbb{Z}} \mathbb{Z}\left[x_0, \dots, x_n, \frac{1}{x_j}\right]_0$

$ab \longleftarrow a \otimes b$

The ring map is surj; the scheme map is a closed immersion. $\boxtimes$

3. Projective and quasi-projective schemes. (SP 01W7)(27.43)
01VV (27.41)

Def. Let $f: X \to S$ be a morphism of schemes

1. We say f is H-projective if $\exists\, n \in \mathbb{Z}_{\geq 0}$ and a closed immersion $X \to \mathbb{P}^n_S$ over S. (H for Hartshorne).
2. We say f is H-quasi-projective if $\exists\, n \in \mathbb{Z}_{\geq 0}$ and a quasi-compact immersion $X \to \mathbb{P}^n_S$ over S.

Rem. 1. So an S-scheme is H-projective if it is isomorphic to a closed subscheme of some $\mathbb{P}^n_S$.
For A a ~~noth~~ noetherian ring all closed subschemes of $\mathbb{P}^n_A$ are of the form $V_+(f_1, \ldots, f_r)$, $f_i \in A[x_0, \ldots, x_n]_{d_i}$.

2. An S-scheme is H-quasi-projective if it is isomorphic to an open subscheme of ~~an~~ a closed subscheme of ~~some~~ $\mathbb{P}^n_S$ for some n, with the open immersion quasi-compact.

Exercise: give an example of an open imm. that is not q-compact.
Exercise: show that for S locally noetherian, and $f: X \to S$ locally of finite type, ~~every~~ every open immersion $j: U \to X$ is quasi-compact.

Prop. Let S be a scheme,
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & S \end{tikzcd}$$
a morphism of H-quasi-projective S-schemes.

Then: f is proper $\iff$ f is H-projective.

We need a few lemma's.

Lemma Let $X \xrightarrow{f} Y \xrightarrow{g} S$ in (Sch). If f & g are separated, then so is $g \circ f$.

Proof.
$$\begin{tikzcd} X \times_S X \arrow[r, "(f\circ p_1, f\circ p_2)"] & Y \times_S Y \\ X \times_Y X \arrow[u, "(id_X \circ p_1, id_X \circ p_2)"] \arrow[r, "f\circ p_1 = f \circ p_2"'] & Y \arrow[u, "(id_Y, id_Y)"'] \end{tikzcd}$$
is Cartesian (use a test-scheme $T \to S$),

hence:
$$\begin{tikzcd} & Y \times_S Y \\ X \arrow[ur, "\Delta_{X/S}"] \arrow[r, hook, "\Delta_{X/Y}"'] & X \times_Y X \arrow[u] \end{tikzcd}$$

- closed imm. stable by base change,
- cl. imm. stable under composition

Cor. Let $X \xrightarrow{f} Y \xrightarrow{g} S$ in (Sch) with f & g proper. Then $g \circ f$ is proper.
Proof. It is universally closed, of finite type, and separated. $\boxtimes$

Proof of "$\Longleftarrow$" in the Proposition. We have f H-projective, hence a comm. diagram:

$$\begin{tikzcd} & \mathbb{P}^n_Y \arrow[d, "p"] \\ X \arrow[ur, "i"] \arrow[r, "f"'] & Y \end{tikzcd}$$

with i a closed immersion. Then i and p are proper, hence f is proper. $\boxtimes$

The other implication of the Proposition is proved using the following very useful Lemma, which is the analog of: $f: X \to Y$ in Top with X compact and Y Hausdorff, then f is closed.

Lemma Let
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h=gf"'] & Y \arrow[d, "g"] \\ & S \end{tikzcd}$$
in Sch with h proper and g separated. Then f is closed.

Proof

$$\begin{tikzcd} X \arrow[r, "(id_X, f)"] \arrow[d, "f"'] & X \times_S Y \arrow[d, "(f \circ p_1, id_Y \circ p_2)"] \\ Y \arrow[r, "\Delta_g"'] & Y \times_S Y \end{tikzcd}$$

is Cartesian (use $T \to S$ a test scheme), hence (id_X, f) is a closed immersion, hence is closed.

Also:
$$\begin{tikzcd} X \times_S Y \arrow[r, "pr_2"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "h"'] & S \end{tikzcd}$$
is Cartesian, hence pr_2 is proper, hence closed.

So:
$$\begin{tikzcd} & X \times_S Y \arrow[dr, "pr_2"] & \\ X \arrow[ur, "(id_X, f)"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$
, f is closed. $\boxtimes$

Proof of "$\Longrightarrow$" of the Proposition. $h: X \to S$ is H-quasi-projective, so we can take

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "h"'] & \mathbb{P}^n_S \arrow[d, "p"] \\ & S \end{tikzcd}$$

with i a quasi-compact immersion.

This gives:
$$\begin{tikzcd} X \arrow[r, "(i,f)"] \arrow[dr, "f"', "\text{proper}"] & \mathbb{P}^n_S \times_S Y = \mathbb{P}^n_Y \arrow[d, "pr_2"] \\ & Y \end{tikzcd}$$
p separated

Hence (previous lemma) (i,f) is closed. It is injective because

$$\begin{tikzcd} & \mathbb{P}^n_S \\ X \arrow[ur, "i"] \arrow[r, "(i,f)"'] & \mathbb{P}^n_S \times_S Y \arrow[u, "pr_1"'] \end{tikzcd}$$

, hence it is closed imm. ~~of~~ in Top, and surjective on stalks bec. i is.

$\boxtimes$