Advanced Algebraic Geometry, week 6; 2013/10/15.

Two topics today: 1. derived functors, 2. cohomology of sheaves. 12.10,
General references: [H]: III.1 and 2 and 4; SP 13.17 – 13.21, 20.3–4, 20.10–13
Remark: double & total complexes, spectral sequence are not treated in [H].
SP starts with derived categories, it is the right approach but we have no time.

Motivation. Let X be a closed subscheme of $\mathbb{P}^r_k$, $i: X \to \mathbb{P}^r_k$ the closed imm., and $I \rightarrowtail \mathcal{O}_{\mathbb{P}^r_k} \twoheadrightarrow i_*\mathcal{O}_X$ the exact sequence. For $n \in \mathbb{Z}$ we have the exact sequence $0 \to I(n) \to \mathcal{O}_{\mathbb{P}^r_k}(n) \to (i_*\mathcal{O}_X)(n) \to 0$, but the linear map $\Gamma(\mathbb{P}^r_k, \mathcal{O}(n)) \to \Gamma(\mathbb{P}^r_k, (i_*\mathcal{O}_X)(n))$ is not always surjective.
The obstructions for that lie in $H^1(\mathbb{P}^r_k, I(n))$. Etc: long exact cohom. sequence, extremely useful. Note: $\Gamma(X,-): Ab(X) \to Ab$ is left-exact.

1. Derived functors. (and hyperderived functors) (Ref. for homological algebra: Weibel's book)

Let $\mathcal{A}$ be an abelian category.
A cochain complex $A^\bullet$ in $\mathcal{A}$ is a collection of A^i in $\mathcal{A}$, $i \in \mathbb{Z}$, and $d^i: A^i \to A^{i+1}$, such that $d^{i+1} \circ d^i = 0$ for all i.
A morphism of cochain complexes $f: A^\bullet \to B^\bullet$ is a collection $f^i: A^i \to B^i$ commuting with the d's.
Category of cochain complexes: $CoCh(\mathcal{A})$. This is again an abelian category; ($(\ker f)_i = \ker(f_i)$, etc.)
Similarly: a chain complex $A_\bullet$ in $\mathcal{A}$ is a family $(A_i)_{i\in\mathbb{Z}}$, $d_i: A_{i+1} \to A_i$, etc.
Both notions come from algebraic topology, but have been adopted in homological alg.

Let $f, g: A^\bullet \to B^\bullet$ in $CoCh(\mathcal{A})$. A homotopy h from f to g is a family of $h^i: A^i \to B^{i-1}$, $i \in \mathbb{Z}$, s.t. $\forall i \in \mathbb{Z}$: $f^i - g^i = d^{i-1}\circ h^i + h^{i+1} \circ d^i$.
If h is a homotopy from f to g, and $k: C^\bullet \to A^\bullet$ and $l: B^\bullet \to D^\bullet$, then $(l^{i-1} \circ h^i \circ k^i)_{i\in\mathbb{Z}}$ is a homotopy from $l\circ f\circ k$ to $l\circ g\circ k$.
(Think of analogous statement in Top). (homotopy compatible with composition)
This makes it possible to define the homotopy category SP 13.7 (013H):
$K(\mathcal{A})$ objects: same as $CoCh(\mathcal{A})$
morphisms: $\mathrm{Hom}_{CoCh(\mathcal{A})}(A^\bullet, B^\bullet)$ / homotopy equivalence
composition: works, indeed. So $CoCh(\mathcal{A}) \to K(\mathcal{A})$ is a quotient category.
Full subcategories: $K^-(\mathcal{A})$: $A^\bullet$ s.t. $\forall i \gg 0$, $A^i = 0$
$K^+(\mathcal{A})$: $A^\bullet$ s.t. $\forall i \ll 0$, $A^i = 0$
$K^b(\mathcal{A})$: $A^\bullet$ s.t. for almost all i, $A^i = 0$.

Def. $H^i(A^\bullet) := \ker(d^i)/\mathrm{im}(d^{i-1})$, i-th cohom. grp. of $A^\bullet$,
functor $\mathrm{CoCh}(\mathcal{A}) \to \mathcal{A}$.
$f: A^\bullet \to B^\bullet$ is a <u>quasi-isomorphism</u> if $\forall i \in \mathbb{Z}$, $H^i(f): H^i(A^\bullet) \to H^i(B^\bullet)$ is an isom.

<u>Lemma</u>
- If f & $g: A^\bullet \to B^\bullet$ are homotopic, then $H^i(f) = H^i(g)$, $\forall i \in \mathbb{Z}$.
- If $f: A^\bullet \to B^\bullet$ becomes an isom. in $K(\mathcal{A})$, then f is a quasi-isomorphism.
- For $A^\bullet \rightarrowtail B^\bullet \twoheadrightarrow C^\bullet$ a short exact sequence of cochain complexes, the snake lemma induces $\delta^i: H^i(C^\bullet) \to H^{i+1}(A^\bullet)$, and the sequence: $\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \xrightarrow{\delta^i} H^{i+1}(A^\bullet) \to \cdots$ is <u>exact</u>.

An object I of $\mathcal{A}$ is <u>injective</u> if $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact; equivalently; if $\forall\ 0 \to A \xrightarrow{i} B$ exact, $\mathrm{Hom}(B, I) \xrightarrow{-\circ i} \mathrm{Hom}(A, I)$ is surjective
Dual notion: <u>projective</u>. (Example: free modules in $\mathrm{Mod}(R)$.)

$\mathcal{A}$ is said to have <u>suff. many injectives</u> if $\forall\ A$ in $\mathcal{A}$, $\exists$ injection $A \hookrightarrow I$ with I injective.
If $\mathcal{A}$ has suff. many injectives, then $\forall A \in \mathcal{A}$ $\exists$ inj. resolution: $A \xrightarrow{\varepsilon} I^0$, $I^\bullet \in \mathrm{CoCh}(\mathcal{A})$ with $\forall i: I^i$ injective, $\forall i<0: I^i = 0$, $\forall i>0: H^i(I^\bullet) = 0$, $A \underset{\sim}{\xrightarrow{\varepsilon}} \ker(d^0) = H^0(I^\bullet)$.
In other words: $\exists\ A[0] \to I^\bullet$ a quasi-isom. to $0 \to I^0 \to I^1 \to \cdots$ with all I^i injective. Notation: $I^\bullet \in \mathrm{CoCh}^{\geq 0}(\mathcal{J})$.
$A[0] = \cdots 0 \to A \to 0 \to \cdots$

<u>Lemma</u> Given
$$\begin{tikzcd} A[0] \arrow[r, "q.is"] \arrow[d] & C^\bullet \arrow[dl, dashed, "\exists f"] \\ I^\bullet & \end{tikzcd}$$
with $I^\bullet \in \mathrm{CoCh}^{\geq 0}(\mathcal{J})$, $C^\bullet \in \mathrm{CoCh}^{\geq}(\mathcal{A})$; $\mathcal{J}$ = full subcat. of injective objects in $\mathcal{A}$.
s.t. diagr. commutative, moreover, 2 such f are homotopic.

<u>Proof</u>: a good exercise, it is straightforward from the definitions. ☒

<u>Consequence</u>: we have, if $\mathcal{A}$ has suff. many injectives, a <u>resolution functor</u>
$j: \mathcal{A} \to K^+(\mathcal{J}) \subset K^+(\mathcal{A})$.
$A \mapsto$ any inj. res. $I^\bullet$ of A.

$\forall\, (0 \to A \to B \to C$ exact in $\mathcal{A})$, $0 \to F(A) \to F(B) \to F(C)$ is exact in $\mathcal{B}$.

Let now $F : \mathcal{A} \to \mathcal{B}$ be a <u>left exact</u> additive functor, $\mathcal{A}$ and $\mathcal{B}$ abelian. Then ~~sk~~ we have:

$$A \longmapsto I^\bullet \text{ an inj. res. of } A$$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "j"] \arrow[dr, "RF"'] \arrow[ddr, bend right, "R^iF"'] & K^+(\mathcal{J}) \arrow[d, "F"] & (0 \to I^0 \to I^1 \to \cdots) \arrow[d, mapsto] \\ & K^+(\mathcal{B}) \arrow[d, "H^i(-)"] & (0 \to FI^0 \to FI^1 \to \cdots) \\ & \mathcal{B} & \end{tikzcd}$$

$\left(\text{total right derived functor of } F\right) = RF$

$\left(i\text{th right derived functor of } F.\right) = R^iF$

<u>Correction!</u> Let $M^\vee := \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$, let $F \twoheadrightarrow M^\vee$ with F free $\mathbb{Z}$-module, then $M \hookrightarrow M^{\vee\vee} \hookrightarrow F^\vee$ and $F^\vee$ is injective.

<u>Example</u>. Let $\mathcal{A} = Ab = \mathbb{Z}\text{-Mod}$. (SP 01D7)

Then $M \in Ab$ is injective $\iff$ M is divisible: $\forall n \in \mathbb{Z}_{>0}$ $M \xrightarrow{n\cdot} M$ surject.

$\forall M \in Ab$: $M \to \mathrm{Hom}(\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ is injective into divisible.

<u>Exercise</u>: Let $n \in \mathbb{Z}_{>0}$, $F : Ab \to Ab$, $M \mapsto \mathrm{Hom}_{Ab}(\mathbb{Z}/n\mathbb{Z}, M)$.
Compute $(RF)(\mathbb{Z})$, and the $(R^iF)(\mathbb{Z})$, and the same for $\mathbb{Z}/m\mathbb{Z}$, $m \geq 1$: $(RF)(\mathbb{Z}/m\mathbb{Z})$ etc.

<u>Exercise</u>. Let $R \in \mathrm{Ring}$. Let $F : Ab \to R\text{-Mod} : N \mapsto \mathrm{Hom}_{\mathbb{Z}}(R, N)$, and $G : R\text{-Mod} \to Ab : M \mapsto {}_{\mathbb{Z}}M$ the forget functor.
Show that G is a left-adjoint of F.
Show that F maps injectives to injectives, using the adjunction and the exactness of G.
<u>Remark</u>. This produces many injectives in R-Mod. Read SP 01D8 for the fact that R-Mod has sufficiently many injectives.

<u>Remark</u>. For $F : \mathcal{A} \to \mathcal{B}$ right exact, one uses projective resolutions, in $Ch(\mathcal{A})$: $\cdots \to P_1 \to P_0 \to A \to 0$ $\quad$ (P proj. $\iff$ $\mathrm{Hom}_{\mathcal{A}}(P, -)$ exact)
R-Mod has sufficiently many projectives, because free modules are projective.

2. Cohomology of sheaves.

Let $(X, \mathcal{O}_X)$ be in RSp.

Lemma. (SP01DI) $Mod(\mathcal{O}_X)$ has enough injectives.

Proof. Let $\mathcal{F} \in Mod(\mathcal{O}_X)$. For each $x \in X$, let $\mathcal{F}_x \hookrightarrow I(x)$ be an injection of $\mathcal{F}_x$ into an injective $\mathcal{O}_{X,x}$-module. Then let I be the presheaf on X defined by:

$$\forall U \subset X \text{ open}, \quad I(U) = \prod_{x \in U} I(x).$$

Then I is a sheaf of $\mathcal{O}_X$-modules. Note that $I = \prod_{x \in X} i_{x,*} I(x)$, where $i_x : \{x\} \hookrightarrow X$ is the inclusion.
We show that $\forall x \in X$: $i_{x,*} I(x)$ is an injective $\mathcal{O}_X$-module, then the product is also injective.
We know: $i_{x,*} : \underset{\substack{\| \\ Mod(\{x\}, \mathcal{O}_{X,x})}}{\mathcal{O}_{X,x}\text{-Mod}} \longrightarrow Mod(\mathcal{O}_X)$ has left adjoint $i_x^* : \mathcal{F} \mapsto \mathcal{F}_x$, which is exact. Hence (exercise on page 3), $i_{x,*}(I(x))$ is injective. ☒

Def. For $\mathcal{F} \in Mod(\mathcal{O}_X)$ and $i \in \mathbb{Z}$, $H^i(X, \mathcal{F}) := (R^i \Gamma(X, -)) \mathcal{F} = H^i(I^\cdot(X))$, where $\mathcal{F} \to I^\cdot$ is any injective resolution.

Def. For $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in RSp, and $\mathcal{F}$ in $Mod(\mathcal{O}_X)$, and $i \in \mathbb{Z}$,
$(R^i f_*) \mathcal{F} := H^i(f_* I^\cdot)$, where $\mathcal{F} \to I^\cdot$ is an inj. resolution.
Note: $f_* I^\cdot = 0 \to f_* I^0 \to f_* I^1 \to \cdots$ is a complex of $\mathcal{O}_Y$-modules, of which $H^i(f_* I^\cdot)$ is the i-th cohom. group: $\ker(f_* d_{i+1}) / im(f_* d_i)$.

The $\underbrace{H^i(X, -) : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_X(X))}_{i \geq 0}$ form a universal δ-functor (long exact sequences), and the $R^i f_*$, $i \geq 0$, form a universal δ-functor $Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$.

Nice, such a general definition. But it is a very complicated one.

<u>Exercise</u>. Let $j : Y \hookrightarrow X$ be a closed immersion in Top. Let $\mathcal{F}$ be in $Ab(Y)$. Give an isomorphism from $H^i(Y, \mathcal{F})$ to $H^i(X, j_* \mathcal{F})$.
Hint: let $\mathcal{F} \to I^\bullet$ be an inj. res. in $Ab(Y)$. Then consider $j_*\mathcal{F} \to j_* I^\bullet$.

For certain topological spaces, including manifolds, $H^i(X, \mathbb{Z}_X) = H^i_{sing}(X, \mathbb{Z})$ defined as in algebraic topology, but you have to be careful.
Here is an example: let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}_{>0}\}$. Then $H^0(X, \mathbb{Z}_X)$ is <u>not</u> the same as $H^0_{sing}(X, \mathbb{Z})$ (<u>exercise!</u>).

In general, $H^1(X, \mathcal{F})$ has a useful concrete interpretation as the group of $\mathcal{F}$-torsors on X up to isomorphism (SP 02FN). No time!
But in particular this gives: $H^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}(X)$.

In this course, we are interested in separated schemes and quasi coherent modules.
Then, cohomology can be computed by Čech cohomology w.r.t. an affine covering.
Read SP section 01E9 on Mayer-Vietoris.

SP 01FG, [H] III.4.

To keep things as finite as possible, we use the ordered ~~alternating~~ Čech complex.
We follow [H] here

<u>Def</u>. Let $X \in \mathrm{Top}$, $\mathcal{F} \in Ab(X)$, I a set, with a total ordering $<$, and $\mathcal{U} := (U_i)_{i \in I}$ an open cover of X.

Then for $p \geq 0$ $C^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0, \ldots, i_p})$ where $U_{i_0, \ldots, i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$.

And: $d^p : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$

$$\alpha \longmapsto \left((i_0, \ldots, i_{p+1}) \mapsto \sum_{k=0}^{p+1} (-1)^k \alpha(i_0, \ldots, \hat{i}_k, \ldots, i_{p+1}) \right)$$

meaning: leave out.

Then $(C^\bullet(\mathcal{U}, \mathcal{F}), d^\bullet)$ is a complex, and for $i \geq 0$:

$$\check{H}^i(\mathcal{U}, \mathcal{F}) := H^i(C^\bullet(\mathcal{U}, \mathcal{F})).$$

Be careful: the $\check{H}^i(\mathcal{U}, \mathcal{F})$ do not always form a δ-functor (no long exact sequences). But they do for presheaves. See SP 01EH.

Here is an important result: (the proof uses a bit more homological algebra than we have treated)

<u>Lemma</u> (01ET + 01FM) Let $(X, \mathcal{O})$ in RSp, $(U_i)_{i \in I}$ an open cover of X, $<$ a total ordering on I, and $\mathcal{F}$ an $\mathcal{O}_X$-module.
Assume that for all $i > 0$, $p \geq 0$, $i_0 < \cdots < i_p \in I$: $H^i(U_{i_0, \ldots, i_p}, \mathcal{F}) = 0$.
Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$ as $\mathcal{O}(X)$-modules.

And here is another important result.

<u>Lemma</u> (01XB) Let $(X, \mathcal{O}_X)$ be a scheme, $\mathcal{F}$ in $QCoh(\mathcal{O}_X)$, and $U \subset X$ an affine open. Then $H^i(U, \mathcal{F})$ is zero for all $i > 0$.

Together they give:

<u>Lemma</u> (01XD). Let X be a scheme, $\mathcal{F} \in QCoh(\mathcal{O}_X)$, $(U_i)_{i \in I}$, $<$ a (totally) ordered open cover of X s.t. $\forall p \geq 0$, $\forall i_0 < \cdots < i_p$, $U_{i_0, \ldots, i_p}$ is affine.
Then $\forall p \geq 0$: $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$.

<u>Exercise</u>. Let A be a ring, $X = \mathbb{P}^1_A$, $\mathcal{F} = \mathcal{O}(n)$ (some $n \in \mathbb{Z}$), $I = \{0, 1\}$, $0 < 1$, $U_0 = D_+(x_0)$, $U_1 = D_+(x_1)$.
Write down $C^\bullet(\mathcal{U}, \mathcal{F})$, give A-bases of the terms, give d, compute $H^p(X, \mathcal{F})$ for all p.

<u>Exam</u>: there will be a take home assignment,
and an oral exam in January,
proposed dates: 28, 29.