Advanced Alg. Geom., week 5: 2013/10/08.

1. We begin with push & pull of sheaves, sheaves of modules, tensor-products of sheaves of modules, Picard group.

Let $f\colon X \to Y$ in Top.
For $\mathcal{F}$ in $\mathrm{Sh}(X)$ we have $f_*\mathcal{F}$ in $\mathrm{Sh}(Y)$ given by: $\forall\, U \subset Y$ open, $(f_*\mathcal{F})U = \mathcal{F}(f^{-1}U)$. This is indeed a sheaf.
For $\mathcal{G}$ in $\mathrm{Sh}(Y)$, we have $f^{-1}\mathcal{G}$ on X, $f^{-1}\mathcal{G} = \left(U \mapsto \operatorname*{colim}_{\substack{f^{-1}V \supset U \\ \Updownarrow \\ V \supset fU}} \mathcal{G}(V)\right)^{\#}$.
The colimit and sheafification are annoying,
but at least $\forall x \in X$: $(f^{-1}\mathcal{G})_x = \mathcal{G}_{fx}$.

$$\text{Adjointness: } \underbrace{\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F})}_{\{f\text{-maps } \varphi:\, \mathcal{G} \to \mathcal{F}\} =} = \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F})$$

<u>Tensor product.</u> Let $(X, \mathcal{O})$ be in RS, $\mathcal{M}$ and $\mathcal{N}$ in $\mathrm{Mod}(\mathcal{O})$.
Then we have ~~the~~ presheaf $(U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{N}(U))$, of $\mathcal{O}$-modules, and $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ is defined as its sheafification.
It has the usual universal property: $\forall\, \mathcal{P} \in \mathrm{Mod}(\mathcal{O})$:
$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}, \mathcal{P}) = \mathrm{Bilin}_{\mathcal{O}}(\mathcal{M}, \mathcal{N}; \mathcal{P})$$

And for stalks: $\forall x \in X$: $(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N})_x = \mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{N}_x$.

Let $(X, \mathcal{O}_X) \xrightarrow{(f,\varphi)} (Y, \mathcal{O}_Y)$ be in RSp.

For $\mathcal{F}$ in $\mathrm{Mod}(\mathcal{O}_X)$: $f_*\mathcal{F} = (V \mapsto \mathcal{F}(f^{-1}V))$ is an $\mathcal{O}_Y$-module via φ.

For $\mathcal{G}$ in $\mathrm{Mod}(\mathcal{O}_Y)$: $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module, and we have $\varphi\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, so we can define:
$$f^*\mathcal{G} = \mathcal{O}_X \otimes_{\varphi, f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}.$$
This is an annoying construction, but we will understand it much better for quasicoherent sheaves on schemes.
At least something nice: $\forall x \in X$: $(f^*\mathcal{G})_x = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,fx}} \mathcal{G}_{fx}$.
And: $\mathcal{G}$ loc. free of rank $n \Rightarrow f^*\mathcal{G}$ loc. free of rank n.

"sheaf Hom"

Let $(X,\mathcal{O})$ be in RSp.
For $\mathcal{F}$ and $\mathcal{G}$ in $\mathrm{Mod}(\mathcal{O})$: $\mathcal{H}om_{\mathcal{O}}(\mathcal{F},\mathcal{G}) : U \mapsto \mathrm{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$, is in $\mathrm{Mod}(\mathcal{O})$.
Special case: $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}}(\mathcal{F},\mathcal{O})$.

We have $\mathcal{F}^\vee \otimes_{\mathcal{O}} \mathcal{G} \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F},\mathcal{G})$, isomorphism if $\mathcal{F}$ loc. free of finite rank: $\forall x \in X$, $\exists\, U \subset X$ open with $x \in U$, $\exists\, n \in \mathbb{Z}_{\geq 0}$, s.t. $\mathcal{F}|_U$ is isomorphic to $(\mathcal{O}|_U)^n$.
<u>Def.</u> $\mathcal{F}$ is <u>invertible</u>: $\mathcal{F}$ is locally free of rank 1.
<u>Picard group</u>. $\mathrm{Pic}(X,\mathcal{O}) := \{\text{invertible } \mathcal{O}\text{-modules}\}/\cong$.
Operations: $\mathcal{L}_1 \otimes_{\mathcal{O}} \mathcal{L}_2$, $\mathcal{L}^\vee$; $\mathcal{L}^\vee \otimes_{\mathcal{O}} \mathcal{L} = \mathcal{H}om_{\mathcal{O}}(\mathcal{L},\mathcal{L}) \xleftarrow[\text{isomorphism}]{} \mathcal{O}$

So: $\mathrm{Pic}(X,\mathcal{O})$ is a commutative group, unit element $[\mathcal{O}]$.

<u>Functoriality</u>: $(X,\mathcal{O}_X) \xrightarrow{(f,\varphi)} (Y,\mathcal{O}_Y)$ induces $\mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X)$,
a morphism of groups. $[\mathcal{L}] \longmapsto [f^*\mathcal{L}]$.

2. Quasicoherent $\mathcal{O}$-modules.

We follow David's notes of Lecture 4, §11 and further.
In Def. 11.1 be careful with $\bigoplus_{j \in J} \mathcal{O}_U = \left(V \mapsto \bigoplus_{j\in J} \mathcal{O}_U(V)\right)^{\#}$.

For J finite the $(-)^{\#}$ is not necessary.
At least some good news: if V quasi-compact, then
$\left(\bigoplus_{j\in J} \mathcal{O}_U\right) V = \bigoplus_{j \in J} \mathcal{O}_U(V)$. (SP lemma 01AI).

<u>Proposition</u>. Let $(f,\varphi): (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ in RSp, $\mathcal{G}$ in $\mathrm{Mod}(\mathcal{O}_Y)$ quasicoherent.
Then $f^*\mathcal{G}$ is quasicoherent.
<u>Proof</u>. f^* is right exact, and commutes with $\bigoplus_{j \in J}$. $\square$

David's § 12. : Quasicoh. sheaves on affine schemes.

In Def. 12.1 we should also define the restriction morphisms.

So let $f, g \in A$ s.t. $D(g) \subset D(f)$. Then $V(g) \supset V(f)$, hence $\sqrt{(g)} \subset \sqrt{(f)}$, hence $\exists\, n \geq 1$ and $a \in A$ s.t. $g = f^n \cdot a$.

Then
$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_g \\ A_f \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$
bec. f becomes invertible in A_g: $f \cdot f^{n-1} \cdot a \cdot g^{-1} = 1$ in A_g.

Hence also:
$$\begin{tikzcd} M \arrow[r] \arrow[d] & M_g \\ M_f \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$
These are the restriction maps.

Thm 12.2: $\mathrm{Mod}(R) \rightleftarrows \mathrm{QCoh}(\mathcal{O}_{\mathrm{Spec} R})$, $M \longmapsto \widetilde{M}$, $\mathcal{F}(X) \longleftarrow\!\shortmid \mathcal{F}$

are an equivalence of categories. $(X = \mathrm{Spec} R)$

In particular: $\widetilde{M}(X)$ is isom. to M, functorially.

$\forall\, \mathcal{F}$ in $\mathrm{QCoh}(\mathcal{O}_X)$: $\exists\, M \in \mathrm{Mod}(R)$, $\mathcal{F} \cong \widetilde{M}$

bijective on Hom.

Moreover: (Hartshorne, II.5.2)

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} = \widetilde{M \otimes_R N}, \qquad \bigoplus_{j \in J} \widetilde{M}_j = \Big(\bigoplus_{j \in J} M_j\Big)^{\sim}$$

and for $\varphi: A \to B$, $f := \mathrm{Spec}\, \varphi : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$, $M \in \mathrm{Mod}(A)$, $N \in \mathrm{Mod}(B)$: $f^* \widetilde{M} = (B \otimes_{\varphi, A} M)^{\sim}$, $f_* \widetilde{N} = ({}_A N)^{\sim}$.

Consequence For X a scheme, $\mathrm{QCoh}(\mathcal{O}_X)$ is abelian.

Proof. It suffices to show ~~that~~ for $\mathcal{F}, \mathcal{G}$ in $\mathrm{QCoh}(\mathcal{O}_X)$, and $f: \mathcal{F} \to \mathcal{G}$ $\mathcal{O}_X$-linear, $\ker(f)$ and $\mathrm{coker}(f)$ are in $\mathrm{QCoh}(\mathcal{O}_X)$. But that is a local question, and we may (and do) assume that X is affine. Then Thm. 12.2 settles it. $\boxtimes$

David's §13: Example, Jordan canonical form.

Let k be a field, then we have equivalences of categories:

$$\left\{\begin{array}{l}(V,a): V \text{ fin. dim.} \\ k\text{-vect. space,} \\ a \in \mathrm{End}_k(V)\end{array}\right\} \begin{array}{c}\longrightarrow \\ \longleftarrow\end{array} \left\{\begin{array}{l}\text{fin. gen. } k[x]\text{-} \\ \text{modules } M \text{ with} \\ k(x) \otimes_{k[x]} M = 0\end{array}\right\} \begin{array}{c}\longrightarrow \\ \longleftarrow\end{array} \left\{\begin{array}{l}\text{q. coh. } \mathcal{O}\text{-modules} \\ \mathcal{F} \text{ on } \mathbb{A}^1_k \text{, fin.} \\ \text{gen. and } \mathcal{F}_\eta = 0\end{array}\right\}$$

η = generic point.

$$(V,a) \longmapsto V + \text{let } x\cdot = a \qquad\qquad M \longmapsto \widetilde{M}$$

$$(M, x\cdot) \longleftarrow\!\shortmid M \qquad\qquad \mathcal{F}(\mathbb{A}^1_k) \longleftarrow\!\shortmid \mathcal{F}$$

this is an isomorphism of categories.

Why is this interesting/useful: geometry tells us that $\mathcal{F} = \bigoplus_{\substack{t \in \mathbb{A}^1_k \\ \text{closed}}} i_{t*} \mathcal{F}_t$, this is the decomposition of V into its generalised eigenspaces (if $k = \bar{k}$).

So we get a geometrical explanation of why $\mathrm{Hom}((V,a),(V',a')) = 0$ if the modules have disjoint support. Same for $\otimes$, Ext ... .

3. QCoh(Proj S). (Hartshorne II §5.)

Let $S = \bigoplus_{i \geq 0} S_i$ be a graded ring.

Then $\mathrm{Proj}(S) = \bigcup_{f \in S_{hom}} D_+(f)$, $D_+(f) = \mathrm{Spec}(S_{(f)})$, $D_+(fg) = D_+(f) \cap D_+(g)$.

Let M be a graded S-module: $M = \bigoplus_{i \in \mathbb{Z}} M_i$ (note: $i \in \mathbb{Z}$!)
$S_i \times M_j \xrightarrow{\cdot} M_{i+j}$.

For $f \in S_{hom}$ we have the $S_{(f)}$-module $M_{(f)}$, hence $\widetilde{M_{(f)}}$ on $D_+(f)$.
These glue, and give us $\widetilde{M}$ on $\mathrm{Proj}(S)$, quasi-coherent by constr.

<u>Example</u> Serre's twisting sheaves $\mathcal{O}(n)$, $n \in \mathbb{Z}$.
For $n \in \mathbb{Z}$, let $S(n)$ the graded S-module: $S(n)_i = S_{n+i}$,
we shifted the grading of S by n.
Define: $\mathcal{O}(n) := \widetilde{S(n)}$ on $\mathrm{Proj}(S)$.

<u>Proposition 3.1.</u> (II.5.12(a) in [H]). If S is generated by S_1 as algebra over $A := S_0$, then all $\mathcal{O}(n)$ are loc. free of rank 1, and $\mathcal{O}(n) \otimes_{\mathcal{O}} \mathcal{O}(m) = \mathcal{O}(n+m)$.

<u>Proof.</u> We have $\mathrm{Proj}(S) = \bigcup_{f \in S_1} D_+(f)$.

Let $n \in \mathbb{Z}$. For $f \in S_1$: $\mathcal{O}(n)|_{D_+(f)} = \widetilde{S(n)_{(f)}}$; $S(n)_{(f)} = (S_f)_n \xleftarrow[\sim]{f^n \cdot} \underbrace{(S_f)_0}_{= S_{(f)}}$,

so $\mathcal{O}(n)|_{D_+(f)}$ is free as $\mathcal{O}_{D_+(f)}$-module, with basis f^n.
Moreover, on each $D_+(f)$ we have ~~an~~ unique isomorphism $\mathcal{O}(n) \otimes_{\mathcal{O}} \mathcal{O}(m) \to \mathcal{O}(n+m)$ that sends $f^n \otimes f^m$ to f^{n+m}. These glue: <u>exercise</u>.

4. Exercises.

4.1. Let A be a ring, $r \geq 1$, $n \in \mathbb{Z}$. Let $\mathcal{O}(n) := \mathcal{O}_{\mathbb{P}^r_A}(n)$: Serre's twisting sheaves on $\mathbb{P}^r_A$.

a. For $0 \leq i \leq r$, show that $(\mathcal{O}(n))(D_+(x_i))$ is a free A-module and give a basis.

b. Same for $(\mathcal{O}(n))(D_+(x_0 x_r))$.

c. Show that $(\mathcal{O}(n))(D_+(x_0) \cup D_+(x_r))$ is a free A-module and give a basis.

d. Same for $(\mathcal{O}(n))(\mathbb{P}^r_A)$.

4.2 Let A be a ring, X the affine line with doubled origin.

In other words:

$$\begin{tikzcd} & \mathbb{A}^1_A \arrow[dr, hook, "j_1"] & \\ D(x) \arrow[ur, hook, "j"] \arrow[dr, hook, "j"'] & & X \\ & \mathbb{A}^1_A \arrow[ur, hook, "j_2"'] & \end{tikzcd}$$

with j the inclusion, is a push-out; X with ~~the~~ j_1 and j_2 is a colimit of the diagram without X.

Compute $\mathcal{O}(X)$.

4.3. Let k be a field, P_1, P_2, P_3 in $\mathbb{P}^2(k)$ distinct. Let Z be the reduced closed subscheme of $\mathbb{P}^2_k$ with underlying space $\{P_1, P_2, P_3\}$, and let $i: Z \hookrightarrow \mathbb{P}^2_k$ be the closed immersion.

For which $n \in \mathbb{Z}$ is the map

$$\Gamma(\mathbb{P}^2_k, \mathcal{O}(n)) \longrightarrow \Gamma(Z, i^* \mathcal{O}(n)) \quad \text{surjective?}$$

Does it depend on the position of the P_i?

Remark: The aim of this exercise is not so much the outcome, but more that one has to make sense of it, what is everything in it? How to make this linear map of k-vector spaces explicit? It is nice that there is at least a little bit of geometry ....