Mastermath, Geometry, Lectures 8–11

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1 Planning

There will be 4 lectures, on December 13 an "overview and questions" session, and on December 20 the 2nd partial exam. If necessary, there will be a resit on January 24 (for both Jeroen's and my parts of this course).

These notes, and the geogebra files and other files that they refer to, can be downloaded from `http://www.math.leidenuniv.nl/~edix/teaching/2013-2014/geometry/` at least shortly after each lecture.

I prefer to write some notes like this and using chalk and blackboard over making "powerpoint presentations", because I feel that just clicking to the next pause is somehow cheating in showing how to construct some mathematics. It is more honest to "create" the material in real-time, by thinking, speaking and writing. Of course, this is only a personal opinion of a university professor who has never had any education in teaching. It is not meant to convince you to do the same. And I will use the video-projector to show pictures.

These notes are not intended to be complete. Their purpose is to make it possible for the lecturer to skip some parts if there is not enough time, and to have an overview of what was treated. Students should make their own detailed notes during the lectures.

2 Lecture 8

2.1 From last week

- Go through Jeroen's slides.
- Explain slide 10 if necessary (see Kindt's "Lessen in de projectieve meetkunde", Epsilon uitgaven, Utrecht, page 23, for example).

- Union Jack construction: show it again (`union-jack-constr.ggb`), and show by moving that the result only depends on the point of intersection with the horizon.
- Then the homework given on Jeroen's slides 15–17. I am very pleasantly surprised by this, I have learned a lot from it myself (`quadrangle-square.ggb`).
- He also gave homework Stilwell 5.1–5.3. So let us also go through that.

Then it is now time for new stuff.

2.2 Axioms for projective geometry

Think of Hilbert's approach: just some data satisfying some conditions. A projective plane is a triple (Π, Λ, I) where Π is a set, called the set of Points, Λ is a set called the set of Lines, and I is a subset of $\Pi \times \Lambda$, called the incidence relation ($(P, L) \in I$ means: P is contained in L). The axioms are then:

P1 : every two distinct Points are contained in a unique Line,

P2 : every two distinct Lines contain a unique Point,

P3 : there are 4 distinct Points of which no 3 are collinear (that is, lie on one Line).

2.3 A model for $\mathbb{P}^2(\mathbb{R})$

We put the "all-seeing eye" where it belongs, at the origin in $\mathbb{R}^3$:

$$O = (0, 0, 0).$$

See figure 5.10 in Stillwell.

The idea is now to consider the set of all lines through O and *not* to make a choice of a plane not containing O to project to. I see two good reasons for this. The first reason is simply that it is better to avoid choices; choices imply loss of symmetry, and we want to profit from all the symmetry that projective geometry offers. The second reason is that for every plane H not through O we get trouble with the plane H' through O that is parallel to H: points in H' have no projection to H.

For Π, the set of Points, we take the set of lines in $\mathbb{R}^3$ that contain O, and for Λ, the set of Lines, we take the set of planes in $\mathbb{R}^3$ containing O. And for I we take the set of (P, L) with $P \subset L$.

Then this triple (Π, Λ, I) satisfies the axioms P1–P3 (apply some linear algebra). We denote this projective plane by $\mathbb{P}^2(\mathbb{R})$.

In fact, we could have used any field F instead of $\mathbb{R}$. That gives $\mathbb{P}^2(F)$.

2.4 Embedding of $\mathbb{R}^2$ in $\mathbb{P}^2(\mathbb{R})$

For this we must choose a plane H not containing O. We choose $H = \{(x, y, z) \in \mathbb{R}^3 : z = 1\}$. A point (x, y) in $\mathbb{R}^2$ gives a point $(x, y, 1)$ in H, and that point gives us the Point $\{t{\cdot}(x, y, 1) : t \in \mathbb{R}\}$, also denoted $\mathbb{R}{\cdot}(x, y, z)$, in $\mathbb{P}^2(\mathbb{R})$. Note that this map $(x, y) \mapsto \mathbb{R}{\cdot}(x, y, 1)$ is injective: if $\mathbb{R}{\cdot}(x, y, 1) = \mathbb{R}{\cdot}(x', y', 1)$, then indeed $(x, y) = (x', y')$.

Hence we have extended the euclidean plane $\mathbb{R}^2$ to the projective plane $\mathbb{P}^2(\mathbb{R})$.

2.5 Homogeneous coordinates

We introduce a notation called "homogeneous coordinates" that makes it easier to work with $\mathbb{P}^2(\mathbb{R})$.

For $(x, y, z) \in \mathbb{R}^3 - \{O\}$ we put:

$$(x : y : z) := \mathbb{R}{\cdot}(x, y, z) \quad \text{(the notation suggests ratios!).}$$

So, then, $(x : y : z)$ is a Point. And for every nonzero t in $\mathbb{R}$ we have:

$$(tx : ty : tz) = (x : y : z).$$

Formally speaking, the map $q\colon \mathbb{R}^3 - \{O\} \to \mathbb{P}^2(\mathbb{R})$ that sends (x, y, z) to $(x : y : z)$ is the quotient map for the equivalence relation:

$$(x, y, z) \sim (x', y', z') \text{ if and only if } \exists t \in \mathbb{R}^* : (tx, ty, tz) = (x', y', z').$$

We also introduce similar notation for Lines. For $(a, b, c) \in \mathbb{R}^3 - \{O\}$ we put:

$$(a : b : c)^\perp := \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\}.$$

Then, $(a : b : c)^\perp$ is a Line.

The exterior product $(a, b, c) \times (d, e, f) = (bf - ce, -(af - cd), ae - bd)$ gives the Line through two distinct Points, and also the intersection Point of two Lines.

Do exercises 5.4.1—5.4.3 of Stillwell.

2.6 Projective spaces

For every natural number $n \geq 1$ we have the set of lines through 0 in $\mathbb{R}^{n+1}$, the set of Points of $\mathbb{P}^n(\mathbb{R})$. The set of Lines are the planes in $\mathbb{R}^{n+1}$ through 0, the Planes are the 3-dimensional subvectorspaces of $\mathbb{R}^{n+1}$, etc. This also works with $\mathbb{R}$ replaced by any field F.

Linear algebra in F^4 gives: for any 3 distinct Points in $\mathbb{P}^3(F)$ there is a unique Plane that contains them, and the intersection of 2 distinct Planes is a Line, etc. Even: it is sometimes good to replace F^{n+1} by any F-vectorspace V, and denote the result by $\mathbb{P}(V)$. This occurs naturally if we consider a Line V in $\mathbb{P}^2(\mathbb{R})$: recall that V is a plane in $\mathbb{R}^3$ with $0 \in V$, that is, V is a 2-dimensional subspace of $\mathbb{R}^3$, but it does not come with a preferred basis. Choosing a basis of V gives an identification of V with $\mathbb{P}^1(\mathbb{R})$. But another choice of basis of V gives another identification of V with $\mathbb{P}^1(\mathbb{R})$, leading to the group of projective transformations of $\mathbb{P}^1(\mathbb{R})$ that we will study next week.

So, the case $n = 1$ is also important to us, it gives us the *projective line* $\mathbb{P}^1(\mathbb{R})$. In terms of homogeneous coordinates, the embedding above of $\mathbb{R}^2$ into $\mathbb{P}^2(\mathbb{R})$ is the map:

$$\mathbb{R}^2 \to \mathbb{P}^2(\mathbb{R}), \quad (x, y) \mapsto (x : y : 1)$$

The points of $\mathbb{P}^2(\mathbb{R})$ that are not in the image of this map are the $(x : y : 0)$. The set of these is in bijection with the set of Points $(x : y)$ of the projective line $\mathbb{P}^1(\mathbb{R})$. This is the set of "directions" in $\mathbb{R}^2$, and indeed these directions correspond to the points on the "horizon" in $\mathbb{P}^2(\mathbb{R})$.

2.7 An important principle

This is *not* contained in Stillwell. It leads to a very simple proof of some incidence theorem in euclidean geometry.

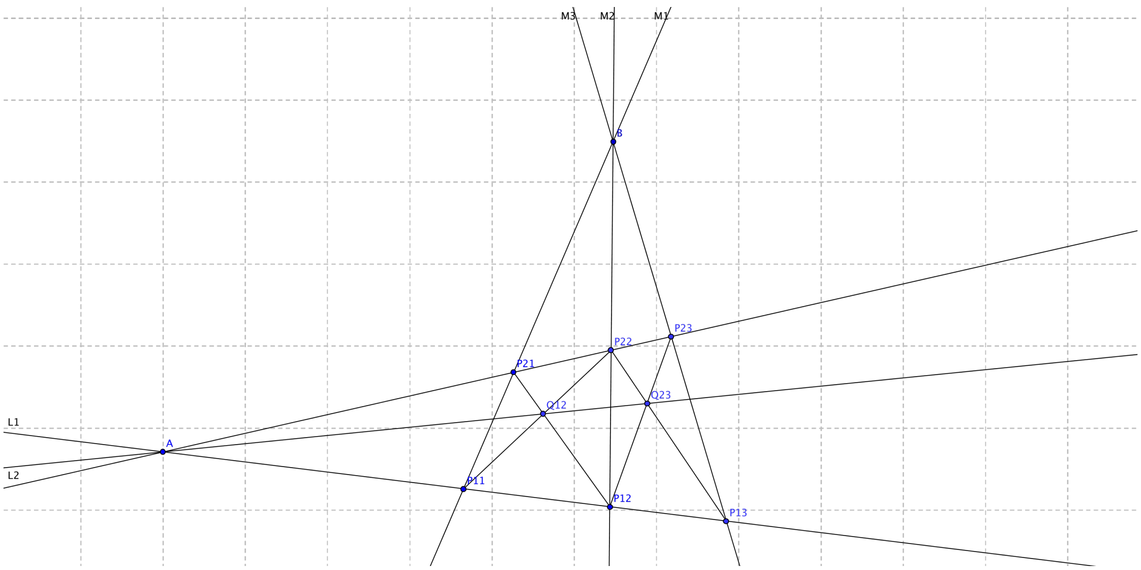
2.7.1 Theorem. *Let P, Q, R and S be in $\mathbb{P}^2(\mathbb{R})$, not any three of them on a Line. Then, after a suitable choice of basis (v_1, v_2, v_3) of $\mathbb{R}^3$, the homogenous coordinates are as follows: $P = (1 : 0 : 0)$, $Q = (0 : 1 : 0)$, $R = (0 : 0 : 1)$ en $S = (1 : 1 : 1)$. The basis (v_1, v_2, v_3) is unique up to simulaneous scaling $(v_1', v_2', v_3') = (tv_1, tv_2, tv_3)$ with $t \in \mathbb{R} - \{0\}$.*

Proof. We must take v_1 on P, v_2 on Q, and v_3 on R, all three non-zero. As P, Q and R are not contained in one Line, v_1, v_2 and v_3 are linearly independent, hence a basis of $\mathbb{R}^3$. Then $S = a_1v_1 + a_2v_2 + a_3v_3$ with all $a_i \neq 0$ (because S does not lie on any of the lines containing two of P, Q and R). Now replace v_1 by a_1v_1, etc. Done. □

2.7.2 Remark. The choice of such P, Q, R and S involves $4 \cdot 2 = 8$ degrees of freedom, whereas a choice of origin in $\mathbb{R}^2$ plus a choice of basis has only 6 degrees of freedom (affine geometry). A choice of origin and an orthonormal basis has only 3 degrees of freedom (euclidean geometry).

2.8 A funny construction

2.8.1 Theorem. *Let A and B in $\mathbb{P}^2(\mathbb{R})$ be distinct. Let L_1 and L_2 be distinct lines through A, not containing B, and M_1, M_2 en M_3 distinct lines through B not containing A. Let $P_{i,j}$ be the intersection point of L_i with M_j. Let $Q_{i,j}$ be the intesection point of the line containing $P_{i,i}$ and $Q_{j,j}$ with the line containing $P_{i,j}$ and $Q_{j,i}$. Then A is on the line containing $Q_{1,2}$ and $Q_{2,3}$.*



Proof. We choose a basis of $\mathbb{R}^3$ such that $A = (0:1:0)$, $B = (1:0:0)$ and $P_{1,3} = (0:0:1)$. Then the line containing A and B is the line with equation $z = 0$, the horizon of the euclidean plane $\mathbb{R}^2$ embedded by $(x, y) \mapsto (x : y : 1)$, L_1 is the line with equation $x = 0$ and L_2 the line with equation $y = 0$. In the euclidean plane, our configuration consists of two rectangles with a common side, and the line containing $Q_{1,2}$ and $Q_{2,3}$ is horizontal, hence meets A on the horizon. □

This proof illustrates well how one can use the extra freedom in projective geometry to reduce the proof of some incidence results in the projective plane to simpler cases in euclidean geometry. It also gives something back to euclidean geometry, namely, such incicende results.

2.9 Homework

1. Use Theorem 2.8.1 to show that in the euclidean plane one can draw the segment containing two distinct points, even if the distance between them is larger than the length of the ruler you have.

2. Read Stillwell §5.5-5.6 and do the exercises of those two sections.

3. Let $\mathbb{P} = (\Pi, \Lambda, I)$ be a projective plane (that is, satisfying the three axioms P1, P2 and P3). Assume that Π is a finite set. Then show that all lines in P have the same number of points, that through every point the same number of lines passes, and that there are as many points as lines. Hint: use projections, show that they give bijections.

4. Let (Π, Λ, I) be a projective plane. Show that (Λ, Π, I^t), with

$$I^t = \{(l, p) \in \Lambda \times \Pi : (p, l) \in I\}$$

is also a projective plane.

3 Lecture 9

3.1 Remarks, and homework from lecture 8

3.1.1 Remark. Last week it was hard to understand that a *Point* of $\mathbb{P}^2(\mathbb{R})$ is a *line* in $\mathbb{R}^3$ through 0, even though the projection from 0 of such a line should of course be a point. The problem is that we have to *define* that projection. That projection should be a map from $\mathbb{R}^3 - \{0\}$ to some set, but the question is then what that set should be. The answer to that question is: it should be the projective plane. So we are turning around in a circle, until we realise that nowadays we live in Cantor's paradise of set theory[1], and we can just consider the set whose elements are the lines in $\mathbb{R}^3$ through 0. In set theoretical notation:

$$\mathbb{P}^2(\mathbb{R}) = \{P : P \text{ is a line in } \mathbb{R}^3 \text{ through } 0\}.$$

One attempt to make the concept of Points more concrete was to use homogeneous coordinates for them:

$$\text{for } (x, y, z) \in \mathbb{R}^3 - \{0\}: \quad (x : y : z) = \mathbb{R}\cdot(x, y, z) = \{(tx, ty, tz) : t \in \mathbb{R}\}.$$

Then we have, for all $(x, y, z) \in \mathbb{R}^3 - \{0\}$ and for all t in $\mathbb{R} - \{0\}$:

$$(tx : ty : tz) = (x : y : z),$$

and this is consistent with the suggestion that the colons instead of commas indicate ratios.

I have tried, during last week, to find a simpler presentation of $\mathbb{P}^2(\mathbb{R})$, and have not succeeded. The best that I can come up with, is to go down by 1 in dimension: $\mathbb{P}^1(\mathbb{R})$ is the set of lines in $\mathbb{R}^2$ passing through 0. The advantage in this case is that the homogeneous coordinates are of the form $(x : y)$, and that this we are used to identify with the real number x/y, unless $y = 0$, in which case we can call it ∞. Indeed, the embedding $\mathbb{R} \to \mathbb{P}^1(\mathbb{R})$, $x \mapsto (x : 1)$, misses only one Point: $(1 : 0)$, and that point we call "the point at infinity". In the higher dimensional case of $\mathbb{P}^2(\mathbb{R})$, we have a "line at infinity". I hope that this helps. Time will also help (at least if you spend some of your time thinking about it, and working with it). Kindt's book, section 30 gives some more details, but no better ideas.

3.1.2 Homework

1. The exercises of Stillwell, §5.5–5.6, questions?

[1] a famous quote of Hilbert: "No one shall expel us from the Paradise that Cantor has created."

2. Who wants to construct the segment AB with a ruler that is too short?

3. The exercise with the finite projective plane: questions?

   Here is a sketch of a solution. Let l and l' be in Λ. Because of the axioms there is a $P \notin l \cup l'$ (fill in the details!). We get a map $f\colon l \to l'$ by sending $Q \in l$ to $PQ \cap l'$, the unique intersection point of the line through P and Q with l'. This map is bijective (that is, injective and surjective) because of axioms P1 and P2. The inverse of f is the projection $g\colon l' \to l$ from P. Conclusion: all lines have the same number of points. Let us denote it by $n+1$.

   Let now P be a point. We take a line l with $P \notin l$ (see exercise 5.3.4 in Stillwell). Then we get a map from the set of points on l to the set of lines through P by sending Q on l to the line PQ. This map is bijective. The inverse map sends a line m through P to $m \cap l$. Conclusion: through every point pass exactly $n+1$ lines.

   How many points are there in $\mathbb{P}$? Choose a point P, and consider the lines through P. There are $n+1$ of these, and they intersect only at P. Hence there are $1+(n+1)n = n^2+n+1$ points.

   The number of lines. Well, do the "dual argument" (exchange the notions of lines and points). Let l be a line. Every line $m \neq l$ meets l in one point. Through every point on l pass $n+1$ lines, among which l itself. That gives $1+(n+1)n$ lines.

   3.1.3 Example. Let p be a prime number. Then $\#\mathbb{P}^2(\mathbb{F}_p) = p^2+p+1$. In homogeneous coordinates: there are p^2 of the form $(x : y : 1)$, p of the form $(x : 1 : 0)$ and there is $(1 : 0 : 0)$.

3.2 Today's program

We treat Stillwell, 5.5, 5.6 and 7.3: projective transformations. The main points are:

- projective transformations are *fractional linear* in inhomogeneous coordinates,
- projective transformations are *linear* in homogeneous coordinates.

3.3 Affine transformations of lines

Let L be a line in $\mathbb{R}^2$ that does not contain 0. Then L has many parametrisations $f\colon \mathbb{R} \to L \subset \mathbb{R}^2$ (choice of a vector in L (steunvector in Dutch) and choice of a directional vector (richtings-vector)). For f_1 and f_2 parametrisations of L: $f_2^{-1} \circ f_1 \colon \mathbb{R} \to \mathbb{R}$ is an affine transformation, that is, of the form $x \mapsto ax+b$, with a and b in $\mathbb{R}$ and $a \neq 0$.

If we want that the parametrisation is an isometry, then the directional vector must have length 1. That means that there are exactly two directional vectors that differ by a factor -1. Then we find as $f_2^{-1} \circ f_1$ the isometries of $\mathbb{R}$: $x \mapsto ax + b$ met $a^2 = 1$.

We can also do this in higher dimension. Let V be a plane in $\mathbb{R}^3$. A parametrisation then corresponds to the choice of a vector v_0 in V, and a basis (v_1, v_2) of the plane $V - v_0$ parallel to V that passes through 0. Two parametrisations then differ by an affine map of $\mathbb{R}^2$: $x \mapsto Ax + b$, met $A \in \mathrm{GL}_2(\mathbb{R})$ en $b \in \mathbb{R}^2$. The matrix A is the matrix that expresses one basis in terms of the other. These transformations form a group, $\mathrm{Aff}_2(\mathbb{R})$.

The principle here is: groups occur as differences between parametrisations (isomorphisms) that preserve a certain "structure". For isometries it is "distance". Sometimes it is not so easy to say what that structure exactly is, for example for affine transformations.

If we consider planes in $\mathbb{R}^3$ containing 0, then we preserve the vector space structure, and we find the group $\mathrm{GL}_2(\mathbb{R})$ of automorphisms of the $\mathbb{R}$-vectorspace $\mathbb{R}^2$. If we want moreover to preserve distances, then…

3.4 Projective transformations of $\mathbb{P}^1(\mathbb{R})$

We have already seen that in $\mathbb{P}^2(\mathbb{R})$ projections between lines give bijections. Now we are interested in the formulas that we obtain in that way.

There are at least two ways to do this: from $\mathbb{R}^2$, with inhomogeneous coordinates, as in Stillwell, §5.5, and with $\mathbb{P}^2(\mathbb{R})$ itself, using homogeneous coordinates, as in, §7.3.

The first way. Consider the lines L_1 en L_2 given by the equations $x = 1$ en $y = 1$, and the projection $p\colon L_1 - \{(1, 0)\} \to L_2$ from 0. Then $p(1, t) = (1/t, 1)$. If we parametrise L_1 and L_2 by $f_1\colon \mathbb{R} \to L_1$ and $f_2\colon \mathbb{R} \to L_2$, by choosing vectors and directions vectors, then we find a map ($f_1(x) = (1, cx + d)$ and $f_2^{-1}(y, 1) = ay + b$):

$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto (1, cx + d) \mapsto (1/(cx + d), 1) \mapsto a/(cx + d) + b$$

that is, a *fractional linear transformation* as in Stillwell 5.5 and 5.6, a map of the form:

$$x \mapsto \frac{ax + b}{cx + d}, \quad \text{met } ad - bc \neq 0.$$

These transformations are invertible, and closed under composition: they form a *group*, the group of projective transformations of $\mathbb{P}^1(\mathbb{R})$. But there is a problem: we do not want to divide by 0, so we have to specify each time the domain of our map. The second approach solves this problem (it makes it disappear).

Second way. We use that $\mathbb{P}^1(\mathbb{R})$ is the set of lines through 0 in $\mathbb{R}^2$. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}_2(\mathbb{R})$ we have the map:

$$\mathbb{R}^2 \to \mathbb{R}^2, \quad x \mapsto Mx$$

that sends lines through 0 to lines through 0, and hence gives us a map from $\mathbb{P}^1(\mathbb{R})$ to $\mathbb{P}^1(\mathbb{R})$. Written out in detail:

$$(x_0 : x_1) \mapsto (ax_0 + bx_1 : cx_0 + dx_1).$$

In inhomogeneous coordinates this is:

$$x = (x : 1) \mapsto (ax + b : cx + d),$$

where we can write:

$$x = (x : 1) \mapsto (ax + b : cx + d) = \frac{ax + b}{cx + d}, \quad \text{if } cx + d \neq 0.$$

We see: these are the same transformations as the fractional linear transformations from $\mathbb{R}$ to $\mathbb{R}$, but in homogeneous coordinates they are linear.

We also see that M and M' in $\mathrm{GL}_2(\mathbb{R})$ give the same projective transformation on $\mathbb{P}^1(\mathbb{R})$ precisely when there is a k in $\mathbb{R}^*$ with $M' = kM$. So the group of projective transformations of $\mathbb{P}^1(\mathbb{R})$ is the same as the *quotient group* $\mathrm{PGL}_2(\mathbb{R})$ of $\mathrm{GL}_2(\mathbb{R})$ by the subgroup of scalar matrices $\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)$ (this is just for information to those who know about groups and subgroups and quotients).

Now we can write the transformations as fractional linear, and do computations with them, if we define: $x = (x : 1)$, for $x \in \mathbb{R}$, $\infty = (1 : 0)$, and, for $ad - bc \neq 0$, $(a\infty + b)/(c\infty + d) = a/c$ if $c \neq 0$ and ∞ if $c = 0$.

Finally: every $x \mapsto (ax + b)/(cx + d)$ can be obtained by composition of transformations of the form

- $x \mapsto x + b$, translation;
- $x \mapsto ax$, scaling;
- $x \mapsto 1/x$, inversion.

3.5 Stillwell, 5.7–5.9, the cross ratio

Main points:

- the cross ratio is invariant under projective transformations of $\mathbb{P}^1(\mathbb{R})$;
- a permutation σ of $\mathbb{P}^1(\mathbb{R})$ is a projective transformation if and only if σ preserves the cross ratio;

- every invariant of 4 points on $\mathbb{P}^1(\mathbb{R})$ is a function of the cross ratio.

Compare this to the role of distance in the context of isometries of $\mathbb{R}^2$.

Stillwell defines the cross ratio by a formula, without much motivation (like a magician, he pulls it out of a hat). Then he deduces properties of it, and then uses those for example for the "3 points map theorem". We are going to do this is the opposite order.

Still, let us first look at the definition. For p, q, r, s in $\mathbb{P}^1(\mathbb{R})$, distinct:

$$[p, q; r, s] := \frac{r-p}{r-q} \cdot \frac{s-q}{s-p} = \frac{(r-p)/(r-q)}{(s-p)/(s-q)} \in \mathbb{R} - \{0, 1\}$$

This is a ratio of ratios: that explains the name "dubbelverhouding" in Dutch. Notice: *one* of p, q, r, s may be ∞.

3.5.1 Theorem. (3 **point map theorem)** *Let p, q, r in $\mathbb{P}^1(\mathbb{R})$ be distinct. Then there is a unique projective transformation f of $\mathbb{P}^1(\mathbb{R})$ with $f(p) = \infty$, $f(q) = 0$ and $f(r) = 1$.*

Proof. We follow the proof of the theorem that every isometry in $\mathbb{R}^2$ is the composition of at most three line-reflections.

First the existence of f. If $p = \infty$, then let $f_1 = \mathrm{Id}$. Otherwise, let $f_1 \colon x \mapsto 1/(x-p)$. Then $f_1(p) = \infty$. Now the problem is reduced to an affine transformation! Let $q_1 = f_1(q)$ and $r_1 = f_1(r)$. Let $f_2 \colon x \mapsto x - q_1$. Then, under $f_2 \circ f_1$: $(p, q, r) \mapsto (\infty, 0, r_2)$, with $r_2 = f_2(r_1)$. Let $f_3 \colon x \mapsto x/r_2$, en $f = f_3 \circ f_2 \circ f_1$. This one works.

Now the uniqueness: … $\square$

3.5.2 Remark. We can also prove Theorem 3.5.1 in homogeneous coordinates: there is a basis (v_1, v_2) of $\mathbb{R}^2$, unique up to a common scaling factor, such that the homogenous coordinates of p, q, r with respect to this basis are $(1 : 0)$, $(0 : 1)$ en $(1 : 1)$.

3.5.3 Definition. Let p, q, r, s in $\mathbb{P}^1(\mathbb{R})$ be distinct. Let f be the unique projective transformation of $\mathbb{P}^1(\mathbb{R})$ that sends (p, q, r) to $(\infty, 0, 1)$. Then we define:

$$[p, q; r, s] = f(s).$$

Note that $[p, q; r, s]$ is an element of $\mathbb{P}^1(\mathbb{R}) - \{\infty, 0, 1\}$, hence we can write it as $(x : 1)$ for a unique x in $\mathbb{R} - \{0, 1\}$.

We can now *compute* $[p, q; r, s]$, by following the proof of Theorem 3.5.1:

$$x \mapsto 1/(x-p) \colon \quad (p, q, r, s) \mapsto (\infty, 1/(q-p), 1/(r-p), 1/(s-p)),$$

$$x \mapsto x - 1/(q-p) \colon \quad (\infty, 1/(q-p), 1/(r-p), 1/(s-p)) \mapsto (\infty, 0, 1/(r-p) - 1/(q-p), 1/(s-p) - 1/(q-p)),$$

and do the last step yourself.

3.5.4 Theorem. (invariance) *Let* g *be a projective transformation of* $\mathbb{P}^1(\mathbb{R})$*, and* p, q, r, s *in* $\mathbb{P}^1(\mathbb{R})$*, distinct. Then:*

$$[g(p), g(q); g(r), g(s)] = [p, q; r, s].$$

In words: projective transformations leave the cross ratio invariant.

Proof. Let f be the unique projective transformation with $(g(p), g(q), g(r)) \mapsto (\infty, 0, 1)$. Then.... □

3.5.5 Theorem. (4th point determination) *Let* p, q, r *in* $\mathbb{P}^1(\mathbb{R})$ *be distinct, and* $y \in \mathbb{R} - \{0, 1\}$*. Then there is a unique* s *in* $\mathbb{P}^1(\mathbb{R})$ *with* $[p, q; r, s] = y$*.*

Proof. We can do this by solving the equation:

$$\frac{r-p}{r-q} \cdot \frac{x-q}{x-p} = y.$$

But then we have to distinguish cases (for example, what if $q = \infty$?).

So I prefer the following way. Let f be the unique projective transformation with $(p, q, r) \mapsto (\infty, 0, 1)$. Then the equation is: $f(s) = y\ldots$ □

3.5.6 Theorem. *Let* $f\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ *be a bijection (this is rather wild: no continuity, not any given form for* f*!) with the property: for all* p, q, r, s *in* $\mathbb{P}^1(\mathbb{R})$ *distinct we have:*

$$[f(p), f(q); f(r), f(s)] = [p, q, r, s].$$

Then f *is a projective transformation.*

Proof. Let g be the unique projective transformation that sends $(f(\infty), f(0), f(1))$ to $(\infty, 0, 1)$. Then:

$$g \circ f\colon \quad (\infty, 0, 1) \mapsto (\infty, 0, 1)$$

and $g \circ f$ preserves cross rations, because.... □

Conclusion. The role played by the notion of distance for isometries is played by the cross ratio for projective transformations of $\mathbb{P}^1(\mathbb{R})$.

3.5.7 Theorem. (fundamental invariant) *Let* X *be a set, and*

$$I\colon \{(p, q, r, s) : p, q, r, s \text{ in } \mathbb{P}^1(\mathbb{R}) \text{ distinct}\} \to X$$

be a function that is invariant under projective transformations. Then there is a unique function $\overline{I}\colon \mathbb{R} - \{0, 1\} \to X$ *such that for all* (p, q, r, s)*:*

$$I(p, q, r, s) = \overline{I}([p, q; r, s]).$$

Proof. Let p, q, r, s in $\mathbb{P}^1(\mathbb{R})$ be distinct. Let g be the unique projective transformation with $(p, q, r) \mapsto (\infty, 0, 1)$. Then, by definition, $g(s) = [p, q; r, s]$. Hence (invariance of I):

$$I(p, q, r, s) = I(g(p), g(q), g(r), g(s)) = I(\infty, 0, 1, [p, q; r, s]).$$

Then $\overline{I}\colon \mathbb{R} - \{0, 1\}, \quad x \mapsto I(\infty, 0, 1, x)$ has the required property. The uniqueness follows from the fact that each x in $\mathbb{R} - \{0, 1\}$ is a cross ratio. □

3.6 Homework

1. Read 5.7–5.9 and 7.3 in Stillwell. Do exercise 5.7.2.
2. Make a geogebra file with a tiled floor and let it compute some cross ratios in it (there is a command in geogebra for cross ratio).
3. For each permutation σ of $\{\infty, 0, 1\}$ (points on $\mathbb{P}^1(\mathbb{R})$) give the unique projective transformation f such that $f(\infty) = \sigma(\infty)$, $f(0) = \sigma(0)$ en $f(1) = \sigma(1)$. Example: $x \mapsto 1/x$ exchanges ∞ and 0 and fixes 1.
4. Do the exercises in Stillwell §5.8, in 2 ways. By computing with the expression of the cross ratio, *and* by applying the definition of the cross ratio of this lecture.
5. Give an invariant for 3 points on $\mathbb{R}$ for affine transformations. Is every bijection $f\colon \mathbb{R} \to \mathbb{R}$ that preserves this invariant an affine transformation?
6. Somewhat deeper (testing your understanding): for how many points in $\mathbb{P}^2(\mathbb{R})$ does it make sense to define an invariant under projective transformations, and how can one then do this?

4 Lecture 10

4.1 Odds and ends from last week

1. Treat the remaining material of last week, starting at Theorem 3.5.5.

2. Go over the homework.

4.2 Today's program

We discuss Chapter 7 of Stillwell. Because of time limitations, we restrict ourselves to §7.1–7.3, but we give some real definitions and theorems, whereas Stillwell does not.

The subject is Felix Klein's "Erlanger Program". Erlangen is the place where he became professor, in 1872 at age 23. In his inaugural lecture ("oratie" in Dutch), he gave a new view on geometry, in terms of symmetry and the theory of groups. Well, this point of view was not really new, it had been "in the air", but he gave a precise formulation (well, precise..., nowadays, the language of set theory makes that easier).

To put the whole idea in a nutshell:

1. each geometry has its transformations that preserve its defining data, we call them the "automorphisms" of the geometry;

2. the set of these transformations is closed under composition and taking inverses (by definition, automorphisms have inverses), and contains the identity map, so they form a *group*;

3. the geometry in question can then be considered as the study of *properties that are invariant under the group of automorphisms.*

Let us give some examples. Before we go into detail in the next sections, les us have a look at the introduction of: `http://robotics.stanford.edu/~birch/projective/`

4.3 Projective geometry

We consider the real projective plane $\Pi = \mathbb{P}^2(\mathbb{R})$, with its collection Λ of subsets called Lines.

4.3.1 Definition. An *automorphism* of $\mathbb{P}^2(\mathbb{R})$ is a bijective map $g\colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$ such that for all subsets L of $\mathbb{P}^2(\mathbb{R})$:

$$L \text{ is a Line if and only if } g(L) := \{g(P) : P \in L\} \text{ is a Line.}$$

We let $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ denote the set of automorphisms of $\mathbb{P}^2(\mathbb{R})$.

4.3.2 Remark. In other words, an automorphism of $\mathbb{P}^2(\mathbb{R})$ is a bijection $g\colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$ that preserves the collection Λ of Lines.

4.3.3 Proposition. *The identity map* $\mathrm{id}_{\mathbb{P}^2(\mathbb{R})}$ *is in* $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$. *For all* g_1 *and* g_2 *in* $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, $g_1 \circ g_2$ *is in* $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$. *For all* g *in* $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$, g^{-1} *is in* $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$.

Proof. Let us write out a proof of the last statement. So let g be in $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$. As g is bijective, we have the inverse map $g^{-1}\colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$. Let L be a subset of $\mathbb{P}^2(\mathbb{R})$. If L is a Line, then we must show that $g^{-1}(L)$ is a Line, and if $g^{-1}(L)$ is a line, then we must show that L is a line. Note that we have:

$$g(g^{-1}(L)) = g(\{g^{-1}P : P \in L\}) = \{g(g^{-1}P) : P \in L\} = \{P : P \in L\} = L.$$

Now suppose that L is a line. Then $g^{-1}L$ is a Line because of the "if" in Definition 4.3.1, applied to g and $g^{-1}(L)$. Now suppose that $g^{-1}(L)$ is a Line. Then L is a line because of the "only if" in Definition 4.3.1, applied to g and $g^{-1}(L)$. □

4.3.4 Remark. I should admit that I manipulate the term "if and only if" without constantly realising what the meanings of the "if" and especially of the "only if" are. I find it easier to think in terms of implications ("if A then B", "if B then A") written given by arrows $A \Rightarrow B$, $A \Leftarrow B$, and of equivalence $A \Leftrightarrow B$.

4.3.5 Remark. One could wonder if we get the same notion of automorphism if we demand only one implication: if L is a Line then $g(L)$ is a Line. We will see that this is indeed the case. But the reason for stating Definition 4.3.1 as we do is to guarantee that for all automorphisms g, g^{-1} is also an automorphism.

We have many examples of automorphisms of $\mathbb{P}^2(\mathbb{R})$: the *projective transformations* given by multiplication by an invertible 3 by 3 matrix M with real coefficients (notation: $\mathrm{GL}_3(\mathbb{R})$). We write out explicitly what these transformations are. We write

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Then M gives the map $M\cdot$, multiplication by M:

$$M\cdot\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax+by+cz \\ dx+ey+fz \\ gx+hy+iz \end{pmatrix}.$$

Such a map $M\cdot$ maps lines in $\mathbb{R}^3$ through 0 to lines in $\mathbb{R}^3$ through 0. The inverse of $M\cdot$ is $M^{-1}\cdot$, hence the inverse also has this property. So, $M\cdot$ gives us a bijection from $\mathbb{P}^2(\mathbb{R})$ to itself. In homogeneous coordinates it is:

$$M\cdot\colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R}), \quad (x:y:z) \mapsto (ax+by+cz : dx+ey+fz : gx+hy+iz).$$

In inhomogeneous coordinates it is:

$$(x,y) = (x:y:1) \mapsto (ax+by+c : dx+ey+f : gx+hy+i) \text{``=''}. \left(\frac{ax+by+c}{gx+hy+i}, \frac{dx+ey+f}{gx+hy+i}\right),$$

where we should be careful with the last "=" because of the possible division by 0. Especially so because in this case we have not just one point at infinity (as in the case of projective transformations of $\mathbb{P}^1(\mathbb{R})$), but a whole Line.

As $M\cdot$ and its inverse map planes in $\mathbb{R}^3$ containing 0 to planes in $\mathbb{R}^3$ containing 0, these maps $M\cdot\colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$ are in $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$. The following theorem shows that these are in fact all automorphisms. But we will see that this depends on a special property of $\mathbb{R}$, and that with $\mathbb{R}$ replaced by $\mathbb{C}$ the outcome is different.

4.3.6 Theorem. *Let g be in $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$. Then there is an M in $\mathrm{GL}_3(\mathbb{R})$ such that g is the projective transformation given by M.*

Proof. Consider the standard 4 Points $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (0:0:1)$ and $P_4 = (1:1:1)$. Then $g(P_1)$, $g(P_2)$, $g(P_3)$ and $g(P_4)$ are 4 Points of which no 3 are on a Line. Therefore, there exists an M, even unique up to scaling ($M' = kM$ with $k \in \mathbb{R}^*$), such that $M\cdot g(P_1) = P_1$, $M\cdot g(P_2) = P_2$, $M\cdot g(P_3) = P_3$ and $M\cdot g(P_4) = P_4$.

Let $f\colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$ be the map $P \mapsto M\cdot g(P)$. Then $f \in \mathrm{Aut}(\mathbb{P}^2(\mathbb{R})$ (why?), and f fixes the points $P_1, \ldots, P_4$.

Note that $(1:0:0)$ and $(0:0:1)$ lie on the Line $(0:1:0)^\perp$. Hence f fixes the line $(0:1:0)^\perp$, and on that line it fixes $(0:0:1)$, $(1:0:0)$, and also $(1:0:1)$ (draw a picture!). Therefore, for every x in $\mathbb{R}$, there is a unique y in $\mathbb{R}$ such that $f(x:0:1) = (y:0:1)$. This defines a function $\sigma\colon \mathbb{R} \to \mathbb{R}$, bijective.

We will now show that σ is an automorphism of the field $\mathbb{R}$: σ is bijective and for all x and y in $\mathbb{R}$ we have $\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(xy) = \sigma(x)\sigma(y)$. The proof of this is a simple application of the geometric constructions of addition and multiplication as in Stillwell §1.4. These constructions are given in the geogebra files `addition_projective.ggb`, `addition_euclidean.ggb`, `multiplication_projective.ggb` and `multiplication_euclidean.ggb`. These files probably also help you becoming more familiar with going back and forth between the euclidean and projective perspectives.

How do these constructions show that $\sigma(x+y) = \sigma(x)+\sigma(y)$ and $\sigma(xy) = \sigma(x)\sigma(y)$? Well, apply f to all the Points and Lines in the constructions, and use the definitions. Here are some details. By definition, $f(x:0:1) = (\sigma(x):0:1)$, and $f(y:0:1) = (\sigma(y):0:1)$. We see that $f(0:x:1) = (0:\sigma(x):1)$ and that $f(y:x:1) = (\sigma(y):\sigma(x):1)$, and finally that $(\sigma(x+y):0:1) = f(x+y):0:1)$ is the intersection of the Line through $(1:-1:0)$ and $(\sigma(y):\sigma(x):1)$ and the line $(0:1:0)^\perp$, which is the Point $(\sigma(y)+\sigma(x):0:1)$.

In the previous paragraph we have also seen that for all x and y in $\mathbb{R}$ we have $f(x:y:1) = (\sigma(x):\sigma(y):1)$. It is a good exercise to show that we have, for all $(x:y:z)$ in $\mathbb{P}^2(\mathbb{R})$, that $f(x:y:z) = (\sigma(x):\sigma(y):\sigma(z))$

Finally, we conclude by applying the following theorem. It gives us that $f = \mathrm{id}_{\mathbb{P}^2(\mathbb{R})}$, hence that $M\cdot$ is the inverse of g. But then $g = M^{-1}\cdot$. □

4.3.7 Theorem. *The identity is the only automorphism of the field* $\mathbb{R}$.

Proof. Let σ be an automorphism of the field $\mathbb{R}$. Then $\sigma(0) = \sigma(0+0) = \sigma(0)+\sigma(0)$, hence (subtract $\sigma(0)$ from both sides) $\sigma(0) = 0$. We have $\sigma(1) = \sigma(1\cdot 1) = \sigma(1)\cdot\sigma(1)$, hence (subtract $\sigma(1)$ from both sides): $\sigma(1)\cdot(\sigma(1)-1) = 0$, hence $\sigma(1) = 0$ or $\sigma(1) = 1$. As σ is bijective, and already $\sigma(0) = 0$, we conclude that $\sigma(1) = 1$. Then, $\sigma(2) = \sigma(1+1) = \sigma(1)+\sigma(1) = 1+1 = 2$. By induction, we get $\sigma(n) = n$ for all $n \in \mathbb{Z}_{\geq 0}$. For all x in $\mathbb{R}$ we have $0 = \sigma(0) = \sigma(x+(-x)) = \sigma(x)+\sigma(-x)$, hence $\sigma(-x) = -\sigma(x)$. Hence we have, for all $n \in \mathbb{Z}$, $\sigma(n) = n$. For all x in $\mathbb{R}$ with $x \neq 0$ we have $1 = \sigma(1) = \sigma(x\cdot x^{-1}) = \sigma(x)\cdot\sigma(x^{-1})$, hence: $\sigma(x^{-1}) = \sigma(x)^{-1}$. For all a and b in $\mathbb{Z}$ with $b \neq 0$ we find: $\sigma(a/b) = \sigma(a\cdot b^{-1}) = \sigma(a)\cdot\sigma(b)^{-1}) = a/b$. So σ fixes all elements of $\mathbb{Q}$.

Let us now show that σ preserves the ordering of $\mathbb{R}$. So, let x and y be in $\mathbb{R}$, with $x \leq y$. Then $y - x \geq 0$, hence there is a z in $\mathbb{R}$ with $y - x = z^2$. Then we have $\sigma(y) - \sigma(x) = \sigma(y-x) = \sigma(z^2) = \sigma(z)^2$, hence we see that $\sigma(x) \leq \sigma(y)$. So, if $x \leq y$, then $\sigma(x) \leq \sigma(y)$. As for each x and y in $\mathbb{R}$ precisely one of the three statements $x < y$, $x = y$, $x > y$ holds, we have $\sigma(x) \leq \sigma(y) \Leftrightarrow x \leq y$.

Now let $x \in \mathbb{R}$ such that $x \notin \mathbb{Q}$. Then x divides $\mathbb{Q}$ into two subsets: $\{y \in \mathbb{Q} : y < x\}$ and $\{y \in \mathbb{Q} : y > x\}$, and x is the unique real number that is greater than each element of $\{y \in \mathbb{Q} : y < x\}$ and smaller than each element of $\{y \in \mathbb{Q} : y > x\}$. It follows that $\sigma(x) = x$.
□

4.3.8 Remark. The field $\mathbb{C}$ does have nontrivial automorphisms, for example the complex conjugation $z = a + bi \mapsto \overline{z} = a - bi$ (a and b are in $\mathbb{R}$), and many more, but for showing that one needs more algebra. In the homework you will show that this leads to automorphisms of $\mathbb{P}^2(\mathbb{C})$ that are not given by an element of $\mathrm{GL}_3(\mathbb{C})$.

4.3.9 Klein's program for $\mathbb{P}^2(\mathbb{R})$, cross ratio for 5 points in $\mathbb{P}^2(\mathbb{R})$

Let us continue with Klein's program for $\mathbb{P}^2(\mathbb{R})$. Its group of automorphisms is the group of projective transformations given by $\mathrm{GL}_3(\mathbb{R})$. The next step is then the study of properties that are preserved by the group of projective transformations.

For 2 distinct points P_1 and P_2 in $\mathbb{P}^2(\mathbb{R})$, there is a unique Line L containing them, and therefore, for any projective transformation g of $\mathbb{P}^2(\mathbb{R})$, the Line through $g(P_1)$ and $g(P_2)$ (distincts because g is bijective!), is $g(L)$. There is a projective transformation g such that $g(P_1) = (1 : 0 : 0)$ and $g(P_2) = (0 : 1 : 0)$, so there is no $\mathbb{R}$-valued invariant in this case (except constant ones).

For 3 distinct points P_1, P_2, P_3, there are two possibilities: P_3 is on the unique Line containing P_1 and P_2 (we say that P_1, P_2, P_3 are *collinear*), or not. Collinearity is preserved by projective transformations. If they are collinear, then there is a projective transformation g such that $g(P_1) = (1 : 0 : 0)$, $g(P_2) = (0 : 1 : 0)$, and $g(P_3) = (1 : 1 : 0)$. If they are not collinear, then there is a projective transformation such that $f(P_1) = (1 : 0 : 0)$, $f(P_2) = (0 : 1 : 0)$, $f(P_3) = (0 : 0 : 1)$. Again, no interesting invariants.

For 4 distinct points $P_1, \ldots, P_4$, no 3 of which collinear, there is a unique projective transformation such that $f(P_1) = (1 : 0 : 0)$, $f(P_2) = (0 : 1 : 0)$, $f(P_3) = (0 : 0 : 1)$ and $f(P_4) = (1 : 1 : 1)$. No interesting invariant in this case. However, if they *are* collinear, then there is a projective transformation g such that $g(P_1) = (1 : 0 : 0)$, $g(P_2) = (0 : 1 : 0)$, $g(P_3) = (1 : 1 : 0)$, and a unique x in $\mathbb{R}$ such that $g(P_4) = (x : 1 : 0)$, and we have seen last week that this x is the cross ratio of $(P_1, \ldots, P_4)$. Let us still denote it by $[P_1, P_2; P_3, P_4]$. As we see now, this is the first interesting $\mathbb{R}$-valued invariant of 4 collinear points in $\mathbb{P}^2(\mathbb{R})$.

As suggested in the previous homework assignment, we can attach a "super cross ratio" to 5 points $P_1, \ldots, P_5$ in $\mathbb{P}^2(\mathbb{R})$, of which no 3 are collinear. Let $f\colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$ be the unique projective transformation such that $f(P_1) = (1 : 0 : 0)$, $f(P_2) = (0 : 1 : 0)$, $f(P_3) = (0 : 0 : 1)$ and $f(P_4) = (1 : 1 : 1)$. Then there is a unique (x, y) in $\mathbb{R}^2$ with $f(P_5) = (x : y : 1)$. We call this (x, y) the cross ratio of $(P_1, \ldots, P_5)$. It has the property that x and y are in $\mathbb{R} - \{0, 1\}$, and $x \neq y$ (draw a picture!). By its very definition, it is preserved by all automorphisms of $\mathbb{P}^2(\mathbb{R})$. I should admit here that I came up with this definition myself, and so it should be considered as not more than my own hobby.

4.3.10 Klein's program for $\mathbb{P}^2(F)$ with F any field

This section is *not* part of the material for the 2nd partial exam. It is included for the sake of completeness.

Let F be a field. Then we have the projective plane $\mathbb{P}^2(F)$, because we have the F-vector space F^3, with its one-dimensional subspaces (Points) and two-dimensional subspaces (Lines). Let $\mathrm{Aut}(F)$ be the group of automorphisms of the field F, and let $\mathrm{PGL}_3(F)$ denote the group of projective transformations of $\mathbb{P}^2(F)$. Then the group $\mathrm{Aut}(\mathbb{P}^2(F))$ of automorphisms of $\mathbb{P}^2(F)$ contains both $\mathrm{PGL}_3(F)$ and $\mathrm{Aut}(F)$, where for σ in $\mathrm{Aut}(F)$ and $(x : y : z)$ in $\mathbb{P}^2(F)$ we put $\sigma(x : y : z) = (\sigma(x) : \sigma(y) : \sigma(z))$. Our proof of Theorem 4.3.6 shows that for each element f of $\mathrm{Aut}(\mathbb{P}^2(F))$ there are unique σ in $\mathrm{Aut}(F)$ and g in $\mathrm{PGL}_3(F)$ such that for all P in $\mathbb{P}^2(F)$ we have $f(P) = g(\sigma(P))$. In technical terms, $\mathrm{Aut}(\mathbb{P}^2(F))$ is the semi-direct product of $\mathrm{Aut}(F)$ by $\mathrm{PGL}_3(F)$ (just as isometries are uniquely the composition of an orthogonal linear map and a translation).

The cross ratio of 4 distinct collinear points $P_1, \ldots, P_4$ is then an invariant for $\mathrm{PGL}_3(F)$, but for σ in $\mathrm{Aut}(F)$ we have:

$$[\sigma(P_1), \sigma(P_2); \sigma(P_3), \sigma(P_4)] = \sigma([P_1, P_2; P_3, P_4]).$$

We conclude that $\mathrm{PGL}_3(F)$ is the group of those transformations in $\mathrm{Aut}(\mathbb{P}^2(F))$ that preserve the cross ratio. This is nice, because now we understand more about the group of projective transformations of $\mathbb{P}^2(F)$.

4.4 Euclidean geometry

We consider the plane $\mathbb{R}^2$ with all the extra data that make it into a Hilbert plane: the collection of subsets called lines, the notion of betweenness, congruence of line segments, congruence of angles. We consider bijective maps $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ that preserve all this structure. In particular, such a g has to preserve distances because it preserves congruence of line segments. Hence, such a g is an *isometry*: for all x and y in $\mathbb{R}^2$, we have $\|g(x) - g(y)\| = \|x - y\|$ (here, for $x = (x_1, x_2)$ in $\mathbb{R}^2$, $\|x\| = \sqrt{x_1^2 + x_2^2}$ denotes the length of x). We have seen in Stillwell, §3.7, the "three reflections theorem", that g is then a composition of at most 3 reflections about lines in $\mathbb{R}^2$. But then it follows that g preserves all the other structures (lines, betweenness, angles) as well. So we have proved the following theorem.

4.4.1 Theorem. *The automorphism group of the Hilbert plane* $\mathbb{R}^2$ *is the group* $\mathrm{Isom}(\mathbb{R}^2)$ *of isometries of* $\mathbb{R}^2$.

In the light of Klein's program, this makes us wonder if the notions of lines, betweenness and angles in the Hilbert plane $\mathbb{R}^2$ can be expressed in terms of distance. This is one of the homework exercises.

4.5 Affine geometry

Here we are in an interesting situation, where we have already the notion of "affine transformations", but not yet the notion of a corresponding geometry.

An affine transformation of $\mathbb{R}$ is a map $f\colon \mathbb{R} \to \mathbb{R}$ of the form $x \mapsto ax + b$, where a and b are in $\mathbb{R}$, with $a \neq 0$. Note that such maps are invertible, hence bijective. Note also that a and b are uniquely determined by f.

Of course, one cannot hope to do geometry just in dimension one, we have to consider a plane. An affine transformation of $\mathbb{R}^2$ is a map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that there exist an element A of $\mathrm{GL}_2(\mathbb{R})$ and a b in $\mathbb{R}^2$ such that for all x in $\mathbb{R}^2$:

$$f(x) = Ax + b.$$

In the homework, you will show that such an f is invertible, and that the inverses and compositions of such f's are again affine transformations. Hence they constitute a group of transformations. We call it the *affine group* of $\mathbb{R}^2$ and denote it by $\mathrm{Aff}(\mathbb{R}^2)$.

Now what is the geometry? One notion that is preserved is that of lines. Indeed, let L be a line in $\mathbb{R}^2$, and let f be as above. Then there are x_0 and d in $\mathbb{R}^2$, with $d \neq 0$, such that $L = \{x_0 + td : t \in \mathbb{R}\}$. Then

$$\begin{aligned} f(L) &= \{f(x_0 + td) : t \in \mathbb{R}\} = \{A(x_0 + td) + b : t \in \mathbb{R}\} \\ &= \{Ax_0 + tAd + b : t \in \mathbb{R}\} = \{t(Ad) + (Ax_0 + b) : t \in \mathbb{R}\}, \end{aligned}$$

hence $f(L)$ is the line with "steunvector" $Ax_0 + b$ and direction vector Ad. Just as in the case of $\mathbb{P}^2(\mathbb{R})$, it is a remarkable fact that this notion just by itself defines the notion of affine geometry on $\mathbb{R}^2$.

4.5.1 Theorem. *Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a bijective map that preserves the collection of lines in $\mathbb{R}^2$. Then g is an affine transformation.*

Proof. We argue as in the proof of Theorem 4.3.6. Let g be as in the statement. Consider the 3 points $P_1 = (0,0)$, $P_2 = (1,0)$ and $P_3 = (0,1)$. There is a unique affine transformation $x \mapsto Ax + b$ that maps $f(P_1)$ to P_1, $f(P_2)$ to P_2 and $f(P_3)$ to P_3: first translate by $-f(P_1)$ to move $f(P_1)$ to P_1. Then the images of $f(P_2)$ and $f(P_3)$ form a basis of $\mathbb{R}^2$, and that gives us a unique A that finishes the job.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the composition of g with this affine transformation. It fixes $(0,0)$, $(1,0)$ and $(0,1)$. For all x in $\mathbb{R}$, there is a unique y in $\mathbb{R}$ such that $f(x,0) = (y,0)$. This defines a map $\sigma\colon \mathbb{R} \to \mathbb{R}$ that is an automorphism of $\mathbb{R}$ as a field. One shows that for all (x,y) in $\mathbb{R}^2$, $f(x,y) = (\sigma(x), \sigma(y))$. As σ is the identity automorphism of $\mathbb{R}$ (Theorem 4.3.7), $f = \mathrm{id}_{\mathbb{R}^2}$ and g is an affine transformation. $\square$

4.5.2 Affine transformations of F^2 with F any field

This section is *not* part of the material for the 2nd partial exam. It is included for completeness.

Let F be a field. Then we have the *affine plane* F^2, with its collection of lines. Let $\mathrm{Aut}(F)$ be the group of automorphisms of F. Then the group of bijections of F^2 that preserve collinearity contains the affine transformations but also the automorphisms of F. We want to describe the group of affine transformations as a group that leaves a property invariant. What can that property be? In the case of $\mathbb{P}^2(F)$ we found the cross ratio. In this case we simply find the ratio of three distinct points on a line, as we will now show.

Let P_1, P_2 and P_3 be three distinct points in F^2, that are collinear. Then there is an affine transformation g such that $g(P_1) = (0,0)$, $g(P_2) = (1,0)$, and a unique x in F such that $g(P_3) = (0,x)$. This x does not depend on the choice of g, and we call it the *ratio* of P_1, P_2 and P_3. The group of affine transformations of F^2 is then the group of bijections of F^2 that preserve collinearity and the ratio.

Note: the ratio of P_1, P_2 and P_3 is the element λ of F such that the vector $\overrightarrow{P_1P_3}$ is λ times the vector $\overrightarrow{P_1P_2}$. This is why the group of affine transformations is so closely related to the structure of F-vector space of F^2.

4.5.3 Comparison of some geometries and their groups

Let us now compare the automorphism groups of euclidean geometry, affine geometry and projective geometry, describe a few more groups, and find what geometries they correspond to.

The group of automorphisms of the euclidean plane $\mathbb{R}^2$ is:

$$\mathrm{Isom}(\mathbb{R}^2) = \{x \mapsto Ax + b : A \in \mathrm{GL}_2(\mathbb{R}) \text{ orthogonal, and } b \in \mathbb{R}^2\}.$$

Let us denote by $\mathrm{GL}_2(\mathbb{R})^+$ the group of g in $\mathrm{GL}_2(\mathbb{R})$ with $\det(g) > 0$. These are precisely the g that preserve the standard orientation of $\mathbb{R}^2$. Then $\mathrm{Isom}(\mathbb{R}^2)$ has the subgroup:

$$\mathrm{Isom}(\mathbb{R}^2)^+ = \{x \mapsto Ax + b : A \in \mathrm{GL}_2(\mathbb{R})^+ \text{ orthogonal, and } b \in \mathbb{R}^2\}.$$

which is know as the group of *direct* isometries of $\mathbb{R}^2$. Its elements are of the form $x \mapsto Ax + b$ with A a rotation.

The automorphism group of the affine plane $\mathbb{R}^2$ is:

$$\mathrm{Aff}(\mathbb{R}^2) = \{x \mapsto Ax + b : A \in \mathrm{GL}_2(\mathbb{R}), \text{ and } b \in \mathbb{R}^2\}.$$

This group contains $\mathrm{Isom}(\mathbb{R}^2)$, and this corresponds to the fact that affine geometry is obtained from euclidean geometry by forgetting everything except collinearity. In between is the group

$\mathrm{Sim}(\mathbb{R}^2)$ of *similarities*, that is, affine transformations that preserve angles: the $x \mapsto Ax+b$ with A of the form kB with $k \in \mathbb{R}^*$ and B orthogonal. This group is generated by translations, reflections about lines through 0, and scaling (with respect to 0). We have the subgroups $\mathrm{Aff}(\mathbb{R}^2)^+$ and $\mathrm{Sim}(\mathbb{R}^2)^+$.

We do not want to consider groups smaller than $\mathrm{Isom}(\mathbb{R}^2)^+$ because we feel that for geometry at least all points in $\mathbb{R}^2$ should be equivalent, and also all directions, so we want at least all translations and all rotations.

It is interesting to connect the affine transformations of $\mathbb{R}^2$ with the projective transformations of $\mathbb{P}^2(\mathbb{R})$. Recall that our standard way to view $\mathbb{R}^2$ as a subset of $\mathbb{P}^2(\mathbb{R})$ is by sending (x,y) in $\mathbb{R}^2$ to $(x:y:1)$ in $\mathbb{P}^2(\mathbb{R})$. The complement of $\mathbb{R}^2$ in $\mathbb{P}^2(\mathbb{R})$ is then the line at infinity, the Line $L_\infty = \{(x:y:0) : (x,y) \in \mathbb{R}^2 - \{0\}\}$. Each projective transformation g of $\mathbb{P}^2(\mathbb{R})$ with $g(L_\infty) = L_\infty$ gives an automorphism of $\mathbb{R}^2$ as affine plane. Let us compute what this looks like in terms of matrices. Recall that projective tansformations of $\mathbb{P}^2(\mathbb{R})$ are of the form:

$$M\cdot \colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R}), \quad (x:y:z) \mapsto (ax+by+cz : dx+ey+fz : gx+hy+iz).$$

For $(x:y:0)$ in L_∞ we have:

$$M\cdot \colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R}), \quad (x:y:0) \mapsto (ax+by : dx+ey : gx+hy).$$

We see that g preserves L_∞ if and only if $g = h = 0$. Then $i \neq 0$ and as M and $i^{-1}M$ give the same projective transformation, we may assume that $i = 1$. Then we have:

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix}$$

with $ae - db \neq 0$. We find, for all $(x,y) \in \mathbb{R}^2$:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ax+by+c \\ dx+ey+f \\ 1 \end{pmatrix}.$$

We can write this as:

$$\begin{pmatrix} ax+by+c \\ dx+ey+f \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ f \end{pmatrix}.$$

So we see that the projective transformations that preserve L_∞ correspond exactly to the affine transformations of $\mathbb{R}^2$, and we have found that the affine transformation $x \mapsto Ax + b$

of $\mathbb{R}^2$ can be written as a matrix multiplication in dimension 3. We see that $\mathrm{Aff}(\mathbb{R})$ is contained in $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ in a very precise way. The description of affine transformations by 3 by 3 real matrices is used in many places, for example in the postscript language that most printers use. See for example §4.3.3 of the PostScript Language Reference book, 3rd edition: `http://www.adobe.com/products/postscript/pdfs/PLRM.pdf`. There, the matrix M is the transpose of what we have, because they consider the transformation $\cdot M$ that multiplies row vectors from the right.

4.6 Homework

1. Read Stillwell §7.1–7.3, and do the exercises.
2. Show that the map $g\colon \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$, $(x : y : z) \mapsto (\overline{x} : \overline{y} : \overline{z})$ is bijective and preserves Lines, hence is an automorphism of $\mathbb{P}^2(\mathbb{C})$. Show that it is not given by an element of $\mathrm{GL}_3(\mathbb{C})$.
3. We consider the Hilbert plane $\mathbb{R}^2$. Can one express the the notions of lines, betweenness and angles in the Hilbert plane $\mathbb{R}^2$ in terms of distance?
4. Show that inverses and compositions of affine transformations of $\mathbb{R}^2$ are affine transformations.

5 Lecture 11

5.1 Remaining material from last week

1. Treat the remaining material of last week, starting at Theorem 4.3.7.

2. Go over the homework.

5.2 Today's program

We finally get to hyperbolic geometry, a natural geometry that satisfies all of Hilbert's axioms (pages 43–45 of Stillwell) except I4 on the existence of a unique parallel; for this reason, it is called a *non-euclidean geometry*. The existence of this geometry implies that axiom I4 cannot be derived from the others.

This material is treated in Chapter 8 of Stillwell, but we do not follow his exposition so closely. We first study the complex projective line $\mathbb{P}^1(\mathbb{C})$ and its projective transformations and the complex conjugation, and only then restrict to automorphisms coming from the real projective line, and to the upper half plane.

5.3 The complex projective line, and its Moebius transformations

We have already defined $\mathbb{P}^1(F)$ for any field F, so a special case of this is then $\mathbb{P}^1(\mathbb{C})$: the set of one-dimensional subspaces in the $\mathbb{C}$-vector space $\mathbb{C}^2$. It is not easy to draw pictures of what happens in $\mathbb{C}^2$. But we have homogeneous coordinates:

$$\mathbb{P}^1(\mathbb{C}) = \{(x_0 : x_1) : (x_0, x_1) \in \mathbb{C}^2 - \{(0,0)\}\} = \{(z : 1) : z \in \mathbb{C}\} \cup \{(1 : 0)\} = \mathbb{C} \cup \{\infty\}.$$

Very often, $\mathbb{P}^1(\mathbb{C})$ is called the *Riemann sphere*, because topologically (one has to specify the topology first), it is homeomorphic to the 2-dimensional sphere S^2. It would be nice to treat more details here, but there is no time. A nice reference for this is Niels uit de Bos's bachelor thesis: `http://www.math.leidenuniv.nl/nl/theses/312/`.

It is important to recall that whereas the real projective plane $\mathbb{P}^2(\mathbb{R})$ is obtained from $\mathbb{R}^2$ by adding a whole horizon (a real projective Line), $\mathbb{P}^1(\mathbb{C})$ is obtained from $\mathbb{C}$ by adding the point ∞.

Each invertible complex 2 by 2 matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}_2(\mathbb{C})$ gives a projective transformation:

$$A\cdot \colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}), \quad (x_0 : x_1) \mapsto (ax_0 + bx_1 : cx_0 + dx_1), \quad z \mapsto \frac{az+b}{cz+d},$$

with the usual interpretation of the last fraction regarding ∞. And we also have the complex conjugation:

$$(x_0 : x_1) \mapsto (\overline{x_0} : \overline{x_1}), \quad z \mapsto \overline{z}, \quad \infty \mapsto \infty.$$

Composing complex conjugation with $A\cdot$ (in this order, first complex conjugation) gives:

$$(x_0 : x_1) \mapsto (a\overline{x_0} + b\overline{x_1} : c\overline{x_0} + d\overline{x_1}), \quad z \mapsto \frac{a\overline{z} + b}{c\overline{z} + d}.$$

It is a homework exercise for to show that composing in the other order also gives a transformation of this form. The group of *Moebius transformations* of $\mathbb{P}^1(\mathbb{C})$ is defined to be the set of transformations of $\mathbb{P}^1(\mathbb{C})$ that are of one of the forms $z \mapsto (az+b)/(cz+d)$ or $(a\overline{z}+b)/(c\overline{z}+d)$. We denote it by $M(\mathbb{P}^1(\mathbb{C}))$.

As a complex vector space, $\mathbb{C}$ has dimension one, and there is not really any geometry as intersecting of lines possible. But as a real vector space, it is of dimension two (the complex plane), and we have geometry of lines and circles.

5.3.1 Definition. A *generalised circle* in $\mathbb{P}^1(\mathbb{C})$ is a circle in C in $\mathbb{C}$, or a line in $\mathbb{C}$ together with ∞.

5.3.2 Theorem. *The Moebius transformations of* $\mathbb{P}^1(\mathbb{C})$ *preserve the collection of generalised circles.*

Proof. This is proved (well, in a special case) in §8.4 of Stillwell. So we give just a sketch. We know that the group of Moebius transformations is generated by:

1. the translations ($z \mapsto z + b$),
2. the multiplications (scaling, dilations) ($z \mapsto az$),
3. the complex conjugation ($z \mapsto \overline{z}$),
4. and the inversion ($z \mapsto 1/z$).

So if we show that all these transformations preserve the collection of generalised circles, then we are done. The first three types of transformations clearly do what we claim, and for the last one it is a computation that settles it (Stillwell, page 185).

As the proof has an interesting ingredient: *inversion in a circle*, we say something about it. The transformation $z \mapsto 1/\overline{z}$ is called inversion in the unit circle. For $z \neq 0$ we have: $1/\overline{z} = z/z\overline{z} = z/|z|^2$, hence the argument is unchanged, but the absolute value is inverted. The z with $|z| = 1$ are all fixed.

What we have to show is that this inversion preserves the collection of generalised circles. So let f denote this map, and let C be a circle in $\mathbb{C}$. We may assume that the center a of C is in $\mathbb{R}$ (why?) and that $a \geq 0$ (why?). Let r be the radius. Then $C = \{z \in \mathbb{C} : |z - a| = r\}$. Note that $f^2 = \mathrm{id}$, hence $f = f^{-1}$. Therefore,

$$f(C) = f^{-1}C = \{z \in \mathbb{C} : \left|\frac{1}{\overline{z}} - a\right| = r\}$$

where we are sloppy with what happens with ∞. Then.... What is important is that $f(C)$ is a circle if $0 \notin C$, and $f(C)$ is a line if $0 \in C$, and then it is the line given by the equation $\Re(z) = 1/2r$. Note that this tells us that for L a line in $\mathbb{C}$ with $0 \notin L$, $f(L)$ is a circle containing 0. □

5.3.3 Remark. Inversion in a circle is a great tool to simplify problems about circles, as some circles can be made into lines.

Moebius transformations preserve angles. This is defined and proved in §8.5 of Stillwell. The difficulty here is that we take an infinitesimal perpective on it, as we must not only talk about angles between lines, but about angles between generalised circles (not at ∞, luckily for us). This property is in fact true for all maps $f\colon \mathbb{C} \to \mathbb{C}$ that have a non-zero complex derivative. Such maps are called *conformal*. They scale in a non-uniform way, but preserve angles at all points. For example, see Escher's picture of the portrait gallery and what is written about it and very nice animations at `http://escherdroste.math.leidenuniv.nl`.

Let us do it for our Moebius transformations. Translations, multiplications and complex conjugation preserve (unoriented) angles (complex conjugation is a reflection, it changes the orientation). So, we only need to show that $z \mapsto -1/z$ preserves angles. Let f denote this map, and let $z_0 \neq 0$ be in $\mathbb{C}$. Here is the computation (and implicitly the definition) that f preserves infinitesimally angles at z_0. For h in $\mathbb{C}$ with $|h|$ small:

$$\begin{aligned} f(z_0 + h) &= f(z_0) + f(z_0 + h) - f(z_0) = f(z_0) + \frac{-1}{z_0 + h} + \frac{1}{z_0} = \\ &= f(z_0) + \frac{h}{z_0(z_0 + h)} = f(z_0) + \frac{h}{z_0^2} + \frac{h}{z_0(z_0 + h)} - \frac{h}{z_0^2} \\ &= f(z_0) + \frac{1}{z_0^2}\cdot h - \frac{1}{z_0^2(z_0 + h)}\cdot h^2 \end{aligned}$$

Note that the first term is $f(z_0)$, the image of z_0, the second term is linear in h, and the third term is neglegible compared to h if $|h|$ tends to zero. The complex derivative at z_0 of our map f is $1/z_0^2$, as is to be expected from $f(z) = -1/z$. We have defined the notion of angle and proved the following theorem.

5.3.4 Theorem. *The Moebius transformations of $\mathbb{P}^1(\mathbb{C})$ preserve angles at all points of $\mathbb{C}$ that are not mapped to ∞.*

This is nice, but for euclidean (well, Hilbertian) geometry, we need the concept of Lines, Line segments, and congruence of Line segments, and for that last thing, it is good to have the notion of length of a line segment. We want these notions to be preserved by the transformations of the geometry. But if we have scaling transformations $z \mapsto az$ with $|a| < 1$, then no sensible notion of length can be preserved. So we have to restrict from the group of Moebius transformations to a smaller group. And we also have to restrict the space $\mathbb{P}^1(\mathbb{C})$ to something smaller, because circles can have no "betweenness" notion.

5.4 The upper half plane with its Moebius transformations

Inside $\mathbb{P}^1(\mathbb{C})$ we have the subset $\mathbb{P}^1(\mathbb{R})$. This is maybe hard to see from the definition as complex lines in $\mathbb{C}^2$ through zero, but in terms of inhomogeneous coordinates it is a triviality: $\mathbb{R} \cup \{\infty\}$ is indeed a subset of $\mathbb{C} \cup \{\infty\}$. The complement of $\mathbb{R} \cup \{\infty\}$ in $\mathbb{C} \cup \{\infty\}$ is the union of the *upper half plane* $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ and the lower half plane $\overline{\mathbb{H}} = \{z \in \mathbb{C} : \Im(z) < 0\}$. We will concentrate on $\mathbb{H}$; it is the set of Points for our non-euclidean geometry. Before we discuss its Lines, we decide what should be its transformations, à la Klein.

In the homework you will show that the Moebius transformations of $\mathbb{P}^1(\mathbb{C})$ that preserve $\mathbb{P}^1(\mathbb{R})$ are precisely those of the form $z \mapsto (az+b)/(cz+d)$ with a, b c and d in $\mathbb{R}$, or of the form $z \mapsto (a\overline{z}+b)/(c\overline{z}+d)$ with a, b c and d in $\mathbb{R}$. Such Moebius tranformations either preserve $\mathbb{H}$ and $\overline{\mathbb{H}}$, or exchange them. The ones preserving $\mathbb{H}$ are precisely the $z \mapsto (az+b)/(cz+d)$ with a, b c and d in $\mathbb{R}$ and $ad - bc > 0$, together with the $z \mapsto (a\overline{z}+b)/(c\overline{z}+d)$ with a, b c and d in $\mathbb{R}$ with $ad - bc < 0$. We denote the group of all Moebius transformations of $\mathbb{H}$ by $M(\mathbb{H})$, and the subgroup of all $z \mapsto (az+b)/(cz+d)$ with $ad - bc > 0$ by $M(\mathbb{H})^+$, and the complement of $M(\mathbb{H})^+$ in $M(\mathbb{H})$ by $M(\mathbb{H})^-$.

The fact that ∞ is not in $\mathbb{H}$ is nice, we do not need to worry about it anymore, our transformations of $\mathbb{H}$ do not divide by zero. The next proposition shows that our transformations in $M(\mathbb{H})$ make $\mathbb{H}$ "everywhere the same", just like the isometries in $\mathbb{R}^2$ for euclidean geometry.

5.4.1 Proposition. *Let z be in $\mathbb{H}$. Then there is a g in $M(\mathbb{H})^+$ such that $g(z) = i$.*

Proof. Write $z = x + yi$ with x and y in $\mathbb{R}$. Then $z \mapsto z - x$ is in $M(\mathbb{H})^+$ and it maps z to yi. Then the map $z \mapsto y^{-1}z$ maps yi to i. □

The next proposition about the elements g in $M(\mathbb{H})$ such that $g(i) = i$ will show that we have a good notion of congruence of Line Segments defined in terms of $M(\mathbb{H})$.

5.4.2 Proposition. *For g in $M(\mathbb{H})$ we have that $g(i) = i$ is equivalent to g being of the form $z \mapsto (az+b)/(-bz+a)$ or $z \mapsto (-a\overline{z}+b)/(b\overline{z}+a)$ with a and b in $\mathbb{R}$ and $a^2+b^2 \neq 0$.*

For g in $M(\mathbb{H})$ we have that $g(i) = i$ and $g(\mathbb{R}_{>0}i) = \mathbb{R}_{>0}i$ is equivalent to g being of one of the form $z \mapsto z$, $z \mapsto -1/z$, $z \mapsto -\overline{z}$, $z \mapsto 1/\overline{z}$.

Proof. For the first statement: a simple computation. For the understanding of the second statement, it is useful to note that $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ represents a rotation in $\mathbb{R}^2$, followed by a scaling by $\sqrt{a^2+b^2}$ (it is really multiplication by $a - bi$ in $\mathbb{C}$, with $\mathbb{C}$ viewed as $\mathbb{R}^2$). The derivative of $z \mapsto (az+b)/(-bz+a)$ at $z = i$ is (computation...) equal to $(a+bi)/(a-bi)$, a complex number of absolute value 1, and with argument twice that of $a+bi$. So $z \mapsto (az+b)/(-bz+a)$ acts on the set of infinitesimal directions at i by rotation by twice the argument of $a+bi$. It is clear that one can prove the 2nd statement by a computation. □

5.5 The Lines of hyperbolic geometry

It is time to decide on the Lines of hyperbolic geometry. We present it now without much motivation, but when we define hyperbolic length later (if there is time for it), we can *prove* that these Lines are really the lines in the sense of distance and “triangle equalities”.

We decide that the vertical line $\mathbb{R}_{>0}i = \{yi : y \in \mathbb{R}_{>0}\}$ is a Line. Then, because all g in $M(\mathbb{H})$ are postulated to be transformations, all images of $\mathbb{R}_{>0}i$ are Lines. The g of the form $z \mapsto z + x$ (with x in $\mathbb{R}$) show that all vertical lines $\{x + yi : y \in \mathbb{R}_{>0}\}$ are Lines. Note that the generalised circle $g(\mathbb{R}i)$ that they are half of (up to ∞) intersect $\mathbb{R}$ perpendicularly. This means that there are no more lines that are Lines. Then what circles do we get as images? They are of the form $g(\mathbb{R}i) \cap \mathbb{H}$, and $g(\mathbb{R}i)$. It is not suprising that these are precisely the circles that intersect $\mathbb{R}$ perpendicularly. Equivalently: the circles with center on $\mathbb{R}$. See the pictures in Stillwell. You can also watch `http://www.youtube.com/watch?v=eAn6NHpBn2c`. I do not know how much of hyperbolic geometry is availabe in geogebra.

To summarise: the Lines in $\mathbb{H}$ are the vertical lines $\{x + yi : y \in \mathbb{R}_{>}0\}$ for x in $\mathbb{R}$, and the half circles $\{z \in \mathbb{H} : |z - a| = r\}$ for a in $\mathbb{R}$ and r in $\mathbb{R}_{>0}$.

5.5.1 Theorem. *Let z and w be in $\mathbb{H}$, distinct. Then there is a unique Line containing them.*

Proof. Let us first show existence. If $\Re(z) = \Re(w)$, then there is a vertical line containing them. Now suppose that $\Re(z) \neq \Re(w)$. Then the bisector of z and w (middellloodlijn) is not parallel to $\mathbb{R}$, hence intersects $\mathbb{R}$, and that gives a point a in $\mathbb{R}$ that has equal distances to z and w, hence a Line containing z and w.

Now uniqueness. If $\Re(z) = \Re(w)$ then there is no a in $\mathbb{R}$ with $|z - a| = |w - a|$, hence the only Lines containing z and w are vertical lines, and there is exactly one of them. If $\Re(z) \neq \Re(w)$. Then there is no vertical line containing them, and there is a unique a in $\mathbb{R}$ such that $|z - a| = |w - a|$. □

5.6 Back to Hilbert

Let us now equip $\mathbb{H}$ with all the structure that is needed to make it make it into a Hilbert plane, and let us show that all axioms except I4 are satisfied.

1. We have $\mathbb{H}$ the set of Points.

2. We have the collection of subsets called Lines. The axioms I1, I2 and I3 are satisfied, but I4 is not.

3. On each Line we have a betweenness notion: each Line is a homeomorphic image of $\mathbb{R}$. The axioms B1, B2, B3 and B4 are satisfied (why?). So we have the notions of ray AB for Points $A \neq B$ and of angle $\angle ABC$ for distinct Points A, B and C with C not on the Line containing B and A.

4. We have the notion of congruence of angles $\angle ABC \sim \angle A'B'C'$ because our angles have a *value* in the interval $(0, \pi)$. This is consistent with our transformations (they preserve these values). Axioms C4 and C5 are satisfied.

5. Congruence of Line Segments. We use the transformations, but it would be nicer to have a *value* for the length of a segment; we remedy this in the net section. We say that $AB \cong A'B'$ if there is a g in $M(\mathbb{H})$ such that $g(A) = A'$ and $g(B) = B'$. Then axiom C2 is satisfied because $M(\mathbb{H})$ is a group. It follows from Proposition 5.4.2 that axioms C1 and C3 are satisfied. Axiom C6 (SAS) is satisfied because there is a g in $M(\mathbb{H})$ that sends (A, B, C) to (D, E, F).

6. Axioms E, A and D are also satisfied.

As a consequence we have the following theorem.

5.6.1 Theorem. *Axiom I4 cannot be derived from the other axioms in Hilbert's list of axioms.*

5.7 Hyperbolic distance

We just imagine that we drive our car on $\mathbb{H}$ and that we have to pay toll (rekeningrijden!), and that the toll per meter depends on the position where we are, but not on the direction in which we go: the toll per meter is a function $T\colon \mathbb{H} \to \mathbb{R}$, let us assume that T is continuous. So if our trip is described by a function $f\colon [0,1] \to \mathbb{H}$, say $f(t) = (f_1(t), f_2(t)) = f_1(t) + f_2(t)i$, with f_1 and f_2 differentiable with continuous derivative, then the toll we pay is given by:

$$\mathrm{Toll}(f) = \int_{t=0}^{1} T(f(t)) \cdot \sqrt{f_1'(t)^2 + f_2'(t)^2}\, dt = \int_{t=0}^{1} T(f(t)) \cdot |f'(t)|\, dt,$$

because our speed vector is the vector $f'(t) = (f_1'(t), f_2'(t))$ and so our speed is $\sqrt{f_1'(t)^2 + f_2'(t)^2}$. We want that for any g in $M(\mathbb{H})$, that the trip $g \circ f$ costs the same as f. This is achieved if the integrands for $\mathrm{Toll}(f)$ and $\mathrm{Toll}(g \circ f)$ are equal. Note that $(g \circ f)'(t) = g'(f(t)) \cdot f'(t)$ (chain rule):

$$\mathrm{Toll}(g \circ f) = \int_{t=0}^{1} T(g(f(t))) \cdot |(g \circ f)'(t)|\, dt, = \int_{t=0}^{1} T(g(f(t))) \cdot |g'(f(t)) \cdot f'(t)|\, dt.$$

And that means that we want, for all g in $M(\mathbb{H})$, and all z in $\mathbb{H}$, that:

$$T(g(z)) \cdot |g'(z)| = T(z).$$

So, if we decree that $T(i) = 1$, then there is at most one such a function T by Proposition 5.4.1, and exactly one by (the proof of) Proposition 5.4.2. Using $z \mapsto yz$ and $z \mapsto z + x$, one sees that $T(x + yi) = 1/y$.

We define the hyperbolic distance $d(z, w)$ between z and w in $\mathbb{H}$ as the minimum toll required to drive from z to w. By its construction, this is a distance function (triangle inequality), and for each g in $M(\mathbb{H})$ we have $d(g(z), g(w)) = d(z, w)$. It is also clear that the toll we pay depends only on the route we take, not on our speed (changing the speed gives another function $f \circ s$, with $s\colon [0,1] \to [0,1]$, computation left to th reader). To determine d, it suffices to determine $d(i, yi)$, say for $y > 1$. It is easy to show that the shortest route is via the imaginary axis: $f\colon [1, y] \to \mathbb{H}$, $f(t) = it$. Then we have:

$$d(i, yi) = \int_{t=1}^{y} T(f(t)) \cdot |i|\, dt = \int_{t=0}^{y} \frac{1}{t}\, dt = \ln(y).$$

I stop here, there is still a lot to say, but there is no time now…

5.8 Homework

1. Prove that doing first $z \mapsto (az + b)/(cz + d)$ and then complex conjugation is also a Moebius transformation. Is the group of Moebius transformations commutative?

2. Let g be a Moebius transformation of $\mathbb{P}^1(\mathbb{C})$. Show that it preserves $\mathbb{P}^1(\mathbb{R})$ if and only if it is of the form $z \mapsto (az+b)/(cz+d)$ with a, b, c and d in $\mathbb{R}$, or of the form $z \mapsto (a\overline{z}+b)/(c\overline{z}+d)$ with a, b, c and d in $\mathbb{R}$.

3. Use the invariance of the hyperbolic distance under $M(\mathbb{H})$ to prove give a general formula for the hyperbolic distance $d(p,q)$, for p and q in $\mathbb{H}$, distinct. Here is what you will find: if z and w are the boundary points in $\mathbb{P}^1(\mathbb{R})$ of the Line containing p and q, and the ordering on the Line is (z,p,q,w), then:

$$d(p,q) = |\ln[w,z;p,q]|.$$

5.9 A few exercises for practicing for the 2nd partial exam

1. (a) Explain how to draw the perspective view of a tiled floor with only a straightedge, if one tile is already given and both pairs of its opposite sides are not parallel.

   (b) And what if exactly one pair of opposite sides that *is* parallel? (This one is maybe too hard.)

   (c) And what if *both* pairs of opposite sides are parallel? (This one should be easy.)

2. Let $P = (0:0:1)$, $Q = (6:0:1)$, $R = (6:4:1)$ and $S = (0:4:1)$ be in $\mathbb{P}^2(\mathbb{R})$.

   (a) Compute the equations $ax+by+cz = 0$ (or the homogeneous coordinates $(a:b:c)^\perp$ if you prefer) of the Lines PR and QS, (these Lines are the diagonals of the quadrilateral $PQRS$).

   (b) Compute the intersection Point of the two diagonals.

   (c) Now interpret the quadrilateral in $\mathbb{R}^2$ embedded in $\mathbb{P}^2(\mathbb{R})$ by $(x,y) \mapsto (x:y:1)$, and give a second computationn of the intersection point of the diagonals.

3. The cross ratio of 4 ordered distinct Points p, q, r and s on $\mathbb{P}^1(\mathbb{R})$ is defined to be $g(s)$, where g is the unique projective transformation of $\mathbb{P}^1(\mathbb{R})$ that sends (p,q,r) to $(\infty,0,1)$. Derive the usual formula

$$[p,q;r,s] = \frac{r-p}{r-q} \cdot \frac{s-q}{s-p}$$

   for the cross ratio from this.

4. If you have a photograph with an image of three equally spaced points on a line, how do you know where the image of the fourth equally spaced point in this sequence must be? Apply this to the case where the images of the three given points are at positions 0, 3, 5 on a line.

5. The affine transformations of $\mathbb{R}^2$ are the maps $x \mapsto Ax + b$ with A in $\mathrm{GL}_2(\mathbb{R})$ and b in $\mathbb{R}^2$. The set of them is denoted by $\mathrm{Aff}(\mathbb{R}^2)$.

   (a) Show that $\mathrm{Aff}(\mathbb{R}^2)$ is a group: contains $\mathrm{id}_{\mathbb{R}^2}$, is closed under composition and inverses.

   (b) Let G be the subset of $\mathrm{Aff}(\mathbb{R}^2)$ consisting of those affine transformations that preserve oriented angles. What are then the conditions on A and b for the affine $x \mapsto Ax + b$ to be in G?

   (c) Show that G is a group.

   (d) Try to give, a la Hilbert, a set of notions and axioms such that $\mathbb{R}^2$ with the appropriate structure is a model with G as group of automorphisms.[2]

6. In $\mathbb{P}^1(\mathbb{C})$, we have the generalised circles $\mathbb{P}^1(\mathbb{R})$ and the unit circle C in $\mathbb{C}$. We want to find a g in $M(\mathbb{P}^1(\mathbb{C}))^+$ such that $g(\mathbb{P}^1(\mathbb{R})) = C$.

   (a) Show that for three distinct points in $\mathbb{C}$, there is a unique generalised circle that contains them.

   (b) Find the unique g in $M(\mathbb{P}^1(\mathbb{C}))^+$ that maps $(0, 1, \infty)$ to $(-i, 1, i)$.

   (c) Show that g maps $\mathbb{P}^1(\mathbb{R})$ to C, and maps i to 0.

   (d) Show that $g(\mathbb{H})$ is the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

7. Explain in a few (at most 10, say) lines how we know that the parallel axiom cannot be deduced from the other axioms of Hilbert's list for euclidean geometry.

[2] This is probably a bit hard. If such a questions is in the partial exam, then you will get a list of Hilberts notions and axioms.

2nd partial exam “geometry”, mastermath, Fall 2013

December 20, 14:00–16:45, Bas Edixhoven

You may write your solutions in **Dutch or English**.

1. (a) Draw, with only a straightedge, a perspective view of a floor that is tiled with squares. At least 4 tiles with one common corner should be drawn. Explain the steps in your construction in the “algebra view style” of geogebra.

   (b) Draw, with only a straightedge, a perspective view of a floor that is tiled with regular 6-gons; draw at least 2 adjacent 6-gons. Hint: use the contruction in (a), and subdivide some tiles into two triangles.

2. Let $P = (1 : -1 : 1)$, $Q = (1 : 1 : 1)$, $R = (-1 : 1 : 1)$ and $S = (-1 : -1 : 1)$ in $\mathbb{P}^2(\mathbb{R})$.

   (a) Give the points P', Q', R' and S' in $\mathbb{R}^2$ that correspond to P, Q, R and S if we embed $\mathbb{R}^2$ in $\mathbb{P}^2(\mathbb{R})$ via:

   $$f\colon \mathbb{R}^2 \to \mathbb{P}^2(\mathbb{R}), \quad (x, y) \mapsto (x : y : 1).$$

   And draw them. They are the corners of the square $P'Q'R'S'$.

   (b) Give the points P'', Q'', R'' and S'' in $\mathbb{R}^2$ that correspond to P, Q, R and S if we embed $\mathbb{R}^2$ in $\mathbb{P}^2(\mathbb{R})$ via:

   $$g\colon \mathbb{R}^2 \to \mathbb{P}^2(\mathbb{R}), \quad (u, v) \mapsto (1 : u : v).$$

   And draw them. They are, in *this* order, the corners of the square $P''Q''S''R''$. Note that in the order $P''Q''R''S''$ we do *not* get a square.

   (c) Figure out what is happening here by computing the images of the sides of the square $P'Q'R'S'$ under the map

   $$g^{-1} \circ f\colon (x, y) \mapsto (x : y : 1) = (1 : y/x : 1/x) \mapsto (y/x, 1/x)$$

   that is defined on the set of (x, y) with $x \neq 0$. Draw these images. What is the image of the interior of $P'Q'R'S'$? Can you understand this by looking at what happens in $\mathbb{R}^3$?

3. (a) The cross ratio of 4 ordered distinct Points p, q, r and s on $\mathbb{P}^1(\mathbb{R})$ is defined to be $g(s)$, where g is the unique projective transformation of $\mathbb{P}^1(\mathbb{R})$ that sends (p, q, r) to $(\infty, 0, 1)$. Derive the usual formula

   $$[p, q; r, s] = \frac{r - p}{r - q} \cdot \frac{s - q}{s - p}$$

   for the cross ratio from this.

(b) Use the definition of the cross ratio in (a) to prove that it is invariant under all projective transformations of $\mathbb{P}^1(\mathbb{R})$.

(c) Suppose you have a photograph of a scene in which there are 4 equidistant points P_1, P_2, P_3 and P_4 on a line. Let Q_1, Q_2, Q_3 and Q_4 be the images of these points in the photograph. Explain that these 4 points Q_i are on a line L.

(d) Suppose that, for some coordinate, Q_1, Q_2 and Q_4 are at positions 0, 1 and 4 on L. Then what is the position of Q_3?

4. Consider the following Lines in the hyperbolic plane $\mathbb{H}$: the lines $\{z \in \mathbb{H} : \Re(z) = 0\}$, $\{z \in \mathbb{H} : \Re(z) = 1/2\}$, and the arcs $\{z \in \mathbb{H} : |z| = 1\}$ and $\{z \in \mathbb{H} : |z| = 2\}$.

   (a) Show that there is a 4-gon in $\mathbb{H}$ of which the sum of the angles is less than 360 degrees (you do not need to compute the sum).

   (b) Deduce from this that there are triangles of which the sum of the angles is less than 180 degrees.

5. Explain in a few (at most 10, say) lines why the parallel axiom cannot be deduced from the other axioms of Hilbert's list for euclidean geometry.

6. Recall the 3 axioms for projective geometry:

   P1 every two distinct Points are contained in a unique Line,

   P2 every two distinct Lines contain a unique common Point,

   P3 there are 4 distinct Points of which no 3 are collinear (that is, lie on one Line).

   Explain why these axioms do *not* imply that there are at least 8 Points.

Please fill out the evaluation form on the mastermath website. This feedback is important for the mastermath organisation, and for the teachers.

2e deeltoets "geometry", mastermath, najaar 2013

20 december, 14:00–16:45, Bas Edixhoven

De uitwerkingen mogen in het **nederlands of engels** gedaan worden.

1. (a) Teken, met alleen een latje, een plaatje in perspectief van een vloer die betegeld is met vierkanten. Tenminste 4 tegels met een gemeenschappelijk hoekpunt moeten getekend zijn. Licht de stappen van je constructie toe in de "algebra view style" van geogebra.

   (b) Teken, met alleen een latje, een plaatje in perspectief van een vloer die betegeld is met gelijkzijdige 6-hoeken; teken tenminste twee 6-hoeken met een gemeenschappelijke zijde. Hint: gebruik je constructie in (a), en verdeel een aantal tegels in twee driehoeken.

2. Laat $P = (1 : -1 : 1)$, $Q = (1 : 1 : 1)$, $R = (-1 : 1 : 1)$ en $S = (-1 : -1 : 1)$ in $\mathbb{P}^2(\mathbb{R})$.

   (a) Geef de punten P', Q', R' en S' in $\mathbb{R}^2$ die corresponderen met P, Q, R en S als we $\mathbb{R}^2$ inbedden in $\mathbb{P}^2(\mathbb{R})$ via:

   $$f \colon \mathbb{R}^2 \to \mathbb{P}^2(\mathbb{R}), \quad (x, y) \mapsto (x : y : 1).$$

   En teken ze. Ze zijn de hoekpunten van het vierkant $P'Q'R'S'$.

   (b) Geef de punten P'', Q'', R'' en S'' in $\mathbb{R}^2$ die corresponderen met P, Q, R en S als we $\mathbb{R}^2$ inbedden in $\mathbb{P}^2(\mathbb{R})$ via:

   $$g \colon \mathbb{R}^2 \to \mathbb{P}^2(\mathbb{R}), \quad (u, v) \mapsto (1 : u : v).$$

   En teken ze. Ze zijn, in *deze* volgorde, de hoekpunten van het vierkant $P''Q''S''R''$. Merk op dat we in de volgorde $P''Q''R''S''$ we *geen* vierkant krijgen.

   (c) Vind uit wat er hier gebeurt door de beelden te berekenen van de zijden van het vierkant $P'Q'R'S'$ onder de afbeelding

   $$g^{-1} \circ f \colon (x, y) \mapsto (x : y : 1) = (1 : y/x : 1/x) \mapsto (y/x, 1/x)$$

   die is gedefinieerd op de verzameling van (x, y) met $x \neq 0$. Teken deze beelden. Wat is het beeld van het inwendige van $P'Q'R'S'$? Kun je dit begrijpen door te kijken naar wat er gebeurt in $\mathbb{R}^3$?

3. (a) De dubbelverhouding van 4 geordende verschillende Punten p, q, r en s op $\mathbb{P}^1(\mathbb{R})$ is gedefinieerd als $g(s)$, waarbij g de unieke projectieve transformatie van $\mathbb{P}^1(\mathbb{R})$ is die (p, q, r) naar $(\infty, 0, 1)$ stuurt. Leid hieruit de bekende formule

$$[p, q; r, s] = \frac{r - p}{r - q} \cdot \frac{s - q}{s - p}$$

voor de dubbelverhouding af.

   (b) Gebruik de definitie van de dubbelverhouding in (a) om te bewijzen dat ze invariant is onder alle projectieve transformaties van $\mathbb{P}^1(\mathbb{R})$.

   (c) Stel je hebt een foto van een tafereel waarin 4 punten P_1, P_2, P_3 en P_4 op gelijke afstanden op een lijn liggen. Laat Q_1, Q_2, Q_3 en Q_4 de beelden van deze punten in de foto zijn. Leg uit dat deze 4 punten Q_i op een lijn L liggen.

   (d) Stel dat, voor een coordinaat op L, Q_1, Q_2 and Q_4 de punten 0, 1 en 4 op L zijn. Wat is dan het punt Q_3?

4. Beschouw de volgende Lijnen in het hyperbolisch vlak $\mathbb{H}$: de lijnen $\{z \in \mathbb{H} : \Re(z) = 0\}$, $\{z \in \mathbb{H} : \Re(z) = 1/2\}$, en de cirkelbogen $\{z \in \mathbb{H} : |z| = 1\}$ en $\{z \in \mathbb{H} : |z| = 2\}$.

   (a) Laat zien dat er een 4-hoek in $\mathbb{H}$ is waarvan de som van de hoeken kleiner is dan 360 graden (je hoeft de som niet uit te rekenen).

   (b) Leid hieruit af dat er driehoeken zijn waarvan de som van de hoeken kleiner is dan 180 graden.

5. Leg uit in een paar (ten hoogste 10, zeg) regels waarom het parallel-axioma niet kan worden afgeleid uit de rest van Hilbert's axioma's voor euclidische meetkunde.

6. Hier zijn de 3 axioma's voor projectieve meetkunde:

   P1 ieder tweetal verschillende Punten is bevat in een unieke Lijn,

   P2 ieder tweetal verschillende Lijnen snijdt in een uniek punt,

   P3 er zijn 4 verschillende Punten waarvan geen 3 collineair zijn (d.w.z., op één Lijn liggen).

   Leg uit waarom deze axioma's *niet* impliceren dat er tenminste 8 Punten zijn.

Vul alsjeblieft het evaluatieformulier op de mastermath website in. Deze informatie is belangrijkvoor de mastermath organisatie, en voor de docenten.