Topics in Algebraic Geometry
Week 7

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Quotients of schemes by finite group actions

Let G be a finite group acting (from the left) on a scheme X, meaning that for all $g \in G$ we have an automorphism $\rho(g)$ of X, with $\rho(1_G) = id_G$ and $\rho(gh) = \rho(g)\rho(h)$ $\forall g, h \in G$. We would like to understand whether, or under what condition, we can "pass to the quotient $G\backslash X$" and, in that case, what kind of object we obtain.

To make this precise, by *categorical quotient* of X by the action of G we mean a scheme Y, together with a morphism $q\colon X \to Y$ which is *G-invariant*, i.e. $q \circ \rho(g) = q$ $\forall g \in G$, and such that every G-invariant map $q'\colon X \to Y'$ factors uniquely through q (i.e. there exists a unique map $\bar{q}\colon Y \to Y'$ such that $q' = \bar{q} \circ q$).

It may well happen that such a categorical quotient exists, but with very little to do with what we expect from our geometric idea of quotient; for instance, it might not separate the orbits. The idea, then, is to construct an object in the bigger category of locally ringed spaces, which will be the categorical quotient of X by G *in this category*; in other words, we build a locally ringed space Y satisfying the same universal property as above, but for morphisms between locally ringed spaces. This object will have the desired geometric properties and, if it turns out to be a scheme, then we call it the *geometric quotient* of X by the action of G. When this is the case, since the category of schemes is a full subcategory of that of locally ringed spaces (by definition of a morphism of schemes), Y will also be the categorical quotient of X by G in the category of schemes. The construction goes as follows.

- Let $Y := G\backslash X$ as sets, with the quotient topology, and let $q\colon X \to G\backslash X = Y$ be the canonical projection.
- For $V \subseteq Y$ open, let $U := q^{-1}(V) \subseteq X$; U is open in X and it is *G-stable* (i.e. $\rho(g)(U) \subseteq U$ $\forall g \in G$), hence G acts (from the right) on $\mathcal{O}_X(U) = q_*\mathcal{O}_X(V)$, via $\rho(g)^\#\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$, for $g \in G$.

  Let $\mathcal{O}_Y(V) := \mathcal{O}_X(U)^G := \left\{ f \in \mathcal{O}_X(U) \;\middle|\; \rho(g)^\#(f) = f \ \forall g \in G \right\}$, the subring of *$G$-invariants*.

Then, $\mathcal{O}_Y$ is a subsheaf of rings of $q_*\mathcal{O}_X$ (mainly because $\rho(g)^\#$ commutes with the restrictions of $\mathcal{O}_X$, for $g \in G$). Moreover, $(Y, \mathcal{O}_Y)$ is a locally ringed space, with q a morphism of locally ringed spaces (on the sheaves it is just the inclusion), and $q\colon X \to Y$ is the categorical quotient of X by G, in the category of locally ringed spaces (this is an exercise, see the hint at the end). We will denote Y by $G\backslash X$.

In the affine case, this construction gives a particularly nice result; namely, it gives back an affine scheme.

Proposition 1. *With notation as above, if $X = \operatorname{Spec} A$ is an affine scheme, then $Y = G\backslash X \cong \operatorname{Spec} A^G$, where A^G is the subring of invariants for the (right) G-action on A (via $\rho(g)^\# =$, $g \in G$). The quotient map $q\colon X \to Y$ is the one induced by the inclusion $A^G \subseteq A$ and it is integral, closed and surjective.*

Proof. For all $a \in A$, let $a \cdot g$ denote $\rho(g)^\#(a)$; the polynomial $\prod_{g\in G}(X - a\cdot g) \in A^G[X]$ is monic and vanishes at a. Thus, q is integral. The fact that q is also closed and surjective is a general fact about integral morphisms (see [3, 5.10]).

If $\mathfrak{p}, \mathfrak{p}' \in \operatorname{Spec} A$ lie in the same orbit, then $q(\mathfrak{p}) = \mathfrak{p} \cap A^G = \mathfrak{p}' \cap A^G = q(\mathfrak{p}')$, hence the continuous map $|q|\colon |\operatorname{Spec} A| \to |\operatorname{Spec} A^G|$ factors as $|\operatorname{Spec} A| \to G\backslash|\operatorname{Spec} A| \xrightarrow{|\bar{q}|} |\operatorname{Spec} A^G|$. On the other hand, if $\mathfrak{p}, \mathfrak{p}' \in \operatorname{Spec} A$ are such that $\mathfrak{p} \cap A^G = \mathfrak{p}' \cap A^G$, then, for all $x \in \mathfrak{p}$, we have $N(x) := \prod_{g\in G} x \cdot g \in \mathfrak{p} \cap A^G = \mathfrak{p}' \cap A^G \subseteq \mathfrak{p}'$, hence $\mathfrak{p} \subseteq \bigcup_{g\in G} g \cdot \mathfrak{p}'$, so $\mathfrak{p} \subseteq g\cdot\mathfrak{p}'$ for some $g \in G$ (see [3, 1.11(i)]), but then $\mathfrak{p} = g\cdot\mathfrak{p}'$. Thus, $|\bar{q}|$ is a homeomorphism (the inverse is continuous because q is closed).

Finally, for all $f \in A^G$, we have $(A^G)_f \cong (A_f)^G$ (this is an exercise, namely [3, ex5.12]), hence $\mathcal{O}_Y \cong \mathcal{O}_{\operatorname{Spec} A^G}$. ☐

The way to generalize the above result to arbitrary scheme is given in the following corollary.

Corollary 2. *Let X be a scheme with an action by a finite group G, such that every orbit is contained in an open affine subset of X. Then $G\backslash X$ is a scheme.*

Proof. The trick is to find, for all $x \in X$, a G-stable open affine subset $U \subseteq X$ containing the orbit $G\cdot x$. Then, for $y \in G\backslash X$, if we take $U = \operatorname{Spec} A$ containing the orbit represented by y as above, we have, by the last proposition, $q(U) \cong \operatorname{Spec} A^G$ (here $q\colon X \to G\backslash X$ is the projection; note that $q(U) = G\backslash U$ by construction), with $q(U)$ open in $G\backslash X$.

As for the first part, let $U_0 = \operatorname{Spec} A$ be an affine open subset of X containing the orbit $G \cdot x$ and define $U_1 := \bigcap_{g\in G} g \cdot U_0$. U_1 is open, G-stable and contains $G \cdot x$. Let $I \subseteq A$ be an ideal defining the closed subset $U_0 \setminus U_1$ of $U_0 = \operatorname{Spec} A$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{n} \in U_0$ be the prime ideals corresponding to the points of $G\cdot x$. Let $a \in I \setminus \bigcup_{i=1}^n \mathfrak{p}_\mathfrak{i}$ (if $I \subseteq \bigcup_{i=1}^n \mathfrak{p}_\mathfrak{i}$, then $I \subseteq \mathfrak{p}_\mathfrak{i}$ for some i, i.e. $\mathfrak{p}_\mathfrak{i} \in V(I) = U_0 \setminus U_1$, contradicting $G \cdot x \subseteq U_1$). Let, finally, $U_2 := D(a) = \operatorname{Spec} A_a \subseteq U_1$, so that $G \cdot x \subseteq U_2$, and define $U_3 := \bigcap_{g\in G} g \cdot U_2$. Since U_2 is affine, as well as every $g \cdot U_2$ (because $U_2 \subseteq U_1 \subseteq U_0$, with U_1 G-stable) and U_0 is separated (because affine), then U_3 is affine. It is also open, G-stable and it contains the orbit $G \cdot x$. ☐

Remark 3. When X is projective over a scheme S and G acts on X over S, then the hypothesis of the preceding corollary is automatically satisfied. This follows from the fact that a finite set of points in the projective space, over any filed, is always contained in an open affine subset.

Example 4. Here we will take for granted some basic notions in invariant theory. Let R be a ring, $n \geq 1$ an integer, $A := R[x_1, \dots, x_n]$, $G := Sn$ the symmetric group on n elements, acting on A via permutations of the x_i's (for $\sigma \in G$, $\sigma(x_i) = x_{\sigma(i)}$). Then $B := A^G = R[e_1, \dots, e_n]$, where $e_1, \dots, e_n$ are the elementary symmetric polynomials ($e_k = \sum_{1 \leq j_i < \dots < j_k \leq n} x_{j_1} \cdots x_{j_k}$). Since $e_1, \dots, e_n$ are algebraically independent over R, we have $B \cong A$. Therefore, if $X := \operatorname{Spec} A = \mathbb{A}^n_R$, then $X/G \cong \operatorname{Spec} B \cong X$.

Remark 5. All the above discussion can be put in the more general context of a group scheme acting on a scheme. Indeed, a finite group G can be seen as a group scheme over a base field k as $\underline{G} := \coprod_{g \in G} \operatorname{Spec} k \cong \operatorname{Spec} \prod_{g \in G} k$, where the group law of G induces maps $m \colon \underline{G} \times_k \underline{G} \to \underline{G}$ (multiplication), $e \colon \operatorname{Spec} k \to \underline{G}$ (neutral element), $i \colon \underline{G} \to \underline{G}$ (inverse), satisfying the group axioms. Since $\underline{G} \times_k X = (\coprod_{g \in G} \operatorname{Spec} k) \times_k X \cong \coprod_{g \in G} (\operatorname{Spec} k \times_k X) \cong \coprod_{g \in G} X$, an action $\rho \colon \underline{G} \times_k X \to X$ is just a set of maps $\rho(g) \colon X \to X$, for $g \in G$, satisfying the axioms of an action.

In the general case of a group scheme G acting on a scheme X over a basis S, the fact that it may happen that, as topological spaces, $|G \times_S X| \neq |G| \times_{|S|} |X|$, forces the construction of the geometric quotient to be done in the category of ringed spaces, where such an equality holds.

Proposition 6. *Let R be a Noetherian ring, X a scheme of finite type over R, G a finite group acting on X over R, in such a way that the conditions of corollary 2 are satisfied. Then $G \backslash X$ is of finite type over R and the projection $\pi \colon X \to G \backslash X$ is finite.*

Proof. First reduce to the case of $X = \operatorname{Spec} A$ affine, with A finitely generated over R (cover X by G-stable affine opens as in the proof of the corollary).

Now let $x_1, \dots, x_n$ generate A over R. Let $C \subseteq A^G$ be the R-algebra generated by the coefficients of the polynomials $\prod_{g \in G} (X - x_i \cdot g)$, for $i = 1, \dots, n$. Since C is finitely generated over R, which is a Noetherian ring, C is Noetherian too. Moreover, A is integral and finitely generated over C, hence it is finite over C. But then $A^G \subseteq A$ is also finite over C, hence finitely generated over R. Moreover, the fact that A is finite over C implies that A is finite over A^G as well. $\square$

Proposition 7. *Let X be a scheme over a ring R, G a finite group acting on X over R, in such a way that the conditions of corollary 2 are satisfied. If R' is a flat R-algebra, then $G \backslash (X \times_{\operatorname{Spec} R} \operatorname{Spec} R') \cong (G \backslash X) \times_{\operatorname{Spec} R} \operatorname{Spec} R'$.*[1]

Proof. For $X = \operatorname{Spec} A$ affine, consider the exact sequence of R-modules:

$$0 \to A^G \to A \to \prod_{g \in G} A$$
$$a \mapsto (a \cdot g - a)_{g \in G}.$$

[1] Note that the action of G on X pulls back to an action on $X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$.

By flatness, the sequence remains exact after tensoring with $\otimes_R R'$, i.e. $A^G \otimes_R R' \cong (A \otimes_R R')^G$ (the G-action on $A \otimes_R R'$ is given by $g \otimes id_{R'}$, for $g \in G$).

In the general case, reduce to X affine as before. □

Symmetric Powers of curves

As an example of the above construction, let k be a field and C a smooth projective geometrically irreducible curve over k. On the product $C^n = C \times_k \cdots \times_k C$, we have an action of the symmetric group S_n, given by permutations of the components ($\sigma \in S_n$ acts via $(pr_{\sigma(1)}, \dots, pr_{\sigma(n)})\colon C^n \to C^n$). As C is projective of finite type over k, the preceding results ensure that $C^{(n)} := S_n \backslash C^n$ is a scheme of finite type over k, called the *n-th symmetric power* of C.

Example 8. We have $(\mathbb{P}^1_k)^{(n)} \cong \mathbb{P}^n_k$.

The idea behind this is that for any k-algebra A we can construct a map $(\mathbb{P}^1_k)^n(A) = (\mathbb{P}^1_k(A))^n \to \mathbb{P}^n_k(A)$ in the following way. For any point $(P_1, \dots, P_n) \in (\mathbb{P}^1_k(A))^n$ we can write, locally on $\operatorname{Spec} A$, $P_i = [a_i, b_i]$ (homogeneous coordinates in A), for $i = 1, \dots, n$.[2] We can then map $(P_1, \dots, P_n)$ to the point of $\mathbb{P}^n_k(A)$ corresponding to the homogeneous coordinates given by the coefficients of the polynomial $\prod_{i=1}^n (b_i X - a_i Y) \in A[X,Y]_n$ (again locally; one has indeed to check that these pieces glue to a map $\operatorname{Spec} A \to \mathbb{P}^n_k$). By Yoneda lemma and the fact that the functor of points of a scheme is determined by its values on affine schemes, this yields a map $(\mathbb{P}^1_k)^n \to \mathbb{P}^n_k$. By construction, this map is S_n-invariant and hence yields a morphism $(\mathbb{P}^1_k)^{(n)} \to \mathbb{P}^n_k$. Then one proves that this is an isomorphism.

Proposition 9. *With notation as above, $C^{(n)}$ is smooth, projective, of dimension n.*

Proof (idea). $C^{(n)}$ is smooth if and only if $(C^{(n)})_{\bar{k}} = C^{(n)} \times_k \operatorname{Spec} \bar{k}$ is regular ($\bar{k}$ an algebraic closure of k). But $k \to \bar{k}$ is flat, hence $(C^{(n)})_{\bar{k}} \cong (C_{\bar{k}})^{(n)}$ by proposition 7. Moreover, regularity can be checked on closed points. However, since $\bar{k}$ is algebraically closed, the closed points of $(C_{\bar{k}})^{(n)}$ are exactely the $\bar{k}$-valued points $(C_{\bar{k}})^{(n)}(\bar{k}) \cong S_n \backslash (C_{\bar{k}})^n(\bar{k})$, where the last isomorphism is because $(C_{\bar{k}})^{(n)} = S_n \backslash (C_{\bar{k}})^n$ is also a topological quotient. Now, if $x = (P_1, \dots, P_1, \dots, P_r, \dots, P_r) \in (C_{\bar{k}})^{(n)}(\bar{k})$, with P_i occurring $m_i \geq 1$ times $(i = 1, \dots, r)$, we have $\widehat{\mathcal{O}_{C_{\bar{k}}, P_i}} \cong \bar{k}[\![t_i]\!]$ and $\widehat{\mathcal{O}_{(C_{\bar{k}})^n, x}} \cong \bar{k}[\![t_{1,1}, \dots, t_{1,m_1}, \dots, t_{r,1}, \dots, t_{r,m_r}]\!]$. One then proves that $\widehat{\mathcal{O}_{(C_{\bar{k}})^{(n)}, y}} \cong \bar{k}[\![e_{1,1}, \dots, e_{1,m_1}, \dots, e_{r,1}, \dots, e_{r,m_r}]\!]$, where y is the image of x in $S_n \backslash (C_{\bar{k}})^n = (C_{\bar{k}})^{(n)}$ and $e_{i,1}, \dots, e_{i,m_i}$ are the elementary symmetric polynomials in $t_{i,1}, \dots, t_{i,m_i}$ $(i = 1, \dots, r)$. Thus, $\widehat{\mathcal{O}_{(C_{\bar{k}})^{(n)}, y}}$ is regular, hence so is $\mathcal{O}_{(C_{\bar{k}})^{(n)}, y}$ (see [3, 11.24]).

The rest of the proposition is proved using the previous example. □

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[2] A map $\operatorname{Spec} A \to \mathbb{P}^1_k$ does not necessarily factor through an affine component of $\mathbb{P}^1_k$, but it does locally on $\operatorname{Spec} A$. Each local piece of our map will then correspond to a k-linear homomorhism $k[x_0/x_1] \to C$ or $k[x_1/x_0] \to C$, with C a localization of A, i.e. to homogeneous coordinates $[c_0 : c_1]$ in C.

Extra

Hint. For $y \in Y$ and $x \in X$ mapping to y, there is a natural map $\mathcal{O}_{Y,y} \to (q_*\mathcal{O}_X)_y \to \mathcal{O}_{X,x}$, which is in fact the map induced by q on the stalks. Let $\mathfrak{m}^c$ be the contraction of the maximal ideal $\mathfrak{m}$ of $O_{X,x}$ in $O_{Y,y}$. If $f \in \mathcal{O}_{Y,y} \setminus \mathfrak{m}^c$, then its image in the residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}$ is non-zero. Say f comes from a section $f \in \mathcal{O}_Y(V) = O_X(q^{-1}(V))^G$ for some $V \subseteq Y$ open. Then $D(f) \subseteq q^{-1}(V)$ is an open neighbourhood of x and, since f is G-invariant, it is G-stable, so $D(f) = q^{-1}(V')$, $V' = q(D(f)) \subseteq Y$ open. Now, $f \in \mathcal{O}_X(D(f))^\times \cap \mathcal{O}_X(D(f))^G = (\mathcal{O}_X(D(f))^G)^\times = \mathcal{O}_Y(V')^\times$, hence $f \in \mathcal{O}_{Y,y}^\times$. Thus, $\mathcal{O}_{Y,y}$ is local with maximal ideal $\mathfrak{m}^c$ and the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings.

Comment. These notes are based on [1] and claim no originality.

References

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[3] M. F. Atiyah, I. G. MacDonald, *Introduction to Commutative Algebra*