Today our goal is to prove the following theorem (thm. 2 in Borel-Serre).

Thm 1. Let A be a noetherian ring such that $\mathrm{Spec}(A)$ is finite dimensional. Let X be a regular, quasi-projective A-scheme. The the morphism $K^0(X) \to K_0(X)$ is an isomorphism.

1. Some properties of regular local noetherian rings.

References for dimension (Krull) theory for rings: Atiyah-Macdonald, Serre's "Algèbre locale, multiplicités", Stacks project.

Let A be a regular noetherian local ring.

Then: A is an integral domain (SP 00NP).

A is a u.f.d. (Serre, IV D Cor. 4; Auslander-Buchsbaum thm).

Let $n = \dim(\mathrm{Spec}\, A)$, $x_1, \ldots, x_n \in m$ s.t. $m = (x_1, \ldots, x_n)$, then $(x_1, \ldots, x_n)$ is a regular sequence, i.e., $\forall i \in \{1, 2, \ldots, n\}$, $x_i \cdot : A/\sum_{j<i} Ax_j \to A/\sum_{j<i} Ax_j$ is injective.

(Proof: $A/\sum_{j<i} Ax_j$ is regular, hence a domain, and $x_i \neq 0$ in it.)

2. Koszul complex.

Reference: Serre IV A, or with more detail: Lang, Algebra, XVI §10.

Let A be a ring.

For $x \in A$ we define the chain complex $K(x)_\bullet$:

$$\begin{array}{ccc} K(x)_1 & & K(x)_0 \\ \| & & \| \\ A & \xrightarrow{x\cdot} & A \end{array}.$$

For $n \geq 0$, $(x_1, \ldots, x_n) \in A^n$, we define the n-complex

$$K(x_1, \ldots, x_n)_{\bullet, \ldots, \bullet} := K(x_1)_\bullet \otimes_A \cdots \otimes_A K(x_n)_\bullet$$

$$K(x_1, \ldots, x_n)_{p_1, \ldots, p_n} = K(x_1)_{p_1} \otimes_A \cdots \otimes_A K(x_n)_{p_n}$$

For $i \in \{1, \ldots, n\}$, $d_{(p_1, \ldots, p_n), i} : K(x_1, \ldots, x_n)_{p_1, \ldots, p_n} \xrightarrow[\;]{\overset{i}{\mathrm{id} \otimes \cdots \otimes d \otimes \cdots \otimes \mathrm{id}}} K(x_1, \ldots, x_n)_{p_1, \ldots, p_i + 1, \ldots, p_n}$

012X.

Note: we follow the SP here. Some references put other signs, to make the n-complex anti-commute.

Then we take the <u>total complex</u>, also called <u>simple complex</u>. Most of the time, this is written for double complexes, and then left to the reader to generalise.

For $C_{\bullet,\dots,\bullet}$ an n-complex we define:

$$(s\, C_{\bullet,\dots,\bullet})_p := \bigoplus_{p_1+\dots+p_n=p} C_{p_1,\dots,p_n} \qquad \text{(in our case the direct sum will be finite).}$$

$$d_p : \underset{\underset{C_{p_1,\dots,p_n}}{\cup}}{(s\, C_{\bullet,\dots,\bullet})_p} \to (s\,C_{\bullet,\dots,\bullet})_{p-1} \quad : \quad \sum_{i=1}^{n} (-1)^{\sum_{j<i} p_j} d_{(p_1,\dots,p_n),i}$$

Then $((s\,C_{\bullet,\dots,\bullet})_\bullet, d_\bullet)$ is a chain complex.

For $(x_1,\dots,x_n) \in A^n$ we define the <u>Koszul complex</u>

$$K(x_1,\dots,x_n)_\bullet := \left(s\ K(x_1)_\bullet \otimes_A \cdots \otimes_A K(x_n)_\bullet\right)_\bullet, \text{ it is a chain complex.}$$

<u>Thm 2.</u> Let $(x_1,\dots,x_n)$ be a regular sequence in A.
Then $K(x_1,\dots,x_n)_\bullet$ is a resolution of $A/x_1A+\dots+x_nA$, that is: for $p \in \mathbb{Z}$, $H_p(K(x_1,\dots,x_n)_\bullet) = \begin{cases} 0 & \text{if } p \neq 0 \\ A/(x_1,\dots,x_n) & \text{if } p = 0. \end{cases}$

The <u>proof</u> is by induction on n.
For $n=1$: obvious: $A \xrightarrow{x_1 \cdot} A \twoheadrightarrow A/x_1A$.
Now assume $n>1$. Then $K(x_1,\dots,x_n)_\bullet = s\left(K(x_1,\dots,x_{n-1})_\bullet \otimes_A K(x_n)_\bullet\right)_\bullet$

Put $L_\bullet := K(x_1,\dots,x_{n-1})_\bullet$. The induction hypothesis says that $H_p(L_\bullet) = 0$ if $p \neq 0$, $A/(x_1,\dots,x_{n-1})$ if $p=0$.

We write out the double complex $L_\bullet \otimes_A K(x_n)_\bullet$:

$$\begin{tikzcd}
\cdots \arrow[r] & L_2 \arrow[r, "d"] & L_1 \arrow[r, "d"] & L_0 \\
\cdots & L_2 \arrow[u, "x_n\cdot"] \arrow[r, "d"] & L_1 \arrow[u, "x_n\cdot"] \arrow[r, "d"] & L_0 \arrow[u, "x_n\cdot"] \\
\cdots \arrow[ru, phantom, ""] & & &
\end{tikzcd}$$

This has

$$\begin{tikzcd}
\cdots & L_2 \arrow[r] & L_1 \arrow[r] & L_0 \\
\cdots & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

as double subcomplex, $= L_\bullet \otimes_A K(x_n)_0$

with quotient double complex :

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
\cdots & \\
\cdots \arrow[r] & L_2 \arrow[uu] \arrow[r] & L_1 \arrow[uu] \arrow[r] & L_0 \arrow[uu]
\end{tikzcd} = L_\bullet \otimes_A K(x_n)_1 .$$

Taking the simple complexes of this short exact sequence of double complexes gives a short exact sequence of chain complexes:

$$\begin{tikzcd}
L_\bullet \otimes_A K(x_n)_0 \arrow[r, tail] \arrow[d, equal] & K(x_1,\dots,x_n)_\bullet \arrow[r, two heads] & L_\bullet \otimes_A K(x_n)_1 \arrow[d, equal] \\
L_\bullet & & L_\bullet[-1]
\end{tikzcd}$$

$L_\bullet[-1] : \; L_p[-1] = L_{p-1}$.

This short exact sequence gives a long exact sequence of homology :

$$\begin{tikzcd}
H_{p+1}(L_\bullet[-1]) \arrow[r] \arrow[d, equal] & H_p(L_\bullet) \arrow[r] & H_p(K(x_1,\dots,x_n)_\bullet) \arrow[r] & H_p(L_\bullet[-1]) \arrow[r] \arrow[d, equal] & H_{p-1}(L_\bullet) \\
H_p(L_\bullet) \arrow[ru, "(-1)^p\cdot x_n\cdot"'] & & & H_{p-1}(L_\bullet) \arrow[ru, "(-1)^{p-1} x_n\cdot"'] &
\end{tikzcd}$$

$\boxed{\text{\underline{Exercise}: prove that this map is } (-1)^p\cdot x_n\cdot}$

We find: $H_0(L_\bullet) \xrightarrow{x_n\cdot} H_0(L_\bullet) \to H_0(K(x_1,\dots,x_n)_\bullet) \to 0$ exact

$0 \to H_1(K(x_1,\dots,x_n)_\bullet) \to H_0(L_\bullet) \xrightarrow{x_n\cdot} H_0(L_\bullet)$ exact

for $p>1$: $H_p(K(x_1,\dots,x_n)_\bullet) = 0$. $\boxtimes$

See Serre IV, C, Thm. 8.

Corollary 3. Let A be a regular local noetherian ring, $n := \dim(A)$.

1. For $x_1, \ldots, x_n$ generators of m: $K(x_1, \ldots, x_n)_\bullet$ is a resolution of $A/mA = \kappa(A)$, this resolution has length n: for $p > n$, $K(x_1, \ldots, x_n)_p = 0$.

2. The cohomological dimension of A is n: $\forall p > n$, $\forall M, N$ finitely generated in A-Mod, $\mathrm{Ext}^p_A(M,N) = 0$, $\mathrm{Tor}^A_p(M,N) = 0$.

3. Let $0 \to N \to F_p \to \cdots \to F_0 \to M \to 0$ be exact in A-Mod with $F_0, \ldots, F_p$ free of finite rank, and $p \geqslant n-1$. Then N is free of finite rank.

Proofs 1: That's the previous thm.
2: results from 3.
3: Write it like this: $0 \to \underset{F_{p+1}}{N} \xrightarrow{d_{p+1}} F_p \to \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} \underset{F_{-1}}{M} \xrightarrow{d_{-1}} 0$,
put $Z_i := \ker(d_i)$ for $i \in \{-1, \ldots, p+1\}$.
Then $Z_{-1} = F_{-1} = M$, $Z_p = N$, $Z_i \hookrightarrow F_i \twoheadrightarrow Z_{i-1}$ exact for $i \in \{0, \ldots, p+1\}$.
So, $\forall p \geqslant 1$; $\forall i \in \{0, \ldots, p+1\}$: $\mathrm{Tor}_{p+1}(Z_{i-1}, k) \xrightarrow{\sim} \mathrm{Tor}_p(Z_i, k)$.

Hence: $\mathrm{Tor}_1(N, \overset{A/m}{k}) = \mathrm{Tor}_1(Z_p, k) = \mathrm{Tor}_2(Z_{p-1}, k) = \cdots = \mathrm{Tor}_{p+2}(Z_{-1}, k)$
This is zero because of 1 and $p+2 > p+1 \geqslant n$.

Lemma 4 Let N be a f.g. A-module, A local noetherian, with $\mathrm{Tor}_1(N,k) = 0$. Then N is free.

Proof Consider $N \twoheadrightarrow N/mN = k \otimes_A N$, let $n_1, \ldots, n_d \in N$ s.t. $(\bar{n}_1, \ldots, \bar{n}_d)$ is a k-basis of $k \otimes_A N$. Then $n_1, \ldots, n_d$ generate N by Nakayama's lemma. So: $K \rightarrowtail A^d \twoheadrightarrow N$. Apply $k \otimes_A -$:

$\underset{0}{\mathrm{Tor}_1(N,k)} \to k \otimes_A K \to k^d \xrightarrow{\sim} k \otimes_A N \to 0$ is exact, hence $k \otimes_A K = 0$.

Hence $K = 0$ by Nakayama's lemma. ∎

3. Back to schemes. We prove Thm 1.

Let X be as in the thm.

Choose $X \overset{j}{\hookrightarrow} \overline{X} \overset{i}{\hookrightarrow} \mathbb{P}^n_A$, j open dense ~~embedding~~ immersion, i closed immersion.

Let $\mathcal{F} \in \mathrm{Coh}(X)$.

Then $\exists\, \overline{\mathcal{F}} \in \mathrm{Coh}(\overline{X})$ s.t. $\overline{\mathcal{F}}|_X = \mathcal{F}$ (Hartshorne, exercise II.5.15)

By David's lecture 7: $\exists\, \overline{\mathcal{E}} \twoheadrightarrow \overline{\mathcal{F}}$, $\overline{\mathcal{E}} \in \mathrm{Vect}(\overline{X})$.

Then let $\mathcal{E} := \overline{\mathcal{E}}|_X$, we have $\mathcal{E} \twoheadrightarrow \mathcal{F}$, $\mathcal{E}$ in $\mathrm{Vect}(X)$.

Let $d := \dim(X)$. Choose $0 \to \mathcal{G} \to \mathcal{E}_{d-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{F} \to 0$ a resolution with all $\mathcal{E}_i$ in $\mathrm{Vect}(X)$.

Then, by Cor. 3, $\mathcal{G} \in \mathrm{Vect}(X)$, write $\mathcal{E}_d := \mathcal{G}$.

So we want to define $\mathrm{res}(\mathcal{F}) \in K^0(X)$ by:

$$\mathrm{res}(\mathcal{F}) := [\mathcal{E}_0] - [\mathcal{E}_1] + \cdots + (-1)^d [\mathcal{E}_d] =: [\mathcal{E}_\bullet]$$

<u>Lemma 5.</u> Let $\mathcal{E}_{1,\bullet} \twoheadrightarrow \mathcal{F}$ and $\mathcal{E}_{2,\bullet} \to \mathcal{F}$ be 2 finite resolutions of $\mathcal{F}$ by vectorbundles. Then $[\mathcal{E}_{1,\bullet}] = [\mathcal{E}_{2,\bullet}]$.

<u>Proof.</u> We first construct a third finite resolution $\mathcal{E}_\bullet$ dominating $\mathcal{E}_{1,\bullet}$ and $\mathcal{E}_{2,\bullet}$:

For the details, see Borel–Serre, lemma 13–14.

$$\begin{tikzcd} \mathcal{E}_{1,\bullet} \arrow[d] & \mathcal{E}_\bullet \arrow[l, two heads] \arrow[r, two heads] \arrow[d] & \mathcal{E}_{2,\bullet} \arrow[d] \\ \mathcal{F} \arrow[r, equal] & \mathcal{F} \arrow[r, equal] & \mathcal{F} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E}_{1,0} \oplus \mathcal{E}_{2,0} \arrow[d, two heads, "\mathcal{E}_{3,1}"'] & \overline{\mathcal{E}}_0 \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\square"] & \mathcal{E}_0 \in \mathrm{Vect}(X) \arrow[l, two heads] \\ \mathcal{F} \oplus \mathcal{F} & \mathcal{F} \arrow[l, hook', "\mathrm{diag}"] & \end{tikzcd}$$

$\overline{\mathcal{E}}_0$ coherent

That gives:

$$\begin{tikzcd} \mathcal{E}_{1,1} \arrow[d] & & \\ \mathcal{G}_1 \arrow[d, hook] & \mathcal{G} \arrow[l, two heads] \arrow[r, two heads] \arrow[d, hook] & \mathcal{G}_2 \arrow[d, hook] \\ \mathcal{E}_{1,0} \arrow[d, two heads] & \mathcal{E}_0 \arrow[l, two heads] \arrow[r, two heads] \arrow[d, two heads] & \mathcal{E}_{2,0} \arrow[d, two heads] \\ \mathcal{F} \arrow[r, equal] & \mathcal{F} \arrow[r, equal] & \mathcal{F} \end{tikzcd}$$

$$\begin{tikzcd} & & \mathcal{E}_2 \arrow[d, two heads] \\ (\mathcal{E}_{1,1} \times_{\mathcal{G}_1} \mathcal{G}) \oplus (\mathcal{E}_{2,1} \times_{\mathcal{G}_2} \mathcal{G}) \arrow[d, two heads] & & \overline{\mathcal{E}}_2 \arrow[ll, hook'] \arrow[d] \\ \mathcal{G} \oplus \mathcal{G} & & \mathcal{G} \arrow[ll, hook', "\mathrm{diag}"] \end{tikzcd}$$

etc.

Now consider

$$\begin{array}{ccccc} \mathcal{E}_{1,\bullet} & \twoheadleftarrow & \mathcal{E}_\bullet & \hookleftarrow & \mathcal{E}'_\bullet \\ \downarrow & & \downarrow & & \\ \mathcal{F} & = & \mathcal{F} & & \end{array}$$

short exact sequence of complexes. $\forall i\ \mathcal{E}'_i \in \mathrm{Vect}(X)$.

The long exact sequence of homology of the s.e.s. of complexes shows that $\mathcal{E}'_\bullet$ is acyclic: $\forall i \in \mathbb{Z}$: $H_i(\mathcal{E}'_\bullet) = 0$.
Hence (exercise, if you want): $[\mathcal{E}'_\bullet] \in 0$ in $K^0(X)$,

and $[\mathcal{E}_\bullet] = [\mathcal{E}_{1,\bullet}] + [\mathcal{E}'_\bullet] = [\mathcal{E}_{1,\bullet}]$ in $K^0(X)$. $\boxtimes$.

So we have a map $\mathrm{Ob}(\mathrm{Coh}(X)) \longrightarrow K^0(X)$,

$$\mathcal{F} \longmapsto [\mathcal{F}]^0 := [\mathcal{E}_\bullet], \quad \mathcal{E}_\bullet \twoheadrightarrow \mathcal{F} \text{ finite res. with all } \mathcal{E}_i \in \mathrm{Vect}(X).$$

It suffices now to show that this map is additive: $\forall\ \mathcal{F}' \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}''$ exact in $\mathrm{Coh}(X)$: $[\mathcal{F}]^0 = [\mathcal{F}']^0 + [\mathcal{F}'']^0$ in $K^0(X)$.
The additivity gives then $K_0(X) \to K^0(X)$.

<u>Lemma 6.</u> $\mathcal{F} \mapsto [\mathcal{F}]^0$ is additive.
<u>Proof</u>. For details, see Borel-Serre, lemma 12, the proof. Well, actually, we give the details here, <u>and</u> I make it a bit simpler. Choose $\mathcal{E}''_0 \twoheadrightarrow \mathcal{F}$, $\mathcal{E}''_0$ in $\mathrm{Vect}(X)$. Then $\mathcal{E}''_0 \twoheadrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}''$.
Choose $\mathcal{E}'_0 \twoheadrightarrow \mathcal{F}'$, $\mathcal{E}'_0 \in \mathrm{Vect}(X)$. Then we have:

$$\begin{array}{ccccc} \mathcal{E}'_0 & \rightarrowtail & \mathcal{E}'_0 \oplus \mathcal{E}''_0 & \twoheadrightarrow & \mathcal{E}''_0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}' & \rightarrowtail & \mathcal{F} & \twoheadrightarrow & \mathcal{F}'' \end{array}$$

, take kernels, continue, it stops at $\dim(X)$ by above.

$\boxtimes$.

<u>Exercise</u> 1. Let $A = \mathbb{Z}/4\mathbb{Z}$. Show that the A-module $\mathbb{F}_2$ does not have a finite free resolution. Hint: compute all $\mathrm{Tor}_i^A(\mathbb{F}_2, \mathbb{F}_2)$.
2. Show that (for k a field) $K_0(\mathbb{P}^1_k) \longrightarrow \mathbb{Z} \times \mathbb{Z}$
($\eta :=$ generic point) $\quad [\mathcal{F}] \longmapsto (\mathrm{rank}_\eta \mathcal{F}, \chi(\mathcal{F}))$ is an isom.
3. Describe $K^0(\mathbb{P}^1_k)$ + ring structure.